17

Appendix B

FRACTIONAL MUTABLE NUMBERS

Thus far mutable numbers have been introduced and developed for the representation of positive whole numbers, and the positive integers are basically all that are required for harmonic analysis. This is because the ear and processes of aural cognition 'read' chords from bottom to top, as (parts of) ascending harmonic series. Even when the information contained within the ascending frequency relationships of a simple common major chord is inverted – as in the process of reflection – the ear still reads the information from bottom to top and so recovers the configuration of a minor chord rather than that of an inverted major chord. For this reason: that the ear always reads harmonic relationships in an upward direction, and that ascending series represent positive whole mutable base numbers, only these digit sequences are directly relevant to the harmonic analysis of tonal compositions.

However, as mutable numbers can easily and naturally be extended to cover fractional values; and perhaps also because they could be of some utility in dealing with minor chords (Chapter 14), plus encapsulating Arthur von Oettingen's conception of *phonics* (notes sharing a common overtone e.g. the minor triad E-h10, G-h12 and B-h15 share the overtone B-h60 – Chapter 11), they are briefly introduced here.

In positional number systems fractional values lying between one and zero, are represented as implicit ratios. For example, in the decimal number system, one-half – one divided by two, the proportion of unity measured by the ratio 1:2 – is represented as 0.5 that is five divided by the column base which is ten in decimal, 5/10. Expressed in the generalised positional notation decimal 0.5 would be written: $0_1.5_{10}$ and this clearly exposes the implicit ratio 5:10 which reduces to 1:2 or 1/2, one-half. Indeed in mathematical terminology, numbers terminating in 'clean' digit patterns, such as decimal 0.5 or decimal 0.333... or decimal 0.090909... are referred to as *rational* numbers, numbers formed from ratios. In practice the representation of fractional values is normally limited to decimal numbers and we use the term decimal point '·' which might more rightly be described as the 'fractional point'. It is perfectly possible to represent factional values in other fixed base systems like binary (base two) or hexadecimal (base sixteen) but generally there appears to be little need to do so.

Another example: one-quarter, decimal 0.25. Here the general notation would be $0_1.2_{10}\,5_{10}$ yielding the ratio 25:100 or the fraction 25/100, which reduces to 1/4, one-quarter. Thus, with the column bases now dividing the column digits (rather than multiplying them as on the whole number side of the decimal/ fractional point), we have two tenths plus five hundredths:

$$2/10 + 5/(10\text{x}10) \;=\; 20/100 + 5/100 \;=\; 25/100 \;=\; 1/4$$

All fixed-base positional systems have difficulty in expressing some simple fractions – this is essentially a feature deriving from the trade-off in having the convenience of a fixed base. A fixed base positional system is easier to use in many respects but the straight-jacket of only having a single base introduces some inflexibility. And because the common fixed base systems use even bases – two, eight, ten and sixteen – they all struggle to express the simple fraction of one-third. For the decimal one-third we write 0.333... a sequence of digits followed by three dots to indicate an indefinite extension of the digits. Three divided into one wont go; three divided into ten-tenth goes three time plus one (tenth) remainder, which is where the statement started but one column to the right, so we must repeat the procedure..., add infinitum. The process produces an unending, predictable sequence. If one asks: What is the ninety-ninth digit? The answer is of course three, there is no need to actually calculate the ninety-nine digits to find out. Notwithstanding this certainty, it is something of a humiliation to have to admit that decimal numbers cannot express the simple magnitude of a third, precisely. However, mutable numbers don't suffer from the inflexibility of fixed base systems and cope with all fractions equally well. Again using the generalised notation:

Decimal: $0_1.3_{10}\,3_{10}\,3_{10}$ ... becomes the tidy MBN: $0_1.1_3$ without an extension.

Most of the theoretically unlimited range of values extending between one and zero, behave like one-third, that is, have indefinite extensions, but crucially the majority of the extensions are not predictable. Mathematicians call these awkward numbers *irrational*, 'not ratios', perhaps the most famous one probably being pi, the relationship between the diameter and circumference of a circle – 3.1415926535... The ten decimal places gives some flavor of the apparently random sequence of digits, which continues on and on. To date the extension has been calculated to many billions of digits without any pattern emerging to indicate a ratio could be extracted from it. Ultimately, numbers with indefinitely extended digit sequences are written as approximations to an actual value, a value which cannot be expressed exactly in the number system being use. For simple ratios like one-third (decimal 0.333...), changing to a numbers system with an appropriate base allows the value to written down precisely. For example in a base six number system one-third is written: 0.2 or explicitly $0_1.2_6$ (using decimal subscripts) – but fixed base systems will always struggle with some factions. Using a base six system to get around the difficulty of 0.333... only transfers the problem to other numbers, e.g. one-fifth, tidy decimal 0.2, becomes $0_1.1_6 1_6 1_6$... However, for the awkward irrational numbers there is no simple fix at all. Even with the flexibility of mutable numbers, pi has no precise expression – and is not a particularly pretty sight either:

MBN $3_1.14159..._{100000...}$

The best that can be managed for irrational values is to write down an exact rational value which is arbitrarily close to the inexpressible irrational magnitude. Thus for pi 3.1415926535... in decimal, we write down the exact rational number three, plus the fraction 1415926535 divided by 10000000000, as an approximation to pi, and append three dots '...' to indicate this is not exactly the value. By adding more digits to the fraction a closer approximation may be obtained but the precise value can never be reached.

Now the question arises (again) what meaning is to be attached to numbers, which even in principle, cannot be expressed exactly. For 'abstract' mathematics this is not such a great problem as the exact irrational value can be postulated to exist amongst an infinitude of other values along the smooth extrapolation of the number line; and a symbol – in this example the Greek letter pi – can used to denote the exact magnitude. But for a physical number system, in a finite context, these irrational numbers cannot be reached, they are inexpressible. Yet on the other hand, in the physical world, there exists circles and radii which exhibit such awkward relationships. This same conundrum faced the Pythagoreans of ancient times, when they discovered the relationship between the diagonal and the sides of a unit square were linked by a similarly irrational relationship – the square root of two: A relationship in music which produces the restless and awkward sounding equal-tempered tritone (C-F# 1:1.414, see file SCALES.PDF in EXTRAS). Perplexing and unresolved, such issues perhaps suggest that geometry might be less fundamental than the arithmetic of whole number relationships: That spatial relationships in the material world in some way partake of the freedom of software – a freedom that emerges from the grainy hardware of whole numbers – or of course, that there are “more things in heaven and earth”, than dreamt of in this chunky philosophy.

In the previous paragraphs of this section you will have already met a few fractional mutable numbers, so it is probably high time they were formally introduced. Fractional values in mutable numbers, are represented as digit sequences beginning with a leading zero (0_1) and continuing with further columns (to the right of a ‘fractional point’) containing zeros and bases, finally terminating with a column digit greater than zero. This is basically the reverse of whole mutable numbers. It may be helpful to associate the base ‘$_1$’ of the leading zero (0_1) in the units column, with the fractional point. The fractional point has been include in examples but is not absolutely necessary. A number of simple examples are given in Figures B.1 and B.2.

Number	Decimal	Mutable	Harmonics (frequency)	Ratios
One-half	Dec: 0.5	MBN $0_1.1_2$	h2, h1	1:2
One-quarter	Dec: 0.25	MBN $0_1.1_4$	h12, 6, 4, h3	3:12
One-third	Dec: 0.33...	MBN $0_1.1_3$	h6, 3, h2	2:6
Three-fifths	Dec: 0.6	MBN $0_1.3_5$	h60, 30, 20, 15, h12, 24, h36	36:60
Five-eighths	Dec: 0.625	MBN $0_1.0_2\,0_2\,5_2$	h8, 4, 2, h1, 2, 3, 4, h5	5:8
Five-quarters	Dec: 1.25	MBN $0_1.5_4$	h12, 6, 4, h3, 6, 9, 12, h15	15:12

Figure B.1 Example fractional values in decimal and mutable numbers, given with harmonic relationships and the ratios between first and terminating frequencies. The harmonics are set at the lowest level which allows all the relationships to be expressed in whole numbers. The bases come first as descending wavelength series terminating with the underlined ratio, from where the final digit counts upward in the form of a normal harmonic series. So in the harmonics column the left (first) entry represents the unit value ‘MBN 0_1’ in each case.

As you would probably by now expect with mutable numbers, there are a range of alternative digit sequences for many of the fractional values – Figure B.2.

Number	Decimal	Mutable	Harmonics (frequency)	Ratios
One-half	Dec: 0.5	MBN: $0_1.2_4$	h12, 6, 4, h3, h6	6:12
One-quarter	Dec: 0.25	MBN: $0_1.0_2\,1_2$	h12, 6, h3	3:12
Five-eighths	Dec: 0.625	MBN: $0_1.5_8$	h840, to h105, to h525	525:840
One + quarter	Dec: 1.25	MBN: $1_1.0_2\,1_2$	h4, h2, h1 + h4	(1+4):4

Figure B.2 A few alternative mutable base digit sequences for the values expressed in Figure B.1. Illustrated by the last example in each table, mutable numbers can distinguish five quarters (5/4) from one and one-quarter (1.25).

Taking the fractional mutable numbers apart: First comes the empty unit column ‘0_1’ to set a reference fundamental frequency a notional ‘H1’ for the number. Next, after the optional fractional point, are zero or more columns containing zero digits and their bases. Finally terminating the sequence, the last (and perhaps only) column contains a digit one or greater and again a base subscript. Essentially, the sequence of fractional column base(s) is relocating the reference fundamental frequency step by step down a wavelength series or sequence of nested wavelength series. For example using the 'arithmonic' notation: a1(h840), a2(h420), a3(h280), a4(h210), a5(h168), a6(h140), a7(120), a8(h105). Here the original reference fundamental – h840 – which represents unity '0_1', the number one, has been transferred down to h105. A process similar to moving the decimal point around, as in floating point arithmetic. On its new starting frequency, h105 in this example, the final column digit constructs an ascending harmonic series: h105, h210, h315, h420, h525. The ratio of the final frequency to the original reference frequency, the beginning and end points, yields the fractional value 525 divided by 840 or five-eighths: MBN $0_1.5_8$ the third example in Figure B.2.

Where there is more than one column to the right of the fractional point, the downward stepping wavelength series has a nested structure. For example, in the fraction five-eighths as represented in Figure B.1, the sequence of base two columns efficiently transfers the reference fundamental from h8 to h1 by three octave steps, before the final digit produces the ascending series h1 through h5. I'm not sure if it makes sense to talk in terms of ground states for fractions, but if it does then this configuration is five-eighths' most economical arrangement: its ground state. Mutable fractions display other interesting features, one of which is to explicitly represent values greater than one, as either fractions or as integers plus fractions, as in the examples of five-quarters in Figure B.1 and one and one-quarter in Figure B.2.

It is a natural step, now that we have mutable digit sequences capable of representing both whole numbers and fractions, to join the two expressions together to form rational values lying between the integers above one. As indeed the last example in Figure B.2 does, here are some more:

Decimal	Mutable	Harmonics (frequency)	Ratios
Dec: 2.5	MBN: $2_1.1_2$	h2, h1 + h2, 4	(1+4):2
Dec: 8.25	MBN: $2_2\,0_2\,0_1.0_2\,1_2$	h4, h2, h1 + h4, h8, h16, h32	(1+32):4
Dec: 8.25	MBN: $4_2\,0_1.1_4$	h12, 6, 4, h3 + h12, h24, 48, 72, 96	(3+96):12
Dec: 40.33...	MBN: $2_2\,0_2\,0_5\,0_1.1_3$	h6, 3, h2 + h6, 12, 18, 24, h30, h60, h120, 240	(2+240):6
Dec: 15.6	MBN: $5_3\,0_1.3_5$	h60, to h12, 24, h36 + h60, to h180, to 900	(36+900):60

Figure B.3 Fractional mutable base numbers greater than one, with harmonic relationships, first, descending (fractional part); and after the plus sign, ascending (whole number part); plus the ratios of unity they represent.

Four closing notes: Firstly, the representation of alternative fractional digit sequences is not quite as straightforward as for whole numbers where the leading digit and base subscripts are entirely interchangeable. Take for example three-fifths and fifteen. Here swapping digit and base subscript works fine for the whole number alternatives MBN $3_5\,0_1 = 5_3\,0_1$ equals fifteen, but performing the same trick on three-fifths, MBN $0_1.3_5$ yields the fraction five-thirds MBN $0_1.5_3$ not the same thing at all. Alternative fractional digit sequences are available from among the column bases and separately in the factors of the terminating digit should there be more than one. It just requires a little more care, whereas multiplication is commutative, division is not, the order of bases and digits matters in fractional digit sequences. When splitting a final digit into factors brackets help to make clear the meaning, for example, ten-elevenths which is decimal 0.909090... or mutable MBN $0_1.10_{11}$ – could be expressed as MBN $0_1.(5_2)_{11}$ giving a nested configuration to the digit.

Secondly, the representation of irrational values in the form of mutable numbers should perhaps normally be avoided, as to do so breaks the connection between physical oscillatory configurations and mutable digit sequences. In principle at least, a mutable number can always be interpreted as a dynamical system of oscillatory relationships – chords in music and perhaps of structures in the world. Although a notation for irrational values comes to hand readily, for example, the rather rough approximation of decimal 3.14...

pi by a few alternative digit sequences: MBN $3_1.14..._{100...}$ or MBN $0_1.314..._{100...}$ or MBN $3_1.0_2\ 0_2\ 0_5\ 14..._{5...}$

it is taking mutable numbers across the divide into abstract mathematics as it seeks to describe an impossible physical system, a chord of unlimited notes that could never be played.

Thirdly as a small amount of multiplication and division are introduced in Example R *Study: The Divisors of Seventy-two* it might be helpful to briefly review this aspect of mutable base numbers. In this example of computation through tonal music the question arises of how the sum MBN $47_1 \div 5_1$ should be handled. Interpreted as a physical structure this sum translates into the factor format:

h1, h2, h3, ... through h47 divided by h1, h2, h3, h4, h5

But as division involves the removal of a bottom level of nesting equal to the divisor harmonic series (or nested series) there is a problem because MBN 47_1 is a prime and cannot be expressed as a nested structure beginning MBN $...0_5 0_1$ (i.e. 47 is not divisible by 5 without remainder). However all is not lost; forty-five plus two equals forty-seven and forty-five – MBN $9_5 0_1$ – does have the required structure to match that of the divisor – MBN $1_5 0_1$. So by removing the column base ‘$_5$’ division of forty-five by five is achieved.

MBN: $9_5 0_1 \div 1_5 0_1 = 9_1$

h1, h2, h3, h4, h5; h10, h15, h20, h25, h30, h35, h40, h45 ÷ h1, h2, h3, h4, h5 = h1, h2, h3, h4, h5, h6, h7, h8, h9

But what of the remaining two leftover? Well, as division by five is the same as multiplication by the fraction one-fifth, multiplication provides an answer.

Number	Mutable	Harmonic Series	Ratios
Mutable Number Two:	MBN 2_1	h1, h2	ascending series 2:1, or 24:12
Mutable Number One-fifth:	MBN $0_1.1_5$	h60, h30, h20, h15, h12	descending series 12:60
Two multiplied by One-fifth:	MBN $2_1 \times 0_1.1_5$	h60, h30, h20, h15, h12 x h1(h12), h2(h24)	
Equals Two-fifths:	MBN $0_1.2_5$	h60, h30, h20, h15, h12, h24	descending/ascending 24:60, 2:5

Therefore Forty-seven ÷ Five: MBN $47_1 \div 5_1 = (45_1 + 2_1) \div 5_1 = 9_1 + 0_1.2_5 = 9_1.2_5$ (Nine and Two-fifths)

Finally, to recap the role fractions in the context of tonal music: Although human ears are unable to grasp directly the descending (wavelength) whole number relationships which fractional mutable base numbers express, we are able to make something of the aural patterns they represent, by reading them in ascending order. In approaching these audible ‘fractions’ the wrong way round, and interpreting them within the context of a scheme of nested harmonic series, our ears discover the sonority of the minor triad and the chord of the added sixth amongst others – truly a fruitful misapprehension.