Journey to the Heart of Music

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Example P
The II–V–I Cadence

MUTABLE NUMBER ANALYSIS

The full, perfect or authentic cadence has featured in almost every chapter of *Journey to the Heart of Music* and, as shown in Chapter 6 Figure 6.14, the nesting of two full cadences within another more fundamental full cadence produces the iconic ‘dominant-of-the-dominant’ ii–V–I close. Here in Example P the nature of this most familiar of chord progressions is further investigated, and, hopefully by means of a MOS analysis it can be demonstrated that the many different flavors of the supertonic chord used in this cadential formula – major, minor and diminished – may be accounted for by different arrangements (aggregations) of one foundational structure. To this end the main body of the document is divided into sections devoted to pairs of aggregations of elements which give rise to the supertonic and dominant chords in this cadential formula, that is, groups of 1, 2, 3, 4, 5, 6, 7, 8, 9 and 10.

In the following Figures, for reasons of brevity and legibility, the harmonic essence of the various forms of the II–V–I progression have been abstracted into the treble clef, and, the underlying modulating oscillatory system structure placed below this in the bass clef. This arrangement allows the fundamental frequency of the entire system to be set at the relatively high level of 16 hertz. (Naturally, were these analyses to cover the music spread over a more normal pitch range, as in the matching examples from chorales set by J.S. Bach, the fundamental frequency of the system would need to be set at a lower level, which in turn would yield larger mutable number values: integral multiples of the values given here.) Below the staves a full MOS analysis is presented consisting of *fundamental series*, *nested series* and *aggregated series*. The topmost aggregated series, which contains the harmonic gist of the objective chords, is printed in predominantly black type, e.g. hn*. Supporting this ‘sunlit harmonic canopy’ lies the ‘branches’ of the middle level nested series and the ‘trunk’ of the fundamental series, both printed in gray type, e.g. hn and Hn respectively. The topmost aggregated series are formed from groupings of elements – aggregations – of the nested series, while the fundamental series emerges from the modulatory steps trod by the nested series. And it should be borne in mind that though the whole apparatus of a MOS model involves harmonic series of considerable extent, the crucial conjunction frequencies lying at or above the chords are normally low order partials of the sounding notes and therefore are objectively present and readily accessible to human ears.

Toward the bottom of the Figures the modulation exchanges of the aggregated series are detailed in black oblique type, for example *(n groups of m)*, whereas the exchanges of the nested series are shown in upright gray type, for example (n x <written-number>). Finally, at the bottom of the Figures, the mutable numbers represented by the objective chords (extended out to their enfolding nested harmonic series) are given with their decimal equivalent below. These values are generally *fecund numbers* – as defined in Chapter 1, page 19.

Groups of Three and Four

The all-major first example of the II–V–I cadential formula – Figure P.1a – is rather less common than its minor sibling ii–V–I; and, somewhat perhaps less final in its effect as it incorporates the leading note to the dominant (here F#) in the approach to the tonic full cadence. Notwithstanding these features, it probably should still be considered the generic form of the progression, as it produces the simplest or most straightforward expression of the structure. This basic structure consists of the nesting of two full cadences within another more fundamental full cadence, or in other words, the nesting of two successive sesquitertia 3:4 exchanges within a single more fundamental sesquitertia exchange. The distinguishing feature of the MOS interpretation of this variant of the cadential formula is that chord II is unambiguously aggregated in groups of three elements of the nested series. That is, all the notes of chord II fall upon the aggregated series.

At the surface level in Figure P.1a, first comes the major dominant-of-the-dominant resolving to the dominant chord, either of which may include a seventh, followed by the dominant or dominant-seventh resolving to the tonic. These two successive (secondary) sesquitertia 3:4 exchanges in the topmost aggregated series invoke or induce the same modulation exchange at the level of the fundamental series. Once the opening dominant-of-the-dominant exchange has taken place in the aggregated/nested pair of series – releasing tension by moving to the dominant chord – the ‘system’ is left in a state where the lower level nested/fundamental pairing can itself likewise release tension through employing same proportioned (sixfold) sesquitertia 3:4 exchange, without causing any disturbance to the aggregated series, i.e. the dominant chord. (The aggregated series consists of groups of nested elements – aggregations in the nested series – the makeup of these aggregations, whether groups of two, three, four, etc. is immaterial, only the number of groups is significant.) Overall this operation is rather like a domino effect of exchanges rippling down from the surface level, inducing the underlying structure to reorder itself.

After the more energetic arrangement of ‘groups of three’ fundamental elements nesting ‘groups of four’ nested elements has been inverted – so that the larger grouping of four elements is transferred to the fundamental series, the system has reached equilibrium and the process of addition can commence. The ‘unchanged’ aggregated series, formerly in groups of four nested elements (each one of which was worth three fundamental elements) is now in groups of three nested elements (each worth four fundamental elements), and, by addition in groups of three nested elements, the aggregated series grows upward to encompass the notes of the seventh (F-h7*) and octave (G-h8*). The upper octave, G-h8* in the dominant chord, marks again a conjunction or point of relaxation (as did the upper D in the previous chord II). Set at this point, the system is primed to execute another secondary sesquitertia 3:4 exchange in the topmost aggregated series. As with chord II, eight groups of three nested elements are exchanged for six groups of four nested elements, so making the final step of the full cadence from dominant to tonic chord.

P.3 – THE II-V-I CADENCE

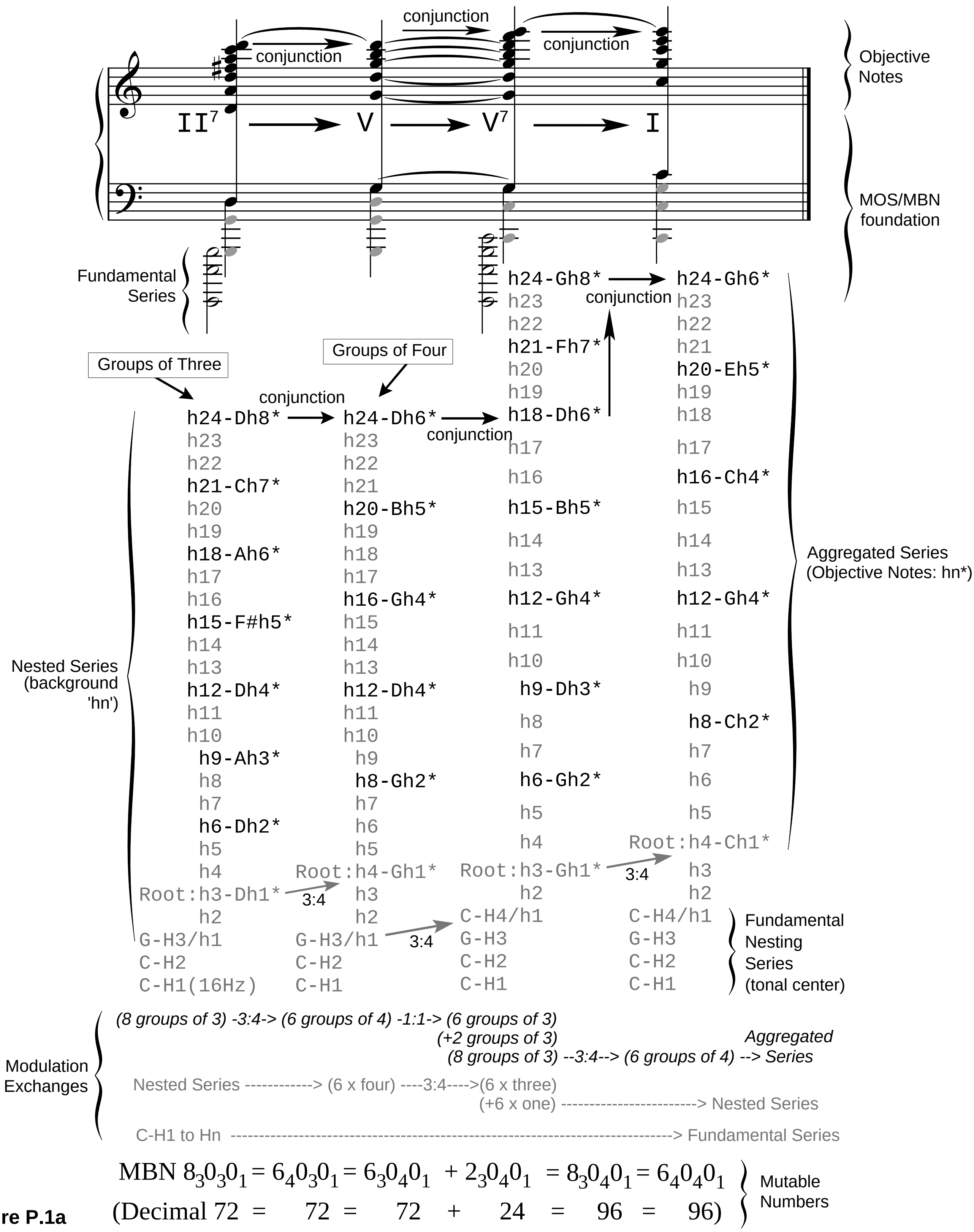


Figure P.1a

Throughout the II–V–I cadential formula the fundamental series – which represents the sense and stability of 'the key' – remains unchanged except for the addition of more elements. It is an extended series based on C-H1(16Hz) ranging up to 'Hn'; and, towards the bottom of Figure P.1a in the section marked

‘Modulation Exchanges’ it is shown in gray type. Above the fundamental series in this section the modulation exchanges of the ‘Nested Series’ and ‘Aggregated Series’ are given – as described previously. Finally, at the bottom of Figure P.1a the whole paraphernalia of the MOS analysis is summarised in a single line of mutable number arithmetic, with decimal translation in brackets below. Ultimately, this iconic tonal progression, the whole II–V–I cadential formula, boils down to a sum: 72 + 24 = 96.

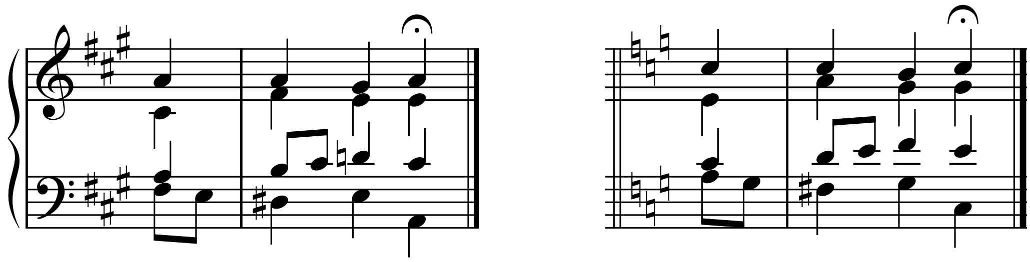
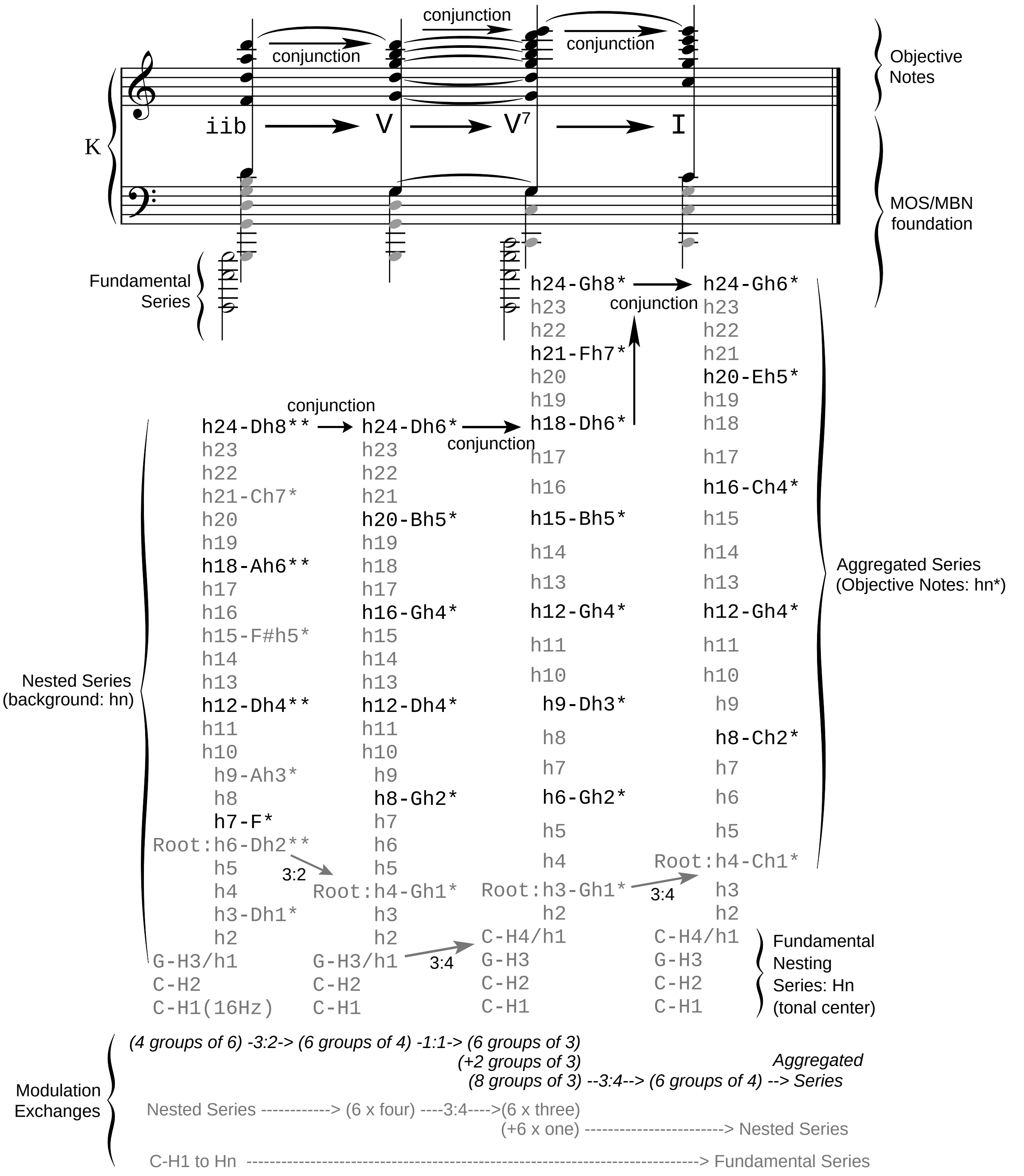


Figure P.1b The final cadence from *Ach Gott, wie manches Herzelied*, Edition Breitkopf Nr. 3765, Chorale No. 8; on the left in the original key and transposed to C major on the right. (See also note 2 below.)

Groups of Six and Four

In contrast to the all-major II–V–I formula, the probably more common minor supertonic form of the cadence, ii–V–I, does not have an aggregated series which totally, and unambiguously, falls into groups in three elements (of the nested series). From the MOS model perspective, the reason for this is twofold: firstly, minor chords gain their minor quality by planting one foot (here F-natural), the minor-third, outside the aggregated series, and conversely, in so doing the minor chord naturally passes over the major-third (here F#h15). That is, the notes of minor chords are shared between the aggregated and nested series, which leaves gaps in the aggregated series, so tending to ‘space-out’ the objective notes across larger groups of nested elements than the equivalent major chord. Thus in Figure P.2a the objective notes held within the aggregated series (D-h12, A-h18 and D-h24) lie on multiples of groups of three – groups of six.

Also, most often in the ii–V–I version of the cadential formula, this blurring of the underlying ‘threeness’ is further emphasised by the minor supertonic chord also being placed in its first inversion, so that the minor-third (F) is the bass note[1]. Looking at Figure P.2a, this distinctive bass note can be seen sitting at element F-h7 in the nested series, and, although the harmonic gist of the chord – the D-rootedness of ii – arises from aggregations of six (2×3) elements of the nested series, similarly to the major II form, there is a certain weakening of effect introduced by such a prominently placed bass note, F-h7, running contrary to groups of three (or six). The bundling up of the basic groups of three, into groups of six, is indicated by a double asterisk ‘hn**’. The structure of the ii–V–I variant is essentially the same as for the all-major formula, with the exception of the downward sesquialtera 3:2 (6:4) exchange replacing the upward sesquitertia 3:4 exchange of the major supertonic chord – and so there is no need to describe the whole sequence again in detail. The mutable numbers represented by the minor ii–V–I cadential formula are the same as in the major form: 72 + 24 = 96.



$$\text{MBN } 4_6 0_3 0_1 = 6_4 0_3 0_1 = 6_3 0_4 0_1 \; + 2_3 0_4 0_1 \; = 8_3 0_4 0_1 = 6_4 0_4 0_1$$

(Decimal 72 = 72 = 72 + 24 = 96 = 96)

Mutable Numbers

Figure P.2a

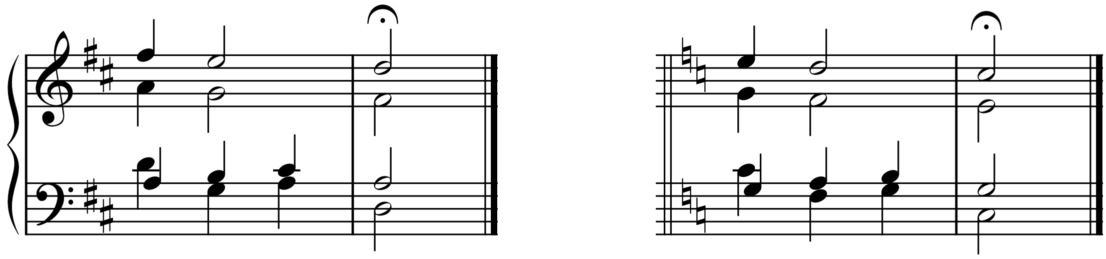


Figure P.2b The final cadence from *Herr Gott, dich loben alle wir*, Edition Breitkopf Nr. 3765, Chorale No. 132.

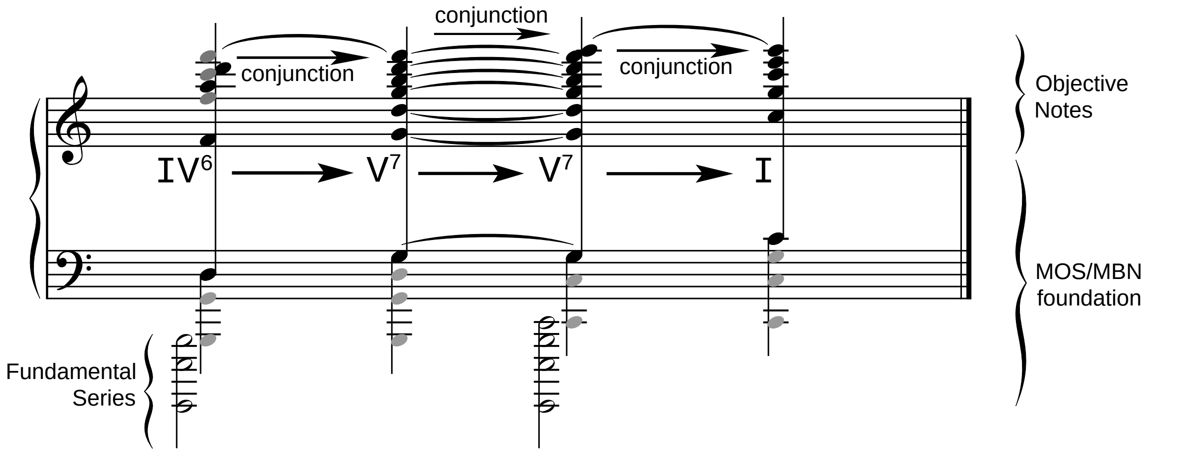
Groups of Seven and Eight

Franko Goldman in Chapter 3 of his book *Harmony in Western Music*[2] observes that the all-major construction of the cadential formula (II–V–I) should be considered the more powerful, as the introduction of F#, the leading note of the dominant key, provides an extra impetus to resolution, and particularly so when it occurs in a seventh chord, with the tritone F# to C resolving forcefully to the third G to B. And, in the previous section the ‘spacing-out’ of notes assigned to the aggregated series in the minor supertonic version of the formula lends support to this view, in that the tighter spacing of groups of three in the major supertonic signals greater tension. Notwithstanding this observation, the MOS model view of the two formulas – i.e. major or minor supertonic – allows a subtly different interpretation of the minor supertonic chord to emerge. A perspective that gives this variant its own impelling dissonance and resolution, though a dissonance of a somewhat more gentle nature. Also, the MOS perspective perhaps sheds some light on Goldman's oft stated observation concerning the (alleged) historical misinterpretation of chord iib as chord IV.

A significant feature of the minor supertonic cadential formula is the frequency with which the supertonic chord occurs in first inversion, i.e. iib–V–I. Indeed, there appears to be a marked preference for this disposition, a preference which goes beyond what might be explained by random variation, coincidence or a particular historical style. Once the pattern had become firmly established in the baroque period, it remained a fixture thereafter. And further, there does appear to be some connection between the rise of instrumental music and the predilection for the supertonic in first inversion, though of course the formula was not unknown or unused beforehand.

The MOS approach offers a possible explanation for the marked preference for the first inversion form of the minor supertonic in the dominant-of-the-dominant cadence in terms of an alternative grouping for the chord’s aggregated series. This alternative grouping of seven is illustrated in Figure P.3a.

The standard disposition of the iib–V–I usually places the minor-third (here F-h7) as a firm low bass note, and such a disposition in instrumental music would normally produce a sound rich in higher partials. Indeed, most of the sonic energy generated in playing this note on an instrument like for example, bassoon, ’cello or keyboard, would reside in the upper harmonics (h2, h3, h4, h5, etc.) rather than the fundamental tone. From the ear’s appreciation of these objective stimuli the minor supertonic chord in first inversion, iib, might sound not unlike a root position subdominant with added sixth – IV^6. That is, the sounding treble notes, with proportionately more energy in their fundamentals, plus the strong partials arising from the bass note, taken in combination, could produce to some degree the effect of a major chord of the added sixth built on the bass note – the subdominant degree.



conjunction → conjunction → conjunction

Nested Series (background: hn)			Aggregated Series (Objective Notes: hn*)
h56-Fh8*	h56-Fh14**	h42-Fh14**	
h49*	h48**	h24-Gh8**	h24-Gh6**
h42*	h40**	h23	h23
h35*	h32**	h22	h22
h28-Fh4*	h28-Fh7*	h21-Fh7*	h21
h26	h26	h20	h20-Eh5*
---	---	h19	h19
h24-D	h24-Dh6**	h18-Dh6**	h18
h23	h23	h17	h17
h22	h22	h16	h16-Ch4**
h21-Ch3*	h21	h15-Bh5*	h15
h20	h20-Bh5*	h14	h14
h19	h19	h13	h13
h18-A	h18	h12-Gh4**	h12-Gh4*
h17	h17	h11	h11
h16	h16-Gh4**	h10	h10
h15	h15	h9-Dh3*	h9
h14-Fh2*	h14	h8	Root:h8-Ch2**
h13	h13	h7	h7
h12	h12-Dh4*	Root:h6-Gh2**	h6
h11	h11	h5	h5
h10	h10	h4	Root:h4-Ch1*
h9	h9	Root:h3-Gh1*	h3
h8	Root:h8-Gh2**	h2	h2
Root:h7-Fh1*	h7	C-H4/h1	C-H4/h1
h6	h6	G-H3	G-H3
h5	h5	C-H2	C-H2
h4	Root:h4-Gh1*	C-H1	C-H1
Root:h3-Dh1	h3		
h2	h2		
G-H3/h1	G-H3/h1		
C-H2	C-H2		
C-H1(16Hz)	C-H1		

7:8 3:4 3:4

Fundamental Nesting Series: Hn (tonal center)

Modulation Exchanges

(8 groups of 7) -7:8-> (7 groups of 8) -1:1-> (7 groups of 6)
(−3 groups of 6)
(4 groups of 6) --3:4--> (3 groups of 8) --> Aggregated Series

Nested Series ------------> (14 x four) ----3:4---->(14 x three)
(−18 x one) -----------------------> Nested Series

C-H1 to Hn ---> Fundamental Series

$$\text{MBN } 8_7 0_3 0_1 = 7_8 0_3 0_1 = 7_6 0_4 0_1 - 3_6 0_4 0_1 = 4_6 0_4 0_1 = 3_8 0_4 0_1$$

$$(\text{Decimal } 168 = 168 = 168 - 72 = 96 = 96)$$

Mutable Numbers

Figure P.3a

The effect is illustrated in Figure P.4. However the situation is not exclusively that of one interpretation or the other, chord iib or IV^6. Depending on the relative strengths of the written notes and partials, the inclination, experience and attention of the listener, etc.; the sense impression which arises from this chord might be one or other of the chords, but more likely a mixture of the two. An attractive harmonic alloy that hints at chord IV^6 within the sound mass of the minor supertonic.

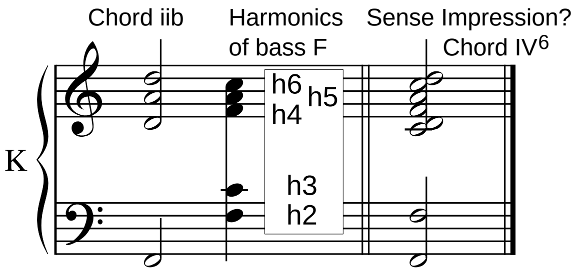


Figure P.4 The strong partials (illustrated in quarternotes) generated by the bass note of a first inversion supertonic chord may give the ear the impression of a chord of the added sixth built on the subdominant, F.

In Figure P.3a the essence of chord IV^6 has been distilled, through the grouping into sevens of the underlying elements of the nested series. These relatively large groups in the aggregated series dictate a rather higher conjunction (F-h56) and thereby a larger mutable number value emerges from the exchange. Notwithstanding, the basic structure of the cadence remains intact, however, looking at Figure P.3a it can be seen that the groups of seven naturally graduate to groups of eight (2×4) for the dominant chord and groups of six (2×3) in the tonic chord. To account for this doubling of groupings over the previous examples, the harmonics of the aggregated series are again marked with either one or two asterisks – h* and h**.

Thus for an exclusive chord IV^6 reading the mutable number sum is amended to $168 - 72 = 96$. However, it is probably better to envisage the progression as containing both interpretations simultaneously and in recognising this feature of harmonic alloy or allotropy some basis for the variable interpretation of chord iib/IV^6, as noted by Goldman, may be found.

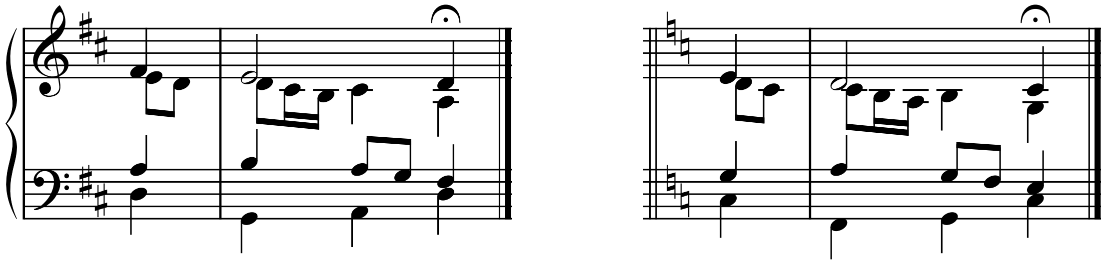


Figure P.3b The final cadence from *Wie schön leuchtet der Morgenstern*, Edition Breitkopf Nr. 3765, Chorale No. 377.

Groups of Three and Five

The II–V–I cadential formula may also be found in the minor key, often with the supertonic in the form of a diminished chord of one type or another. In Figure P.5a the half-diminished triad is illustrated, chord ii^{7}b dim (F#ACE) – a diminished triad built on the second scale degree, in first inversion, with added minor-seventh. In this example a little more context is provided by a lead-in E-minor chord and, as all the examples are built upon the fundamental frequency C-H1, this particular example is transposed to the key of E minor.

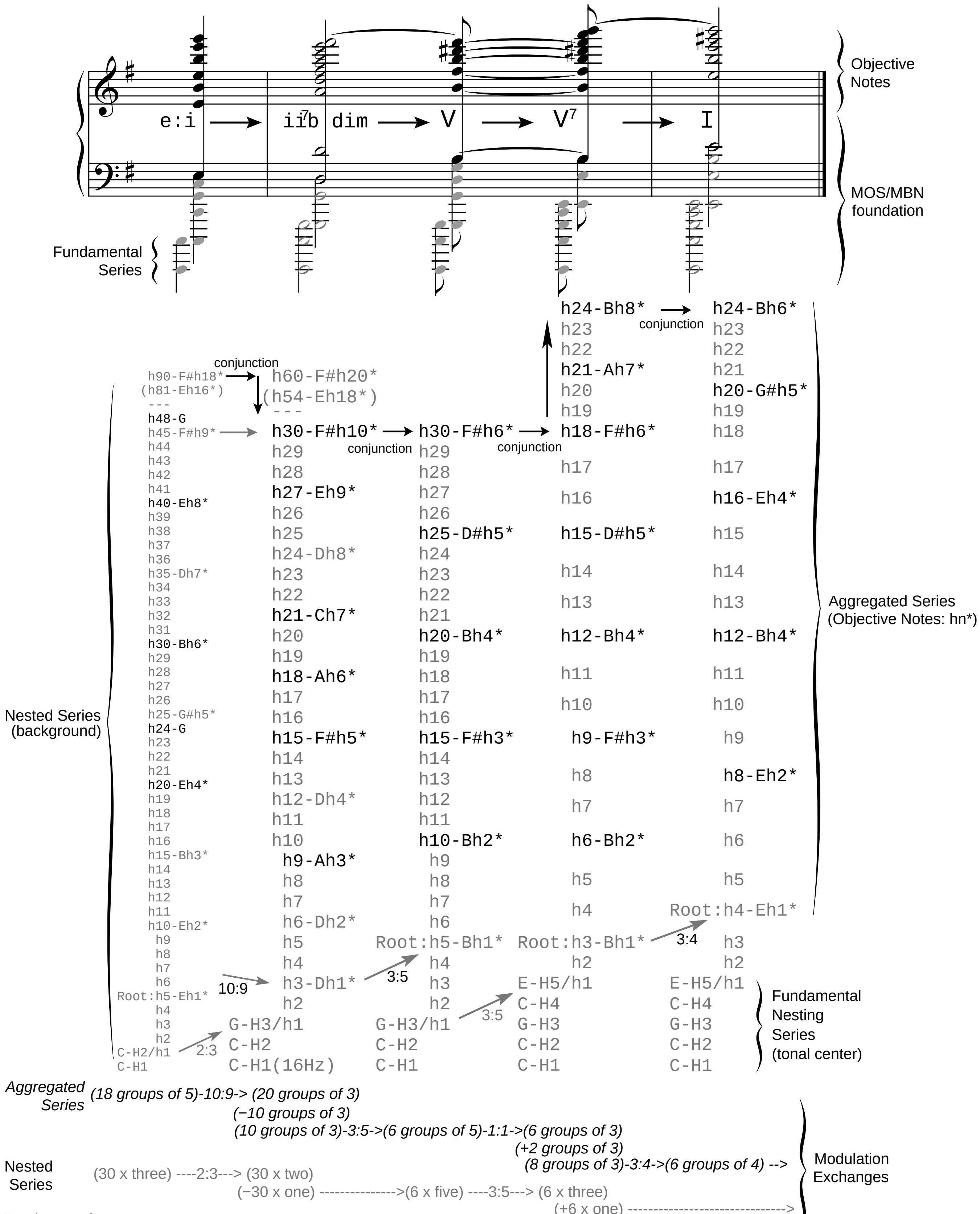


Figure P.5a

MBN $18_5 0_2 0_1 = 20_3 0_3 0_1 - 10_3 0_3 0_1 = 10_3 0_3 0_1 = 6_5 0_3 0_1 = 6_3 0_5 0_1 + 2_3 0_5 0_1 = 8_3 0_5 0_1 = 6_4 0_5 0_1$ Mutable

(Decimal 180 = 180 − 90 = 90 = 90 = 90 + 30 = 120 = 120) Numbers

Diminished triads only have nominal roots. That is, the combination of frequency relationships – here portrayed as part of an aggregated series: F#h5*, A-h6* and C-h7* – does not define an aurally identifiable root because there are no common factors shared between 5, 2×3 and 7. The ear has no common element to grasp on to and so cannot make sense of, or unify, the notes as deriving from one fundamental frequency. Notwithstanding, from a MOS view point, there still is a root, in that there is a fundamental – D-h1*. In the case of the regular diminished triad, the arrangement of notes F#AC could be construed as a seventh chord lacking the root note D and this would, as in the generic example Figure P.1a, result in the underlying nested series being aggregated in groups of three. Similarly in this example, the addition of E-h9* – the minor-seventh – to the basic diminished chord insinuates some degree of 'threeness' into the underlying nested series through the sharing of the factor three between E-h9*(h27) and A-h6*(h18). Yet also, the addition of E-h9* equally runs counter to the emergence of D-h3 as a palpable root, through the dissonance arising between D and E. The overall aural impression is uncertain, and the basically unrooted nature of the diminished chord effectively allows any aggregation of nested series' elements to be postulated. A choice of groups of three is reasonable, and can be further supported as the natural outcome of the transition from, and to, the groupings in the adjacent chords/series. Essentially the disposition of the notes, with their shared conjunction at least allows for, and perhaps encourages, 'the ear' to find an orderly succession from the E-minor chord, through the diminished triad on the supertonic degree and on to the succeeding B-major chord.

The actual exchanges themselves in the minor key are in part different from the generic II–V–I formula in that the initial step is extended from 3:4 to the ratio 3:5. Prior to this, leading into the formula, the E-minor chord is embedded within an aggregated series of groups of five nested elements, built upon the fundamental series consisting of C-H1 and C-H2. The lead-in exchange consists of the nested series stepping from C-H2 up to G-H3 of the fundamental series, while the aggregated series moves simultaneously by means of a reverse or downward sesquinona 10:9 exchange from E to D. This lead-in sets the system up for the cadential formula in the minor mode (ii^{7}dim–V^7–I) with a picardy major-third in the final tonic chord. Although it is not justifiable to call D-h3 in the nested series the 'root' of the half-diminished supertonic chord (i.e. D-h1*), it is possible to aggregate the underlying nested elements in groups of three, as it is consistent with the factor 3 appearing in the objective notes: A-h9, F#h15, C-h21 and E-h27. And, although the conjunction F#h90=F#h60 is the highest used so far, at the level of the musical sound it is only the third harmonic of objective note B-h30 and the second harmonic of F#h30, in the E-minor and F#half-dim. chords respectively.

Once the cadential formula is reached, the conjunction value halves to F#h30 – the highest written note on the treble clef in Figure P.5a – and the progression moves forward as a nesting of one 3:5 exchange within another. The (nominal) groups of three in the supertonic half-diminished chord are exchanged for groups of five in the succeeding dominant chord, which in turn trades its groups of five back into groups of three when the fundamental series steps up by 3:5 from C-H3 to E-H5. The process and symmetry are much the same as in the generic form of the cadence (Figure P.1a) excepting that the ratio of 3:4 is replaced by 3:5 and the mutable numbers are somewhat larger. Also, although not illustrated, the use of groups of five and three in the dominant's aggregated series opens the way for the chord to take the minor form – with D-h12 moving to D-h7 (or their multiples) providing the minor-third component. This minor reading for the dominant chord would not alter the math of the exchanges, which present the essential sum: 90 + 30 = 120.

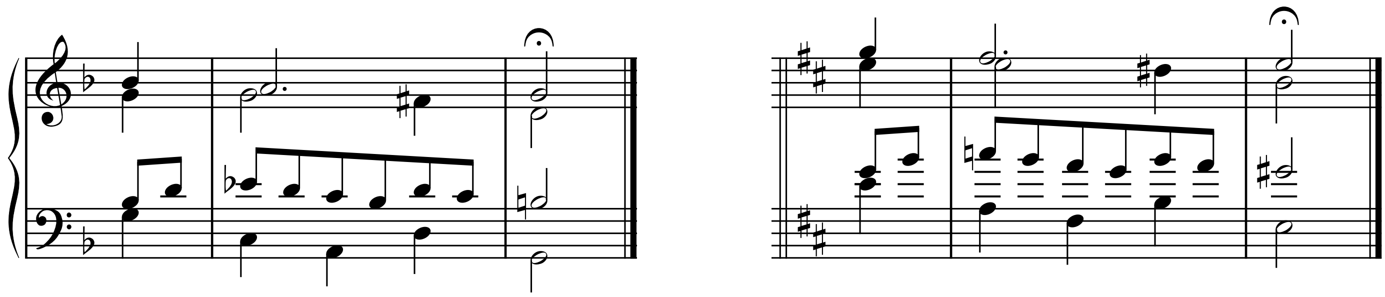


Figure P.4b The final cadence from *Hilf, Gott, dass mir's gelinge*, Edition Breitkopf Nr. 3765, Chorale No. 172.

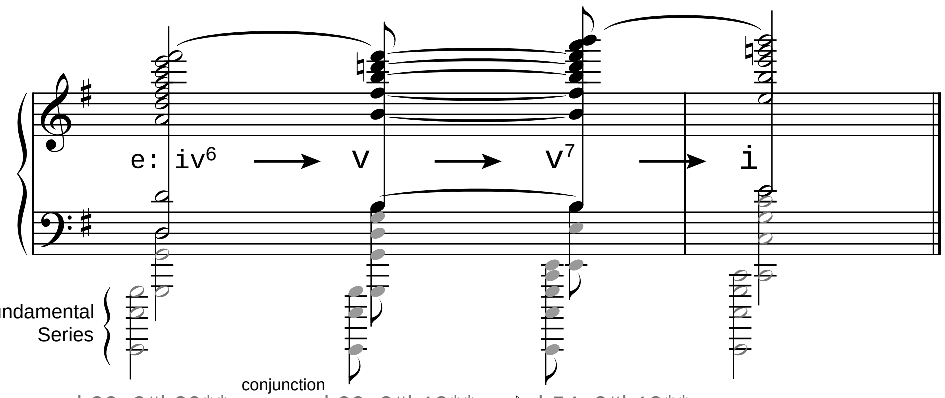
Groups of Nine and Ten

As above with the minor supertonic chord in the cadential formula, the half-diminished form allows an alternative reading – by means of a different aggregation of the elements in the underlying nested series. The combined presence of the minor-seventh E-h9** in the chord together with the first inversion disposition of the chord's notes – so that A-h3** is placed in the bass – could again favor the emergence of a chord of the added sixth, built on the subdominant scale degree. However, this time it is the minor form, the minor chord of the added sixth – iv^6.

The MOS analysis of the iv^6 chord progression is shown in Figure P.6a[3]. Additionally in this figure, the dominant and tonic have been changed into their minor forms, further illustrating the flexibility of the general structure. The alternative rendering of chord iib as iv^6 is achieved through a tripling of the aggregations of the nested series' elements into groups of nine, and thereafter a doubling of the groupings. The conjunction between this minor chord of the added sixth and the succeeding minor dominant chord is positioned even higher, C#h90, than in the previous example. Yet still this conjunction frequency is only the third harmonic of the objective note F#h30 and the fifth harmonic of A-h18, and so falls well within the ear's normal range, though there would inevitably be some interference arising from the proximity of the harmonics generated by the other notes in the chord. At the C#h90 conjunction the iv^6 chord's ten groups of nine nested elements (root A-h9) are exchanged for the dominant's nine groups of ten elements (root B-h10).

Once the aggregated/nested pair of series have moved forward to the dominant chord, the nested/fundamental pairing is then able to execute the lower level adjustment of transferring the more energetic grouping of five elements from the nested series down to the fundamental series. However, before this exchange takes place the conjunction frequency can be reduced by subtracting six (now superfluous) groups of ten. The relatively high conjunction of C#h90 was required to make the crossing to the dominant chord, and this conjunction still remains operable in the new dominant series (i.e. C#h54 shown in Figure P.6a), but where and when lower conjunctions become available, the rule is to choose the lowest frequency. (Implicit in the choice of lowest conjunction is the understanding that all integer multiples of this frequency will also be conjunctions.)

Now returning to the low level adjustment which is made through a modulation exchange of 3:5. The nested series swaps its groups of five for the fundamental's groups of three. The relatively more energetic grouping of five is transferred down to the lowest position in the system, thus reducing the overall energy load of the whole system. Throughout this episode, the top level aggregated series remains constant and is shown as making a una 1:1 exchange of three groups of ten for three groups of six.



Objective Notes

MOS/MBN foundation

Fundamental Series

iv6 (Nested Series, background: hn)

h90-C#h30** —conjunction→ h90-C#h18**
h81-Bh27**
h72-Ah24**
h63-Gh21**
h54-Eh18**
h45-C#h15**
h36-Ah12**
h33-G#h11*
h32
h30-F#h10*
h29
h28
h27-Eh9**
h26
h25
h24-Dh8*
h23
h22
h21-Ch7*
h20
h19
h18-Ah6**
h17
h16
h15-F#h5*
h14
h13
h12-Dh4*
h11
h10
Root:h9-Ah3** —9:10→
h8
h7
h6-Dh2*
h5
h4
Root:h3-Dh1*
h2
G-H3/h1
C-H2
C-H1(16Hz)

V

h90-C#h18**
h80-Bh16**
h70-Ah14**
h60-F#h12**
h50-D#h10**
h40-Bh8**
h35-Ah7*
h33
---
h30-F#h6** —conjunction→
h29
h28
h27
h26
h25-D#h5*
h24-D
h23
h22
h21
h20-Bh4**
h19
h18
h17
h16
h15-F#h3*
h14
h13
h12
h11
Root:h10-Bh2** —1:1→
h9
h8
h7
h6
Root:h5-Bh1*
h4
h3
h2
G-H3/h1 —3:5→
C-H2
C-H1

V7

h54-C#h18**
h24-Bh8** —conjunction→
---
h21-Ah7*
h20
h19
h18-F#h6**
h17
h16
h15-D#h5*
h14-D
h13
h12-Bh4**
h11
h10
h9-F#h3*
h8
h7
h6-Bh2** —3:4→
h5
h4
Root:h3-Bh1*
h2
E-H5/h1 —5:4→
C-H4
G-H3
C-H2
C-H1

i (Aggregated Series, Objective Notes: hn*)

h30-Bh6**
h27
---
h25-G#h5*
h24-G
h23
h22
h21
h20-Eh4**
h19
h18
h17
h16
h15-Bh3*
h14
h13
h12
h11
h10-Eh2**
h9
h8
h7
h6
Root:h5-Eh1*
h4
h3
h2
C-H4/h1
G-H3
C-H2
C-H1

Fundamental Nesting Series (tonal center)

Modulation Exchanges

Aggregated Series — *(10 groups of 9) ---9:10--> (9 groups of 10)*
(−6 groups of 10)
(3 groups of 10) ---1:1--> (3 groups of 6)
(+1 groups of 6)
(4 groups of 6) ---3:4--> (3 groups of 10) -->

Nested Series --------------------------------------> (6 x five) -------3:5------> (6 x three)
(+6 x one)
(6 x four) -------5:4------> (6 x five) ------>

Fundamental Series C-H1 to Hn -->

Mutable Numbers

MBN $10_9 0_3 0_1 = 9_{10} 0_3 0_1 - 6_{10} 0_3 0_1 = 3_{10} 0_3 0_1 = 3_6 0_5 0_1 + 1_6 0_5 0_1 = 4_6 0_5 0_1 = 3_{10} 0_4 0_1$

Figure P.6a (Decimal 270 = 270 − 180 = 90 = 90 + 30 = 120 = 120)

To some extent this low level exchange of 3:5 makes the MOS analysis more complicated than it need be, in that the chord progressions could be adequately processed with the fundamental series passing from groups of three directly to groups of four, rather than with the added passage through groups of five as shown in Figure P.6a. However, the inclusion of the extra exchange allows the example to demonstrate the minor dominant chord embedded within the underlying structure both as groups of five (ten) and groups of three (six) elements.

The final progression to the tonic minor chord involves both pairings of series – aggregated/nested and nested/fundamental – making exchanges, the former a sesquitertia 3:4 exchange and the latter a sesquiquarta 5:4 exchange. The minor status of the tonic chord, as in the dominant, is achieved through the placement of the minor-third interval in the nested series. The numbers computed by this variation of the II–V–I cadential formula is: 270 − 180 = 90 and 90 + 30 = 120.

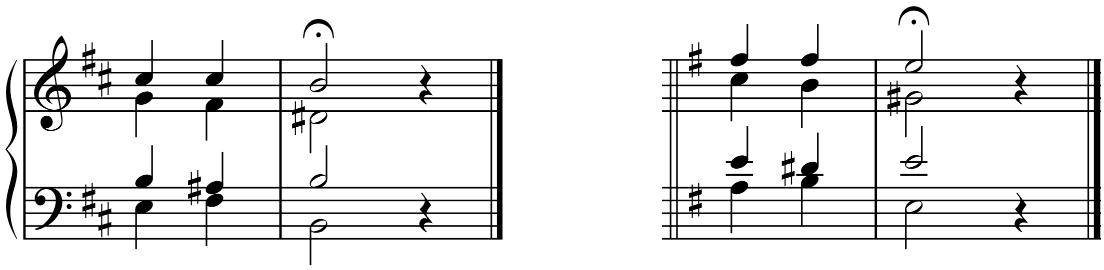
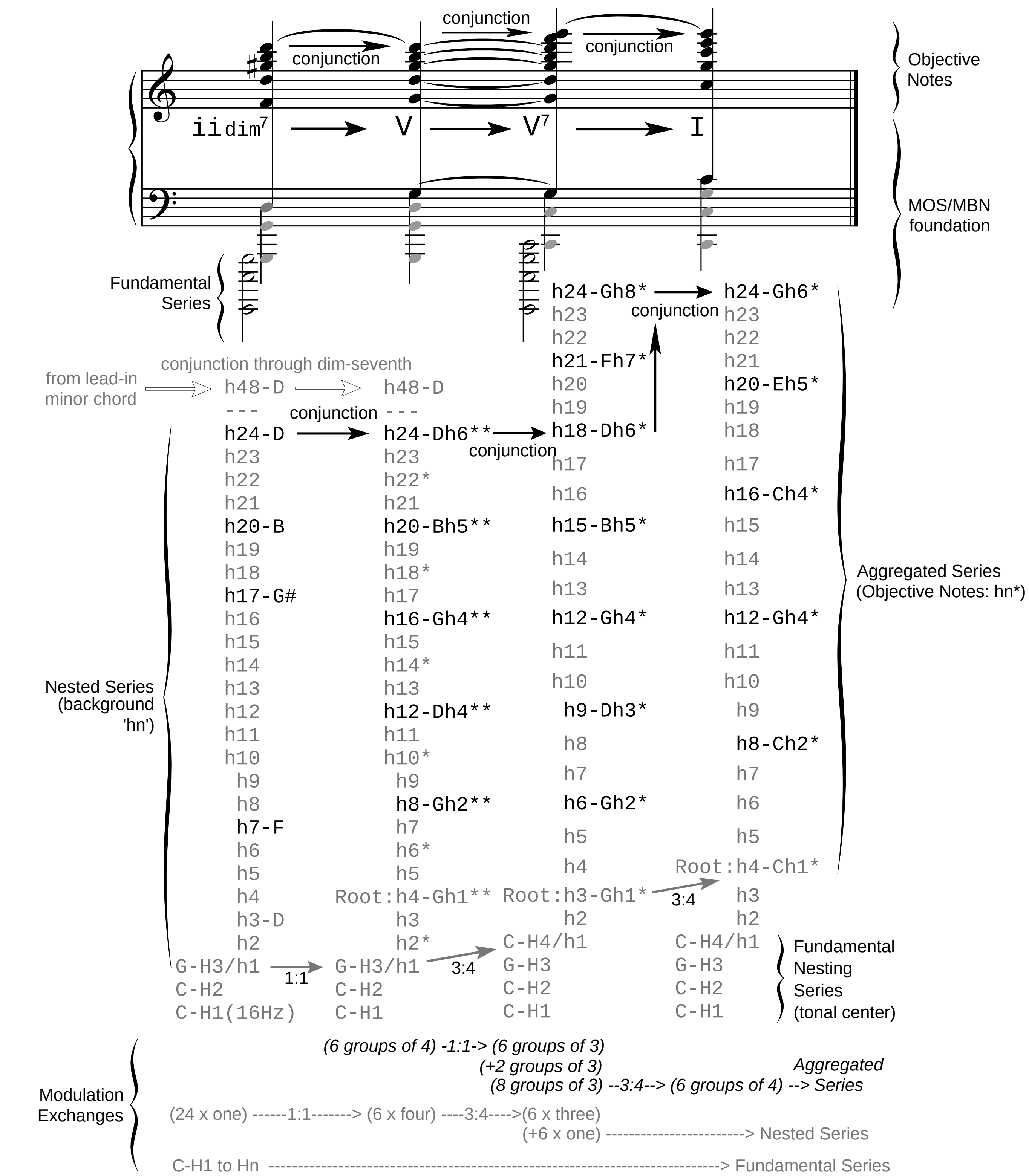


Figure P.6b The final cadence from *Allein zu dir, Herr Jesu Christ*, Edition Breitkopf Nr. 3765, Chorale No. 15.

Groups of One and Two

The closing Figures P.7a/b illustrate the situation where the top level aggregated series – the harmonic series in which normally the majority of the objective notes are embedded – dissolves into its underlying background structure, and, as the dominant and tonic are both major chords the MOS analysis has been rendered in C major. (An E-minor reading of the cadential formula is given in Figure P.7c.)

The objective notes that prevent the aggregated series from forming are those of the diminished-seventh chord. It is the awkwardness of the relationships between the notes of the diminished-seventh chord, the lack of any meaningful common factor shared between the constituent frequencies, which prevents any simple regular structure capable of apprehension by the ear emerging from the harmony. As a result of these angular relationships the ear is unable to grasp or impose a sense of rootedness upon the harmony. This rootless or structure-less effect of the diminished-seventh chord is reflected in the MOS analysis by the disappearance of the aggregated series and the consequent sweeping-up of all the awkwardly proportioned ratios in the finer-meshed net of the underlying nested series[4]. Consequently, the first mutable number shown at the bottom left of Figure P.7a has only two columns – MBN $24_3 0_1$ – and, the leading edge or outer column of this digit sequence is disproportionately enlarged compared to the other numbers.



MBN $24_3 0_1 = 6_4 0_3 0_1 = 6_3 0_4 0_1 + 2_3 0_4 0_1 = 8_3 0_4 0_1 = 6_4 0_4 0_1$

(Decimal 72 = 72 = 72 + 24 = 96 = 96)

Mutable Numbers

Figure P.7a

All the notes of the diminished-seventh find a place within the nested series: F-h7 G#h17, B-h20 and D-h24. But the prime factors of their relative frequencies (7, 17, 2×2×5, 2×2×2×3) reveal there is no common ground between them, upon which a simple and distinguishable 'abbreviated' structure can be built – i.e. a root producing aggregated series. The sole common factor amongst the note frequencies is 2, however, alone amongst the prime numbers, two (and also its higher powers) is incapable of differentiating itself and any structure based upon it, from the background within which it is embedded. The result of this situation, brought about by the diminished-seventh chord, is that 'the ear' is momentarily left adrift, rootless if not rudderless. However, again notwithstanding the ear's inability to fathom a root from the diminished-seventh, from the MOS perspective there is still a 'root', in that the extended nested series has a fundamental frequency – G-h1. But this 'root' is not available to the ear as it is buried too deeply: buried below twenty-four ratios of the harmonic series, when at most the ear can natively untangle somewhat less than half that number of ratios.

As revealed in the MOS analysis in the minor key given in Figure P.7c, by stretching out the ratios of the diminished-seventh chord a little they can be 'encouraged' into supporting an aggregated series based on groups of three: h9-A, h21-C and h30-F# all contain the factor 3. However, the fourth note in the chord h25-D# acts against a D-rooted aggregated series by extreme semitonal dissonance, so that the ear is repelled from hearing a 'groups of three' interpretation. (This is in contrast to the situation of the minor triad where the grouping of five (or three) is not contradicted by dissonance, because the minor-third tone is placed outside the critical band of dissonance on the ear's basilar membrane.)

Probably the best option available to the ear and aural understanding in this situation, is to find a common frequency that leads into the diminished chord, and, holding on to that thread, hope to find that it connects with the chord which succeeds the diminished-seventh. Which indeed is the case in this example (illustrated in Figure 7a), where the shared frequency h48-D would appear as a lower order harmonic of notes found in all three chords: the lead-in minor chord, the diminished-seventh and the succeeding dominant. That is, the third harmonic of G in the lead-in minor chord (see Figure P.7b) and the second harmonic of D in both the diminished-seventh and dominant chords. (In the transposition to an E-minor MOS analysis given in Figure P.7c this shared frequency becomes the actual conjunction h60-F#.)

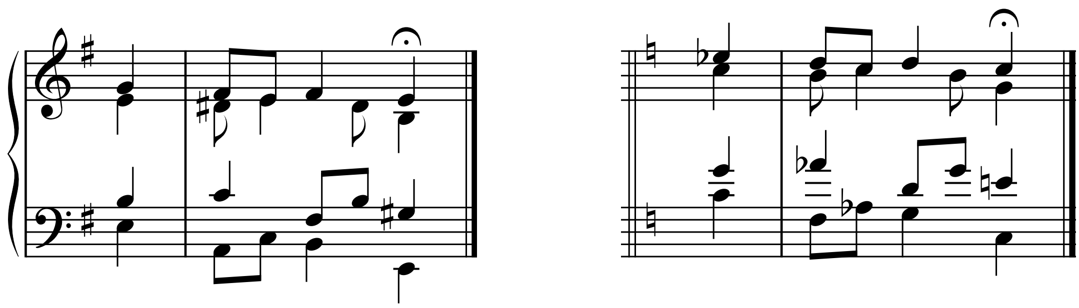


Figure P.7b The final cadence from *Mit Fried' und Freud' ich fahr' dahin*, Edition Breitkopf Nr. 3765, Chorale No. 251.

P.16 – THE II-V-I CADENCE

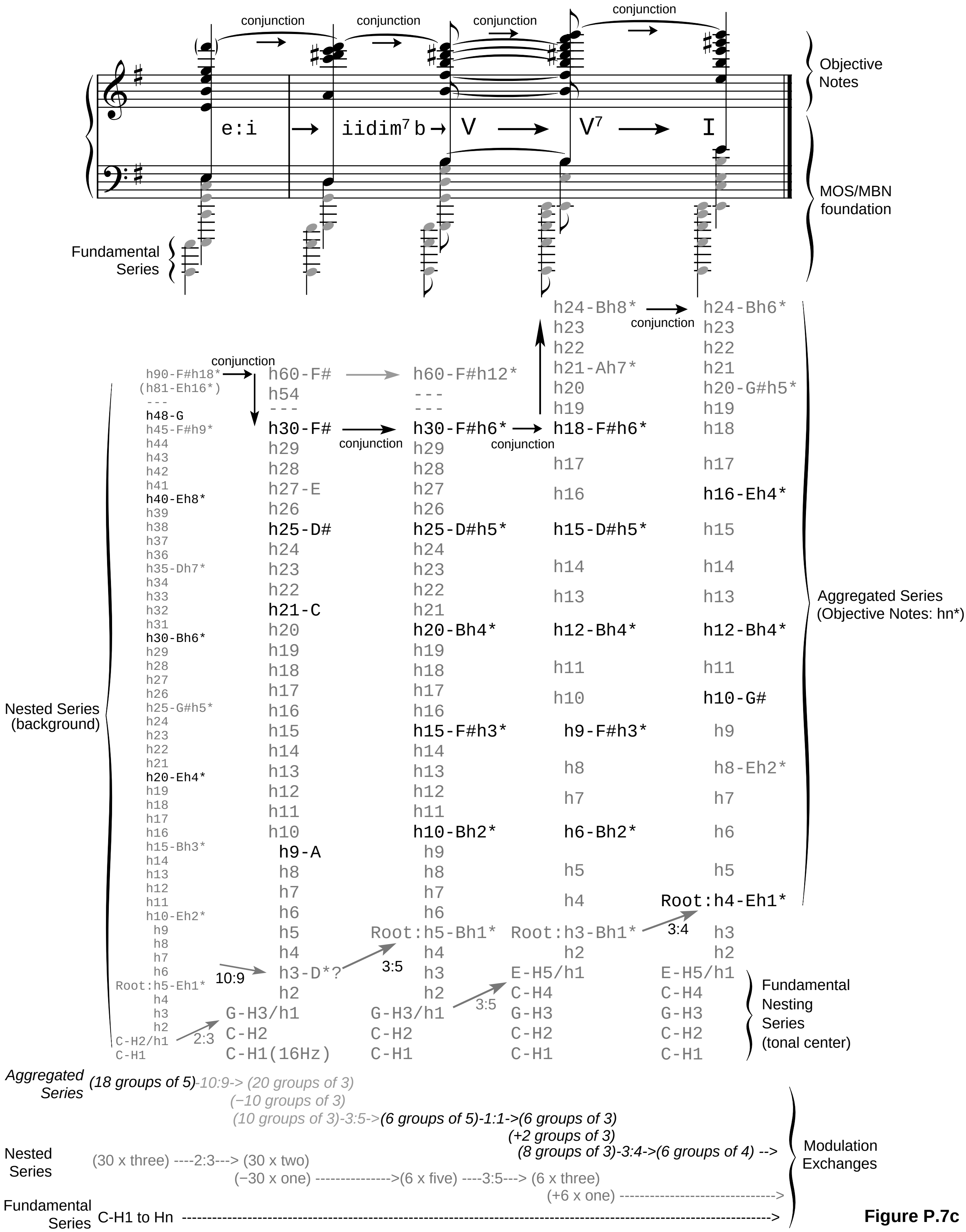


Figure P.7c

MBN $18_5 0_2 0_1 = 20_3 0_3 0_1 - 10_3 0_3 0_1 = 10_3 0_3 0_1 = 6_5 0_3 0_1 = 6_3 0_5 0_1 + 2_3 0_5 0_1 = 8_3 0_5 0_1 = 6_4 0_5 0_1$ Mutable Numbers

(Decimal 180 = 180 − 90 = 90 = 90 = 90 + 30 = 120 = 120)

In the MOS analysis shown in Figure P.7a the diminished-seventh chord yields no aggregated series and this area in the 'modulation exchanges' section of the analysis is left blank. Or perhaps somewhat theoretically, the aggregated series could be considered as stepping in groups of one, having melted into its underlying nested series. Below this blank space the nested series is marked as going through a una 1:1 exchange as it moves from twenty-four ratios encompassing the diminished-seventh chord to twenty-four ratios encompassing the dominant chord. During the nested series' passage between these chords, first the jarring ratios of the diminished-seventh's notes prevent any aggregation emerging, and then, the dominant's notes encourage the formation of groups of four. The ground state arrangement of four elements is 2 × 2, two groups of groups-of-two elements, and so from these somewhat theoretical manoeuvres, groupings of one and two nested elements could be said to arise in the iidim^7–V–I version of the cadence. The sum computed is the same as in the generic form of the II–V–I cadential formula, 72 + 24 = 96 or for the minor key MOS analysis 90 + 30 = 120.

Notes

1. The minor supertonic in first inversion raises the topic of ambiguous Roman numeral symbols. Is ii^6 the supertonic chord in first inversion or a root position supertonic with added sixth? The ambiguity arises from two slightly different perspectives: that of general bass figuring and Weberian chordal analysis. Of course, in almost all circumstances the context makes clear one or other reading; however, for total clarity, the use of a, b, c, d, etc. after the numeral to indicate inversion and the reservation of 'scripts' for intervals beyond the basic triad would make meanings clearer. For example, iib for first inversion, ii^6 for the (minor) chord of the added sixth, ii^{6}b for chord of the added sixth in first inversion, ii^{7}d for the seventh chord in third inversion and Ic for the tonic six-four chord.

2. Goldman, R., *Harmony in Western Music*, (W.W. Norton & Co. Inc, New York, 1965), page 47: *The II and II^7 Chords*. And more generally, this section of the book provided inspiration and guidance for the whole of Example P, including the use of some quoted chorale cadences. Confusingly, my source – Edition Breitkopf Nr. 3765 – disagrees with the numbers given in Goldman's book.

3. The lead-in E-minor has been dropped from this example because the MOS analysis required to conjoin it to the cadential formula would extend the relatively short fundamental series used in these examples and so add unhelpful complications. (A fundamental series extending to A-H13 would create a suitable conjunction at E-h156.)

4. Through the increasingly finely grained structure of each level of nesting within the MOS model any music, no matter how chromatic, or even atonal, can be represented. However below a certain level of tonal organisation, the model gradually ceases to yield meaningful detail – beyond the obvious observation that the music is not tonal – and, at the limit, the model represents a thoroughly non-tonal composition by one super-extended fundamental harmonic series capable of incorporating every note within the piece. Which is to say, the analysis consists solely of single column, prime state, mutable base numbers.