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Appendix E
Quick Start Outline

[An introduction to *Journey to the Heart of Music* composed of selected excerpts from the main text.]

MUSIC'S HIDDEN LANGUAGE

Music has an almost universal appeal, there are few people indeed to whom music communicates little or nothing; and if there is communication there must be a code, a language commonly understood. It is reported by Josef Haydn's contemporary biographer A.C. Dies[1], that at their last meeting in Vienna, on the eve of Haydn departure for London, Mozart remonstrated with his esteemed friend, that England was a faraway and dangerous place, and besides:

"Papa you can't speak English."

To which Haydn made his famous riposte:

"My language is understood throughout the world."

Indeed, though they speak to us with different accents, we all understand Haydn's language, Bach's language and Beethoven's language, Josquin's and Janacek's: it is also the language of Louis Armstrong, Cole Porter and the Beatles. We all understand the *language* of tonality – the common tongue of traditional classical and popular music – sometimes called the music of common practice. And clearly the long established Roman Numeral approach to harmony (first developed by Gottfried Weber, 1779-1839) can provide a high level, descriptive analysis for tonally organised music, somewhat analogous to the grammars of spoken languages. One might characterise the tonic chord (I) as the subject of a phrase and the dominant chord (V^7) as the verb.

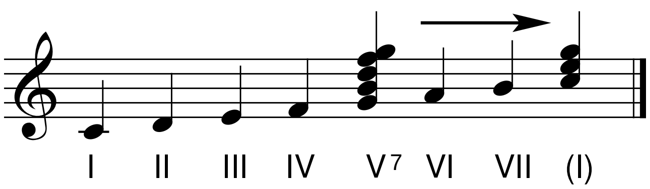

Figure E.1 The notes C, D, E, F, G, A, B, [C] are marked by Roman Numerals I through VII and chords built on these scale degrees can be similarly labelled. Here a dominant-seventh chord on the fifth degree of the C major scale is illustrated resolving to the common major chord on the first degree (tonic) – a full or perfect cadence. Notice that in this configuration both chords share a common top note (G), that 'smoothes' the transition between them.

But could there be a viable low level 'atomist' approach to tonality capable of *explaining* harmonic progression by stripping its mechanisms down to elemental units; rather than simply labelling music at the higher level of chords, with roman numerals – which tells us little about how and why harmony works? Does tonality have an alphabet, a secret code or perhaps, *digits*?

In this short document I would like to presenting a means of construing the physical reality of tonally organised musical sound as a representation of number. Put in more straightforward language, hidden away beyond the surface level of music, lies an implicit number system (with units: 1, 2, 3, etc. and columns) similar though not identical to the decimal system we use in everyday life; that is, viewed from the right perspective, the flow of harmony and changes in meter found in tonal compositions may each be interpreted as a form of number processing in musical sound. In the more elaborate case of chord progression (i.e. harmony rather than meter) this is achieved by extrapolation from the frequency relationships between the written notes, up to common overtones and downward to a unit fundamental, out of which a *positional number system* can be constructed which encapsulates and elucidates the nature harmonic progression – the means by which our ears navigate from one chord to the next. This positional number system, which might potentially play an analogous role for tonal music as the ‘clock’ arithmetic of the integers modulo twelve does for atonal music, has a name: *Mutable Numbers*, or in full, Mutable Base Position-value Number System. It is, I suspect, this number system which provides the ultimate source of structure and coherence underpinning the great flowering of western music, that began to take shape in Europe in the sixteenth century, reached its zenith during the eighteenth century and faded to a mere shadow in years after 1900. However, though many composers were to shunned its use in the twentieth century, it has proved to be a vigorous and hardy species: invasive to foreign music cultures and impossible to restrain in popular genres. There is I believe, something special and distinctive about the music of the tonal era, we revere the works of masters long after the epochs and societies which gave them life and context have disappeared. Indeed we preserve and cherish them often all the more today, ever reluctant to let go of music that is understood, not only as Haydn suggested “throughout the world”, but across the centuries too. Like the Indo-arabic number system which has found near universal application in the modern world, tonal music and the number system embedded within it, is probably here to stay.

In order to easily grasp the concept of mutable numbers and their encapsulation of harmonic motion in western music, we must first review the nature of the harmonic series; and then, thinking back to our early school days, remind ourselves about the structure and mechanics of numbers – how units or ‘digits’ are put together to make multi-column numbers. For it is to the ratios of the harmonic series that we shall need to look, for the digits with which mutable numbers are made. So first we enquire into the harmonic series, then second, examine how positional numbers are made, and finally bring the two topics together in the construction of these mysterious mutable base numbers; which, hopefully, are going to elucidate the nature of harmonic progression in traditional western music; and which, by implication, perhaps also offer some insight into the mechanism through which the processes of aural cognition turn objective musical sound into a meaningful language – *a number system*.

THE HARMONIC SERIES

For traditional tonal music, the basic elements of harmony have long been recognised as deriving from the natural default modes of vibration of a physical object – the harmonic series. The great eighteenth century theorist and composer Jean-Philippe Rameau (using information gleaned from the scientific work of Joseph Sauveur 1653–1716) described this phenomenon as the *corps sonore*, the sounding body, which he described as nature's gift to mankind.

When a physical object is energised by some event, say a piano string struck by a felt-covered hammer, the energy transferred to the string by the hammer blow, causes the object, to vibrate. The string stores the energy it has received as what might be termed 'bound motion' – oscillation. Gradually, a transfer of the energy from the string to the surrounding atmosphere occurs, the energy reaches our ears in the form of pressure waves thrown off by the vibrations of the string and resonant structures connected to it. We hear the note. Over time the note fades away, as more and more of the initial input of energy is lost to the air. Eventually, the string comes to rest, or equilibrium; it has no more excess energy to liberate, and the note falls silent. Significantly, after the initial strike of the hammer, the string settles down to sound one note and not a jumble of sounds. However, this one note that we perceive, is made up of many 'sub-notes' which we rarely separately distinguish, but apprehend as the timbre – the tone quality of the note. And these sub-notes are themselves not jumbled-up either. The note we perceive (under normal circumstances) is the fundamental oscillation of the string. The sub-notes are a range of whole numbered multiples of this fundamental oscillation *'f'*, that is: $2\times f$, $3\times f$, $4\times f$, etc. Together they form a sequence of integer harmonics – *a harmonic series* – which is customarily written: h1, h2, h3, h4, etc. (A few instruments, such as the Xylophone, do not produce neat integer harmonics but as they deposit the majority of their energy in the fundamental tone, these instruments still produce the sensation of a single pitch.) Quite often the intensity of some of the sub-notes/partials generated by musical instruments will be greater than the fundamental tone. The perception of notes – the sensation of individual pitches – is to a large degree a construct of aural cognition; in that although the hearing process collects and examines a wide range of frequencies, both fundamental tones and partials, only an edited summary of this information, which might be described as 'fundamental-with-timbre-appended', impinges upon conscious perception. Another quality attached to the sensation of pitch is that of the direction from which the sound comes, again like timbre, this is the product of considerable amounts of unconscious processing carried out across the frequency range of hearing. There is much more to musical sound and aural cognition than the lonely notes found written in scores, hiding behind each note in performance lies a harmonic series and considerable amounts of unconscious mental processing.

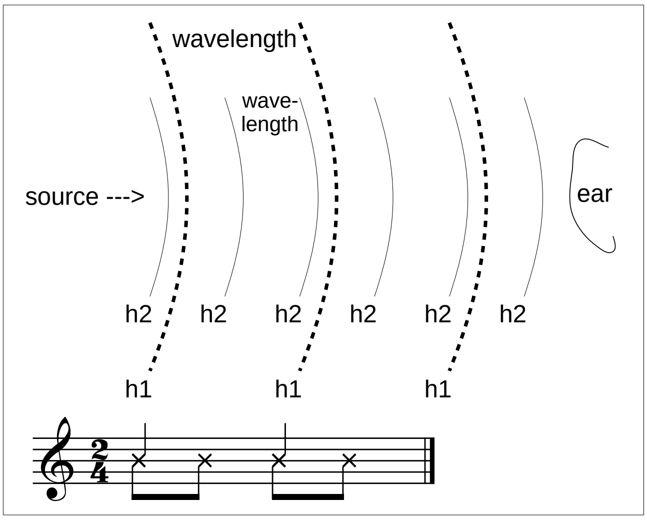

Figure E.2 A schematic diagram of a sequence of waves, showing the wave peaks of frequencies h1 and h2, with below, a metrical interpretation of the waves in combination.

In the form described above (fundamental, h2, h3, h4, etc.) the harmonic series is described in terms of *frequency* – a time based measure. The wavelength of the second harmonic is half that of the fundamental harmonic (Figure E.2) and as both waves travel at the same speed, two 'h2-waves' will reach the ear in the time one fundamental 'h1-wave' passes. Over the time it takes the fundamental tone to move through one complete cycle the second harmonic will complete two cycles. The second harmonic, h2, is twice as 'frequent', the third harmonic, h3, three times as 'frequent', the fourth four times, and so on. For a sound source, like for example a violin string, the fundamental first harmonic vibration (h1) coexists with all the other whole numbered vibrations: h2, h3, h4, h5, etc. yielding in sum a complex pattern of pressure waves travelling away from the instrument, that encodes both pitch and timbre. However, in the strictly mathematical sense, the overtone sequence is a harmonic series of wavelength relationships: one, one-half, one-third, one-quarter, etc. but somewhat paradoxically, its normal written expression takes the form of an arithmetic series of whole number frequency relationships: harmonic one, harmonic two, harmonic three, etc.

The real situation, of course, is much more complex than shown in Figure E.2. However, the relationships that the ear acquires from this input approximate to simple low order ratios, expressed in nerve pulses, which lends a metrical flavor to the frequency information passed down the auditory pathway and on to higher levels of cognition. (Though not pursued in this document, the model of mutable base numbers may be equally applied to the durational dimension of music, as indeed at its core the concept is in essence metrical.)

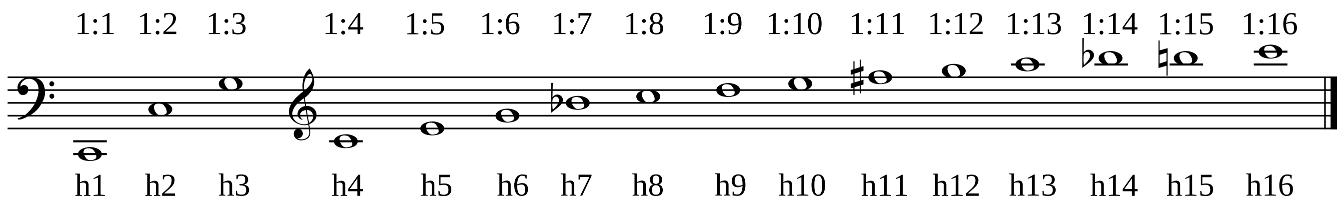

Figure E.3 Harmonic Series: ratios of the first sixteen frequencies of the series are shown above the stave and below the stave the conventional shorthand 'h1' for the fundamental, to 'h16'. (On occasions a capital H may be used to distinguish the lowest or most fundamental of a group of nested series.)

It could be argued that in a sense the harmonic series is the *universal chord*, the statement, in ascending order, of every possible natural interval – the whole number ratio or relationship between any two tones of the series. Figure E.3 illustrates the first sixteen 'notes' of the harmonic series 'chord'. By striking bottom C-h1 on the left of Figure E.3, all the other fifteen harmonic partials, and more, will also be sounded (some very faintly). For art in general and music in particular, a limited selection of the natural elements available to us, has been used as the foundation for creative elaboration; in music, out of the extensive array of intervals offered to us by the harmonic series, *nature's gift* to use Rameau's graceful expression, a few simple ratios from the beginning of the harmonic series form the basis for the harmonies (and rhythms) of tonal music. Beyond the first eight partials of the harmonic series (essentially seventh and perhaps ninth chords), which is to say, harmonies that involve or imply chords stretching from h1 through h9, h10, h11, become increasingly difficult for the ear to decipher and the processes of aural cognition to interpret. The detection mechanism of the ear separates out these components as far as it can and signals their existence and strengths to the automatic processors in the auditory pathway and beyond. But, as can be seen in Figure E.3 the harmonics fall ever closer together – octave, fifth, fourth, major-third, minor-third, etc., – from the major-second onward this causes increasing levels of interference on the ear's detector membrane. Beyond h8, the interference between the harmonics from a single note becomes gradually more and more destructive, and the effect of multiple notes in chords even more so. The mechanisms of hearing place limits upon the range of harmonics that can be distinguished.

Notwithstanding these limitations, the ear is remarkably sensitive to nuances of musical sound, particularly in the range around the top octave of a piano – approximately 2kHz to 4kHz – where our hearing is most acute. Interestingly, this is precisely the hunting ground of the higher resolvable partials generated by notes in the octave above middle C – the cockpit of harmony. Scores are misleading in this regard as they present on the staff only the 'h1' fundamental of each note, when in performance the ear is actually deluged by a swirling sea of unwritten overtones, each note a harmonic series. Though unseen by the eye, the ear unobtrusively sweeps up and sifts all the components of musical sound within its frequency range. And as below a mutable number with twenty-four digits will be examined, to be sure we have sufficient partials to work with, Figure E.4 extends the harmonic series beyond h16 – with their approximate note letters appended. Admittedly there is some disagreement between the notes of the equally-tempered scale and the true whole number relationships of the harmonic series from which they are ultimately derived. However, it is convenient and useful to label the partials of the harmonic series with the note letters of the scale, and generally, to treat the adjusted relationships of keyboard music making, in principle, as if they are the whole relationships of the harmonic series. It is the forgiving tolerance of the ear and processes of aural cognition which allow this extraction of a precise relational 'meaning' from the approximations which are inevitably the objective experience.

Overtone Series of True Harmonic Partials ---->

```
A-h27 -- 26.908Hz -A
A-h26
G#h25 -- 25.398Hz -G#
G-h24
 -h23
F#h22
F-h21 -- 21.357Hz -F
E-h20
D#h19 -- 19.027Hz -D#
D-h18
C#h17 -- 16.951Hz -C#
C-h16
B-h15 -- 15.102Hz -B
A#h14
A-h13 -- 13.454Hz -A
G-h12
F#h11 -- 11.314Hz -F#
E-h10
D- h9 --  8.980Hz -D
C- h8
A# h7 --  7.127Hz -A#
G- h6
E- h5 --  5.040Hz -E
C- h4
G- h3 --  2.997Hz -G
C- h2
C- h1 --  1.000Hz -C
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Twelve Equal-tempered Notes ---->

Figure E.4 The 'true' whole number relationships of the harmonic series compared with the adjusted frequencies of the equal-tempered scale. Both columns can be read as 'h' numbers or hertz (cycles per second). The frequencies of the twelve notes of the equal-tempered scale have been raised to the octave where their equivalent partial first appears in the harmonic series.

Sound in the real world is rarely as straightforward as this outline of the characteristics of the harmonic series might suggest, however, the remarkable acuity, tolerance and filtering processes of the ear and aural cognition fortunately combine to allow us to extract relatively simple whole numbered relationships, ratios, from what are usually extremely complex collections of sound waves. This is the salient point: remarkably, our ears are able to harvest the simple 'relational intent' of the written notes from out of the swirling mass of acoustic chaff that is typical musical sound.

Nesting and Nested Harmonic Series

Having briefly investigated the ratios of the harmonic series, which are to provide the digits for mutable numbers, it must now be considered how the harmonic series might further help by also providing columns within which the individual digits may be placed. This is achieved by nesting one harmonic series within another, the effect of which is multiplication – the same effect as achieved by column shifts in positional numbers.

Illustrated below is a harmonic series, and to keep things simple only the first twelve harmonics of the series are used in this example. Within a harmonic series there are potentially an unlimited number of further, nested, harmonic series. Like a collection of Russian dolls or the complete Tuppaware set, each is of identical form, only different in size. On the left of Figure E.5 is a fundamental nesting series (marked with capital Hs) consisting of the first twelve ratios of the harmonic series built on the root frequency of C-128Hz[2].

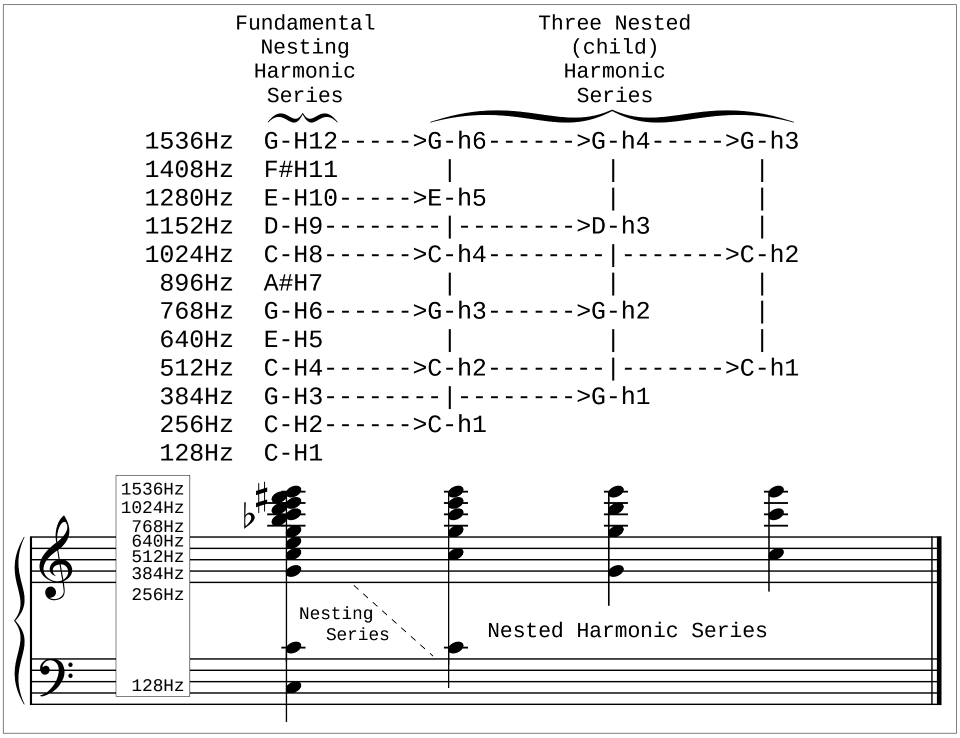


Figure E.5 On the left, is a Fundamental Nesting harmonic series C-H1 through G-H12, with three Nested harmonic series illustrated to the right: C-h1 through G-h6, G-h1 through G-h4 and C-h1 through G-h3.

In Figure E.5, to the right of the twelve ratios, are the beginnings of three *child* or nested series built on the frequencies C-256Hz, G-384Hz and C-512Hz. The nested series are subsets of the nesting series, that is to say, there are no new ratios in the child series that are not also to be found in the *parent* series;

and though we are considering here only twelve elements, however far the parent and child series are extended no ratios will be found in the child series which are not in the parent series. It can also be seen that C-256Hz, the h1 foundation of the first child series is the second harmonic of the fundamental parent series, that G-384Hz, the h1 foundation of the second child series is the third harmonic of the parent series and that C-512Hz the h1 foundation of the third child series is the fourth harmonic of the parent series. Effectively the first child series built on C-256Hz is counting in aggregates of two fundamental harmonics or ratios: C-h1/H2, C-h2/H4. G-h3/H6, C-h4/H8. Similarly the second and third child series count in aggregates of three fundamental ratios and four fundamental ratios respectively. That is, the nested series are formed by multiplying the fundamental series by 2, 3 and 4. Further to which, the third child series built on C-512Hz might equally be interpreted as nesting within the first child series, by counting in aggregates of two, each of which are themselves also stepping in groups of two parent ratios – 2×2.

Now while Figure E.5 doesn't extend to the more familiar mathematical territory of column shifts through multiplication by powers of ten as found in decimal numbers, one can see how an analogous scheme of nested harmonic series, stepping in aggregates of tens, might be constructed; and, as there is no restriction on the number of child series available, this structure can therefore have an unlimited number of ‘columns’ (i.e. nested series) just as ‘normal’ decimals do. There is an inexhaustible supply of nested child series available within a fundamental harmonic series: in addition those illustrated in Figure E.5, there are matching nested series for the tens, hundreds and thousands – i.e. H10, H100 (10×10) and H1000 (10×10×10) – and any other integer one care may to chose. In the discussion below the nested series built on G-H3 (×3) and C-H4 (×4) will feature prominently.

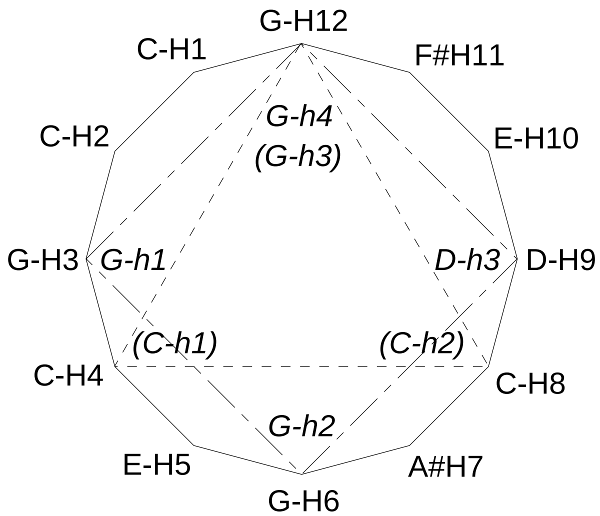


Figure E.6 A geometric interpretation of twelve ratios of a fundamental harmonic series (labelled with upper-case Hs, C-H1 through G-H12) nesting multiples of three harmonics (square: G-h1 through G-h4) and multiples of four harmonics (triangle: C-h1 through G-h3). Notice that the two child series, the square and triangle, share a common vertex, the fundamental harmonic G-H12 which conjoins the two structures.

Significantly, all the harmonic series illustrated in Figure E.5 contain the harmonic G-H12 (1536Hz) and this conjunction of harmonics, creates a commonality or symmetry among the various series, hinting at a potential pathway between one child/nested series and another, when they are set within the context of the parent fundamental series. (Indeed, as suggested by the geometric illustration, Figure E.6, the collection of nested harmonic series could be understood in terms of a mathematical symmetry group.)

POSITIONAL NUMBERS

To reiterate: the main aim of this introduction is to establish the concept that, in principle at least, a tonal composition is effectively a number system – a representation of number relationships in material form. So what precisely comprises a number system?

Fundamentally, a number system is a scheme employing physical tokens, digits, to describe numerical relationships. Remarkably, though mathematicians often think in terms of ‘the Platonistic abstract’, to actually make computations, to do arithmetic, requires the use of material counters of some sort: digits written on paper, electrons in computer circuits, nerve connections in the brain – or something more basic. From the earliest times, the most convenient physical tokens to ‘come to hand’ for counting – keeping track of number relationships – were fingers (and toes) and occasionally the spaces between these digits[3]; also pebbles, beads, seeds, sticks, shells, etc. have been employed. These early, instinctive, counting systems were probably all additive in nature. In such systems ever more fingers, toes, pebbles or beads were required to represent larger numbers, with more sophisticated additive systems developing tokens to represent groups of other tokens. For example V, X, C, D and M in Roman Numerals – the ‘V’ probably began as a picture of an open hand, later stylised, and ten as two hands ‘X’. It was also common for letters of the alphabet to stand double use as numbers: as in the Roman, Greek and Hebrew traditions. Interestingly, the early intuitive counting schemes generally began counting numbers from one rather than zero – a feature (we shall see below) they share with mutable base counting. However, a few societies independently discovered a new and different way of representing number relationships, position-value number systems (often termed place notation or abbreviated to positional notation); and, through the mechanics of these systems, some found a pressing need for a zero token to fill empty positions in columns. Of these various forms of positional counting, one has come to dominate the mathematics of the world.

We are all familiar with this one position-value number system, in fact so familiar that we rarely give it a second thought: it is of course our normal decimal number system, written with the ten units or digits 1, 2, 3, 4, 5, 6, 7, 8, 9 and 0. Though often referred to as Arabic, the system originally developed in India, and later spread to Islamic cultures through trading contacts with the sub-continent, before being passed on to the west. Indeed, the derivation of the word *algorithm* testifies to the interrelated roots of todays system – originating from the name of a great Persian mathematician al-Kuwarizmi working in Baghdad in the ninth century, who brought together the advances of Arab mathematics and astronomy with the existing body of Indian and Greek learning. The translation of his work on arithmetic into Latin: *Algoritmi de numero Indorum*, in the 12th century, helped to make this knowledge available in Europe[4] – as also did the contacts between European and Islamic culture in Spain.

The great advantage of positional notation for the representation of number lies in its inherent flexibility and economy – in essence it is a loss-less compression algorithm. The digits in a position-value number system have two meanings, firstly their *intrinsic (additive) meaning* of oneness, twoness, threeness, etc. plus a second *multiplicative meaning* derived from their *column position*. Thus the threeness of the unit column’s three is multiplied by ten, to become 30, when the digit three appears in the ten’s column; and by ‘ten by ten’ in the hundred’s column to become three hundred, and so on. Each column shift produces a multiplication by ten – from 3 to 30 to 300 to 3000. It is the physical positions of the digits, relative to one another, which is the source of a digit’s second, extra, meaning. Should the physical relationships between material tokens be lost, usually so is the number; thus the zero digit became a vital place-holder, clearly identifying empty columns which otherwise might be a source of confusion. In general additive number systems are not so sensitive to the spacial relationships of their tokens. For example in the ancient Greek additive decimal system, three sequences of nine capital letters, plus three extra symbols, represented 1 through 9, the tens (10 through 90) and the hundreds (100 through 900), which written in any order would represent one and the same value. The great advance made by schemes involving positional notation is to combine the additive principle with that of multiplication, producing a system which can still express every integer through the application of the 'leaden' additive principle while also remaining light enough to fly efficiently to relatively large numbers by means of multiplication.

The decimal system, as its name suggests, is based on ten, or in the jargon, is a *base ten number system*. Here are a few examples written out explicitly showing their base tens, thus:

Number	=	1000-column	100-column	10-column	units
3	=				(3 × 1)
30	=			(3 × 10) +	(0 × 1)
300	=		(3 × 10×10) +	(0 × 10) +	(0 × 1)
3000	=	(3 × 10×10×10) +	(0 × 10×10) +	(0 × 10) +	(0 × 1)
24	=			(2 × 10) +	(4 × 1)
456	=		(4 × 10×10) +	(5 × 10) +	(6 × 1)
7890	=	(7 × 10×10×10) +	(8 × 10×10) +	(9 × 10) +	(0 × 1)
9999	=	(9 × 10×10×10) +	(9 × 10×10) +	(9 × 10) +	(9 × 1)

This is rather cumbersome, a more compact method of explicitly showing the column bases would be to use subscripts:

Decimal Number	=	General Positional-Number Format	
3	=	$3_{\times1}$	
30	=	. . . .	$3_{\times10}0_{\times1}$
300	=	$3_{\times10}0_{\times10}0_{\times1}$	
3000	=	. . . .	$3_{\times10}0_{\times10}0_{\times10}0_{\times1}$
24	=	$2_{\times10}4_{\times1}$	
456	=	. . . .	$4_{\times10}5_{\times10}6_{\times1}$
7890	=	$7_{10}8_{10}9_{10}0_{1}$	
9999	=	. . . .	$9_{10}9_{10}9_{10}9_{1}$

In this compact format all the subscripts up to any given column are multiplied together, rather than writing out, for example, a string like '$3_{\times10\times10\times10}0_{\times10\times10}0_{\times10}0_{\times1}$' full of redundant base tens in each column. For the first five numbers above the multiplication sign (×10, ×1) has been explicitly shown and in the last two omitted. This is the normal usage for this General Positional-Number Format, otherwise called *Subscript Format*, just to show the column bases and column digits as plain decimal numbers (single or multi-column).

Column bases other than ten are perfectly feasible, indeed any base with two or more units can be used to construct a positional number system. Of the non-decimal positional systems in use today the binary, octal and hexadecimal systems, with bases two, eight and sixteen respectively, are perhaps the most familiar – from the domain of computing; and not forgetting the remnants of a base sixty number system evident in our measurement of time, bequeathed to us by the ancient Babylonians. (Theoretically, a *base one* positional system is possible, but it lacks flexibility and is effectively an additive system of the type a convict or castaway might use to record the passage of days.) A short comparative table, is provided in Figure E.7, with both the written *formal notation* and *physical structure* of the equivalent Mutable Number given in the two right hand columns. Mutable numbers have a dual nature, they are both ‘abstract’ artifacts of the human intellect *and* relational oscillatory structures in the material world – musical sound. The digit ranges of the number systems precedes the table.

SYSTEM:	DIGITS
Octal:	0,1,2,3,4,5,6,7
Decimal:	0,1,2,3,4,5,6,7,8,9
Hexadecimal:	0,1,2,3,4,5,6,7,8,9,A,B,C,D,E,F
Mutable:	h1,2,3,4,etc... hn (unlimited)

.... DIGIT SEQUENCES ....

Decimal	Octal	Hex	Mutable Base Numbers	
0	0	0	Formal	Physical
1	1	1	1_1	(h1)
2	2	2	2_1	(h1+h2)
3	3	3	3_1	(h1+h2+h3)
4	4	4	$2_2 0_1$	(h1+h2+h4)
5	5	5	5_1	(h1+h2+h3+h4+h5)
6	6	6	$2_3 0_1$	(h1+h2+h3+h6)
7	7	7	7_1	(h1 through h7)
8	10	8	$2_2 0_2 0_1$	(h1+h2+h4+h8)
9	11	9	$3_3 0_1$	(h1+h2+h3+h6+h9)
10	12	A	$2_5 0_1$	(h1+h2+h3+h4+h5+h10)
11	13	B	11_1	(h1 through h11)
12	14	C	$2_2 0_3 0_1$	(h1+h2+h3+h6+h12)
13	15	D	13_1	(h1 through h13)
14	16	E	$2_7 0_1$	(h1+h2+h3+h4+h5+h6+h7+h14)
15	17	F	$3_5 0_1$	(h1+h2+h3+h4+h5+h10+h15)
16	20	10	$2_2 0_2 0_2 0_1$	(h1+h2+h4+h8+h16)
17	21	11	17_1	(h1 through h17)
18	22	12	$2_3 0_3 0_1$	(h1+h2+h3+h6+h9+h18)
19	23	13	19_1	(h1 through h19)
20	24	14	$2_2 0_5 0_1$	(h1+h2+h3+h4+h5+h10+h20)
21	25	15	$3_7 0_1$	(h1+h2+h3+h4+h5+h6+h7+h14+h21)
22	26	16	$2_{11} 0_1$	(h1 through h11+h22)
23	27	17	23_1	(h1 through h23)
24	30	18	$2_2 0_2 0_3 0_1$	(h1+h2+h3+h6+h12+h24)
25	31	19	$5_5 0_1$	(h1 through h5+h10+h15+h20+h25)
26	32	1A	$2_{13} 0_1$	(h1 through h13+h26)
27	33	1B	$3_3 0_3 0_1$	(h1+h2+h3+h6+h9+h18+h27)
28	34	1C	$2_2 0_7 0_1$	(h1 through h7+h14+h28)
29	35	1D	29_1	(h1 through h29)
30	36	1E	$2_3 0_5 0_1$	(h1+h2+h3+h4+h5+h10+h15+h30)
31	37	1F	31_1	(h1 through h31)
32	40	20	$2_2 0_2 0_2 0_2 0_1$	(h1+h2+h4+h8+h16+h32)
etc.	..	..	......	....
255	377	FF	$3_5 0_{17} 0_1$	(h1-h17+h34+h51+h68+h85+h170+h255)

Figure E.7 Comparative table of Decimal, Octal, Hexadecimal and Mutable Numbers. The letters A through F are used as additional digit symbols in hexadecimal numbers. Mutable numbers are listed in two forms, formal written numbers and as physical entities constructed from the relationships of the harmonic series – that is to say chords. Notice also that in subscript format the decimal numbers used to represent mutable column bases and column digits may themselves be multi-column decimals (e.g. 11_1 or $2_{11}0_1$); it would be cumbersome and inconvenient in the extreme to create a plethora of new digit symbols to cover the unlimited range of the harmonic series (i.e. mutable digits).

Getting to grips with non-decimal systems can be difficult at first, as all our ingrained instincts lead us to treat digit sequences as individual (decimal) numbers. For example, the digit sequence ‘12’ *is* the number twelve, however, this is only true for the decimal digit sequence '12'. In the base eight, octal positional number system, the digit sequence '12' represents the number ten, and in the hexadecimal system, the number eighteen. *Digit sequences and number are not the same thing*. However, if we use the Subscript format this difference is manifest:

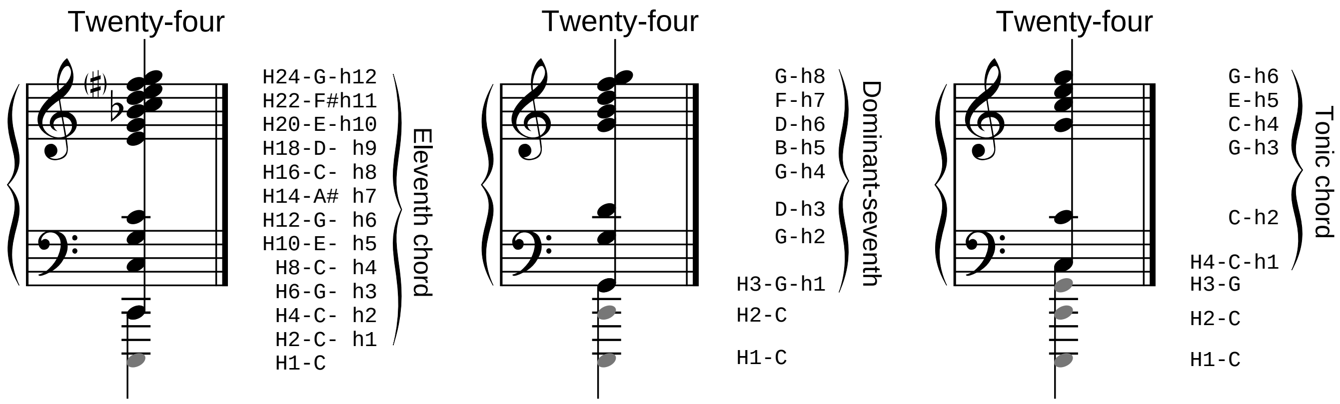
Ten: Octal $1_8\,2_1$ = Decimal $1_{10}\,0_1$
Eighteen: Hex $1_{16}\,2_1$ = Decimal $1_{10}\,8_1$

Digit sequences are material or structural representations of relative magnitudes and vary as to what actual number they represent, depending on the base(s) of the system being employed. Moving between number systems with different bases is endlessly confusing, and for the most part, it is easier to stick to decimal numbers and use a chart of equivalent digit sequences for other bases.

Amongst the positional number systems invented by various civilisations, the one which developed in India had a particular advantage, in that it used separate compact symbols for each digit. The Babylonian, Mayan and Chinese positional systems all economised in this area, using the additive principle to generate some digits from others. For example in the Chinese 'rod' notation, one was written |, two was || and four ||||. The advantage of separate compact symbols or glyphs for a number system lies in the facility it lends to the execution of written computations.

A position-value number system does not have to have a single fixed base, though most do – for quite self-evident reasons of clarity and ease of use. As in the decimal system, where after the units column each and all remaining columns imply a multiplication by ten and only (multiples of) ten. Historically there have been been examples of systems with more than one base. For example the Mayan of central America mixed base eighteen in a predominantly base twenty positional number system – presumably to match the approximate year-days figure of three hundred and sixty. While the Sumerian/ Babylonian system contains traces of parallel bases: ten, twenty and sixty. The old British imperial measures present many other examples like pence, shillings and pounds (bases 12 and 20) or gallons, pecks, bushels and quarters (bases 2, 4 and 8). Generally, the development of number systems emerged from, and were intimately connected with, managing the material environment: harvests, religious festivals, the calendar, taxes and land survey.

Theoretically, but somewhat impracticably, a different, but fixed base could be used for each column. For example, column one would be base one of course, column two base two, column three base three and so on! Or even more confusingly one could go to the limit and have a changing pattern of bases applied to all the columns, after the units column. That is, a positional system where columns can dynamically change their base from number to number, or even within one number! If you are reeling from getting to grips with octal and hexadecimal systems, let alone numbers with shifting bases, do not despair. This last form of positional notation with variable or *mutable* bases – the notation that is needed to model tonally organized music as a positional number system – though inherently the most complex of all, rests on a straightforward natural logic derived from the harmonic series which helps to make it intuitively easy to grasp. And as by this point it might appear that positional number systems are mathematical schemes remote from music, to give a glimpse of how the units and column-digits of mutable base numbers fit together in terms of ratios of the harmonic series, Figure E.8 presents an example that uses the child series illustrated in Figures E.5 and 6. In Figure E.8 the range of the three nested series – counting in multiples of two, three and four harmonics – has been extended out to the twenty-fourth harmonic (H24) of their shared fundamental nesting series.



MBN $12_2 0_1$ MBN $8_3 0_1$ MBN $6_4 0_1$

Figure E.8 The mutable base number twenty-four, in three different configurations. The physical representation of twenty-four consists of the objective chords (black noteheads), with note ratios of the harmonic series given to the right. The stems connecting the notes distinguish the fundamental nesting series in gray with downward pointing stems (H1, H2, etc.), and above the nested series in black (h1, h2, etc.) with upward pointing stems. In the formal written number given below the systems, the unit column '0_1' delineates the absolute fundamental of the system (H1-C) and the subsequent column(s) with subscript(s) the range of each nested harmonic series. In the left-hand example the ratios of the fundamental series (H1 through H24) are shown in full but omitted from the center and right examples. 'Mutable Base Number' is abbreviated to MBN.

Immediately the striking feature of mutable base numbers is that for most values there is a range of different digit sequences all of which represent one and the same value! The mutable number system is rich and subtle, it possesses depth and variety in the way it accesses different magnitudes. Indeed for the number twenty-four there are in total twenty different mutable digit sequences – each of which, considered in terms of physical structure, would imply a different energy profile. (In Figure E.7 the mutable numbers are presented in their ground states.) As can be seen in Figure E.8, two of the digit sequences delineate those familiar chords: the dominant-seventh and the tonic. The full or perfect cadence chord progression is none other than two guises of the mutable number twenty-four expressed in musical sound! Thus do seemingly remote mathematical systems of counting, via the ratios of the harmonic series, enter into the heart of tonal music. Let us enquire a little further into the musical relevance of this prolific mutable number twenty-four.

HARMONY AND MUTABLE NUMBERS

Perhaps the most familiar and recognisable of all chord progressions is that of a *full cadence* (V^7–I) where the dominant or dominant-seventh chord is succeeded by the tonic at the end of a phrase, section or piece. In the key of C major the chord progression consists of a G-major chord (with or without the minor-seventh), followed by the tonic C-major chord. Some other names are also applied to this cadential formula – prefect, authentic or final cadence. Almost every piece of tonal music, from the tritest jingle to the mightiest symphony, closes with some form of this progression, and the majority of phrases within them as well. The chord progression also plays a crucial role in establishing a change of key through the creation a full cadence onto a new tonic chord.

At any point in a tonal composition the harmonic motion of the music can be characterised as the change from one chord to another chord. However many harmonic steps a composition contains, each individual step involves just two chords, a preceding chord and a succeeding chord. The full cadence chord exchange is the very quintessence of harmonic progression, the dominant chord followed by the tonic chord; and, as such it can be take to be the ultimate minimal tonal composition – the encapsulation of the principle of harmonic progression. The full or perfect cadence will be our example tonal exchange and the mutable base number this chord progression delineates is, as we have seen above, twenty-four.

If our minimal piece of music is in the key of C major, the harmonic progression would consist of two chords: a penultimate chord built on the 'root' note G – with an added minor-seventh interval creating a dominant-seventh chord – followed by the tonic, the common C-major chord. The perfect or full close cadence is illustrated in Figure E.9.

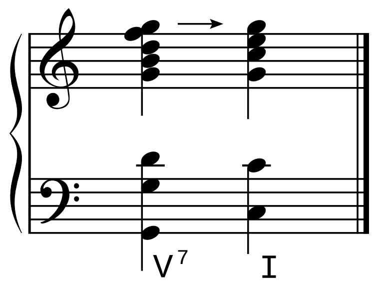


Figure E.9 A full or perfect cadence (V^7–I). Notice that the lowest notes of the two chord, the root tones G and C are separated by a perfect fourth. This interval makes the relationship between the note frequencies the ratio 3:4 (96Hz : 128Hz, at pitch level middle C = 256Hz).

Although the chords in Figure E.9 form probably the most familiar of all chord progressions, it is worth playing them on a piano in the written configuration. The arrangement of notes has some important characteristics:

Firstly, the notes of both chords are in the configuration of harmonic series, that is, the bass or root note is the fundamental tone of each chord's upper notes (considered as overtones).

Secondly, both chords share the same top note: G, which connects or *conjoins* the chords allowing a smooth transition from one to the other.

Thirdly, the G-major chord in this progression contains eight notes while the C-major chord has only six notes, eight notes are exchanged for six – 8:6 notes, which reduces to a ratio of 4:3.

And fourthly, the frequency ratio between the two fundamental root tones (G to C, a perfect fourth) is 3:4, the exact reverse of the above ratio of constituent notes in each chord.

Thus, the perfect cadence chord progression illustrated in the particular arrangement given in Figure E.9, represents a rather special transitional configuration between the two chords, in which they are joined by their common upper note and exchange a ratio of 4:3 elements of the harmonic series (eight for six notes) on fundamental roots which themselves have the inverse frequency relationship of 3:4. This is an example of the *algorithm of symmetrical exchange* at work, the mechanism by which mutable numbers acquire their mutability. Although this might appear rather a restrictive formula at first sight, it is capable of unlimited extension, in that any whole numbers could be substituted for the numbers three and four in the above example, eg. 4:5, 8:9, 3:7, 13:23, etc. And, while most chords do not have an arrangements of objective notes which exactly matches the harmonic series (like those in Figure E.9), they do, for the most part, generate reasonably extensive *configurations of partials* by virtue of the harmonics of timbre contained within the notes themselves, combined with the resonances within the bodies of the instruments being played and rooms being played in. To which may be added the non-objective combination tones produced within the ear itself. In other words, there are frequencies aplenty for the mechanisms of aural cognition to digest. Indeed, with instruments like the bassoon[5] or cello the hearing system actually manufactures the perceived lower register notes almost entirely from higher partials, as very little energy goes into the fundamental frequency, particularly in the lower ranges of these instruments. Further, in controlled acoustic experiments[6] which take this feature of hearing to its limit, notes wholly created from partial frequencies can be perceived that simply don't exist objectively! The processes of aural cognition are flexible, adaptive and creative in their functioning.

A Wider Context

Returning to the arrangement of the notes within the two chords of the full cadence shown in Figures E.9 and E.10, where the notes are not haphazardly placed, but describe harmonic series of their own – one series built on the note G-96Hz and one built on C-128Hz.

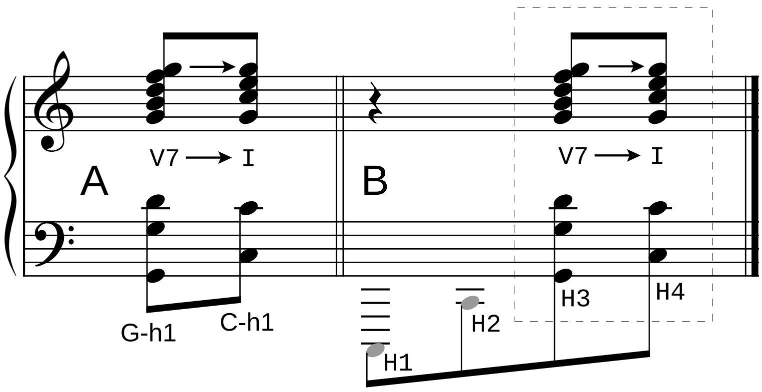


Figure E.10 The wider context of nested harmonic series in the dominant-seventh to tonic chord progression, connected by the common ground of their upper notes, G.

Preceding the dominant and tonic chords in Figure E.10B are two further notes labelled H1 and H2, and the notes that were labelled G-h1 and C-h1 in Figure E.10A are re-labelled H3 and H4! What has changed? The context. In Figure E.10B we are looking at the wider relationship between the two harmonic series which in Figures E.9 and 10A were considered in isolation. These two isolated series have a relationship, their respective fundamental tones G-96Hz and C-128Hz, are related by the proportion 3:4; and this whole number relationship has some interesting implications.

The notes labelled H1 and H2 in Figure E.10B are implied by the full cadence, in that the root notes of the two chords, the bass notes G and C, have the frequency relationship of 3:4; therefore, by actualising the step from G-96Hz to C-128Hz, the potential steps down to C-H2 and C-H1 are implicitly laid out as well. Through creating (or computing) the relationship of 3:4 in the movement from root tone G-H3 to root tone C-H4 of the full cadence chord progression, the value of the unit (H1) is also being delineated. Thus the implications of the relationship of the objective dominant and tonic chords of a full cadence extend further than the notes themselves, describing other potential relationships, in particular the implied fundamental frequency C-H1 (32Hz) and more generally, all frequencies with whole number relationships to G-H3 and C-H4 – like for example G-H24, the top note in both chords. Indeed, what the objective full cadence has defined, is an underlying harmonic series – a *fundamental series*.

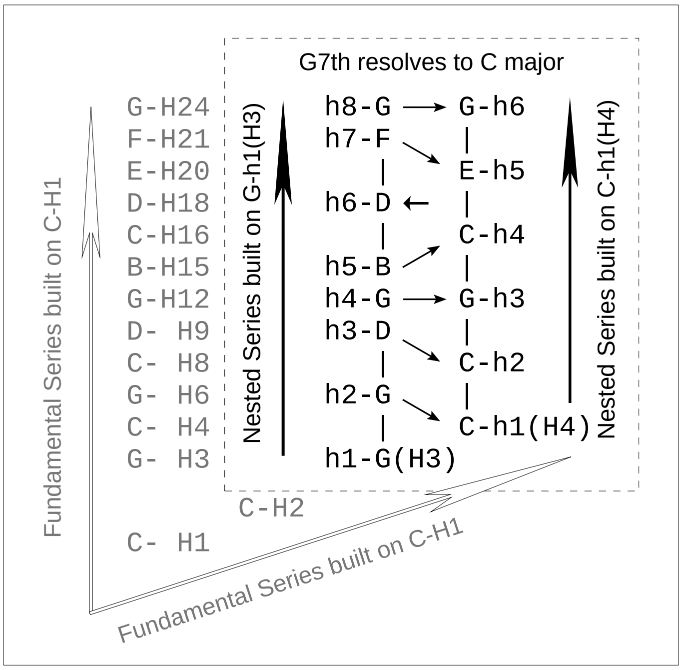


Figure E.11 The two chords of the full cadence V^7–I (dashed line box) are in the configuration of harmonic series. The 3:4 relationship between these two isolated series implies the existence of another harmonic series connecting them together – a fundamental nesting series. Notice that the voice-leading in the upper section of the dashed box leaves no position in the C-major chord for D-h6 to fill; in effect D-h6 is ejected from the system, while G-h1 is left behind in the fundamental series as the progression moves to the C-major chord.

Now, the two isolated series of Figures E.9 and 10A, enclosed in a dashed line box in Figures E.10B and E.11, can be re-interpreted as harmonic series *nesting* within the ratios of a broader underlying series. What makes this re-interpretation possible is their two root/fundamental notes' relationship of 3:4. Any two isolated series with a whole number relationship, be it 4:5 or 13:17 or whatever, will imply the existence of another underlying fundamental harmonic series which connects them together. In Figure E.11 the two isolated series formed from the chords of the full cadence are now illustrated dressed in the full harmonic partials of their 'parent' series (gray text). To distinguish them, the underlying *fundamental nesting series* is marked with capitals – H1, H2, etc. – and the two formerly isolated 'child' series, termed *nested series,* have a lower case 'h' appended to each ratio.

There is, potentially, a great deal going on under the surface in tonally organised music, indeed the objective musical sound is perhaps just the exposed tip of an iceberg of aural relationships; and this broader context of nested harmonic series allows us to connect and interpret the dominant and tonic chords as being (parts of) digit sequences written in the frequency relationships of musical sound, delineating a *single* magnitude – the number twenty-four. Admittedly this example minimal composition has only demonstrated mutable numbers in operation for two chords, important though those chords are, at this point it will have to be taken largely on trust that the principle can be applied consistently, and in general, to tonal music. In Chapter 12, Prelude No.1 from the first book of *The Well-tempered Klavier* by J.S. Bach is analysed in full and other example mutable number analyses are listed in the Examples section of the Contents. Pieces of more normal proportions require mutable numbers with larger values to encapsulate their harmony. Magnitudes in the range from three hundred to three thousand, expressed as three column mutable numbers are more typical; and, such large structures will often require absolute fundamental tones that lie below the audible range.

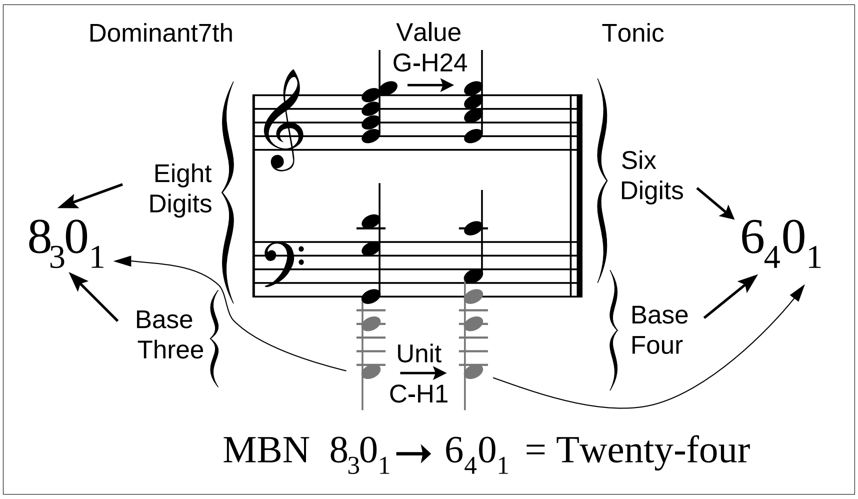


Figure E.12 The full or perfect cadence chord progression of dominant to tonic, encapsulated in two alternative digit sequences of the mutable base number twenty-four.

Now great care is required: in the late nineteenth century a number of music theorists were lead into difficulties pursuing theories about sub-harmonics or ‘undertones’ which proved to be contrary to experience and acoustics. The wider context of mutable numbers, the submerged body of the iceberg, if it has any existence beyond that of a mathematical formalism applied scores, it is to be found within the human ear and mind. This is largely an uncharted ocean, particularly at the higher levels of cognition and at present a comprehensive scientific account of these processes has yet to emerge. Nonetheless, human beings do develop some sense of ‘key’ which is what a fundamental nesting series represents and many low frequency brain rhythms are known to exist – but these are no more than meagre hints.

However, it is not crucial for this model that the aural cognition of music should involve the working out of every detail of formal mutable number exchanges. Rather, it is only necessary that the ear should find common frequencies linking together succeeding chords (conjunctions), because by recognising these linkages, the ear *by implication*, will also have uncovered the common fundamental and nested structure of a *modulating oscillatory system* – the physical expression of formal mutable numbers. In tonal music, most chords do not take the form of complete harmonic series and even less so complete structures of nested harmonic series, but any chord progression that contains a common tone accessible to the ear will also have a common fundamental (though perhaps at a frequency below the range of hearing) upon which can be constructed a valid mutable number digit exchange. The shared frequency may be two succeeding notes (as in the example of the full cadence given above) or more often two succeeding overtones provide the linkage. So, though it is perhaps unlikely that aural cognition would or could explicitly construct such extended structures of nested harmonic series, maybe the ear and aural cognition do something simpler and equivalent to computing these mutable number exchanges, which is, to notice that two contiguous harmonics generated by a chord progression are identical. And under the unifying umbrella of this common frequency, aural cognition may grasp the relationship between the two chords by apprehending the inflection of their harmonic spectra relative to the fixed tone. Which in other words amounts to a *mathematics of aural sensation* capable of extracting the relational essence from musical sound without the necessity of engaging in the full paraphernalia of a mutable base number system. Nature generally achieves its ends with elegance and economy and while this model is only a hypothesis, what is more elegant and economical than a number system?

Undeniably, music in the tonal era largely developed through the agency of human choice, made upon the basis of what delights the ear and aural understanding. So it would seem, if one accepts at least the external mathematical validity of the mutable number approach and discounts coincidence, that whatever else tonal music may be, it is also an exercise in mathematics. Such a conclusion is perhaps not surprising given that ratio, and generally simple ratios at that, form the foundation of the art. What is surprising though, is that the nature of tonal music should be so precisely mathematical at its heart that it produces the structure of a number system; and, one wonders what that character might imply for not only aural cognition but the mechanisms and structures of the mind in general. This however, I must stress, is speculation, and should it turn out to be the case that mutable numbers have no connection at all with the processes of aural cognition, the idea simply falls back to what it essentially is and always has been, *a mathematical model of tonal music*, and hopefully, still a useful analytical tool.

Conclusion

From an overview perspective of world music cultures, the development of western tonal music over the four hundred years from 1500-1900, was atypical. Overall, music cultures throughout the world have tended to evolved structural models based on a vibrant use of pitch and rhythmic elements, often associated with ostinato motives and dance rhythms: creating semi-continuous textured heterophonic strands, over or through which, melodic features of song or chanting are interwoven. The melodic contours frequently include expressive pitch inflection and microtonal intervals lying beyond the whole numbered ratios of diatonic harmony, as for example is used in Arabic cultures. Typically, in the majority of traditions, often complex rhythmic structures are worked out, in the foreground, yielding lively, engaging, energetic music styles – exemplifying a close connection between physical movement, dance and musical sound – as for example is found in many African cultures. Western art music was to develop along a rather different path, with these foreground rhythmic complexities, perhaps one might even call them 'computations of rhythm and meter', being transferred, to a significant degree, to the rarefied frequency level of harmonic progression. Fundamentally similar processes were at work, only the frequency domain was completely different: the focus of activity in the western tradition was sublimated from the frequency range of beats per minute to that of cycles per second. In western music, as the 3:2 hemiola rhythmic proportion became an increasingly rare, though still powerful device, the 3:4 pitch ratio of the dominant-tonic harmonic progression was to become ubiquitous. Of course, the differences are of degree and emphasis, not absolute or exclusive. However, the atypical evolutionary route of western musical practice (engendered by the development of polyphonic music) across a landscape of high frequency harmonic organization, was to open up a significantly wider range of relational exchanges and transitions than it is possible to access at the coarser grained, low frequency level of rhythm and meter. With these enhanced relational resources, western musicians were, intuitively, to forge an extensive and logically consistent system of musical organization – the tonal system. *That this music system is effectively also a number system, is the burden of this document.*

In summary, 1) chords, through extrapolation to related sets of nested harmonic series, form the digit sequences of mutable base position-value numbers; 2) aurally intelligible chord progressions are reconfigurations of mutable number digit sequences representing the one, selfsame, value; and 3) tonal compositions are successions of mutable numbers (i.e. digit sequences) which, in principle, amount to number processing in sound.

Notes

1. Dies, Albert Christoph, *Biographical Accounts of Joseph Haydn* (Vienna, 1810). Eng. trans: Gotwals, V., Haydn: *Two Contemporary Portraits* (Uni. of Wisconsin Press, Milwaukee, 1963).

2. Conventions: A pitch standard of middle C = 256Hz (a little lower that Concert Pitch, A = 440Hz) is used throughout – for the convenience of its concordance with the octave-doubling powers of two frequency sequence, eg. $2^0 = 1$, $2^1 = 2$, $2^2 = 4$, $2^3 = 8$, etc., $2^8 = 256$. Linked in with this choice is the fact that the vast majority of examples and discussions focus on the proto-typical key of C-major. Any major key could have been used – what is true of one is true of all. Usage is International/American English and in matters of typography generally only standard fonts and text characters have been employed; thus some signs are written out (e.g. Aflat) and others substituted. Finally, a nexus is drawn between this proto-key or *tonal center of C* and the ratios of the harmonic series, such that each ratio is associated with a note letter (and generally sharp sign if required, eg. Bflat = A#). Implicit in this convention is the view of the scale tones as a system of *dynamically varying* 'pure' ratios and relationships, rather than a fixed (and 'impure') grid of equally tempered pitches.

3. For a fuller account see: Barrow, J.D., *The Book of Nothing* (Vintage/Random House, London, 2001) and Midonick, H, (Ed) *The Treasury of Mathematics: 1*, p249, (1968, London: Pelican Books).

4. Midonick, H., (Ed) *The Treasury of Mathematics: 1*, p249, (Pelican Books, London, 1968).

5. Taylor, C., *The Science of Musical Sound* in "Music and Mathematics", Eds. Fauvel, J, Flood, R, & Wilson, R, p59, (OUP, 2003).

6. Beament, J., *How We Hear Music*, (The Boydell Press, Woodbridge, Suffolk, UK, 2005). Beament gives an interesting account of notes being apprehended as either pitch or tone in what he calls the 'Three-tone Paradox', section 7.6, p83; and the sensation of a perceived pitch being generated by overtones of a non-existent fundamental.