14

Reflection

HARMONIC INVERSION

In the final analysis, the *dualists* of the later nineteenth and early twentieth centuries were unable to convincingly reconcile their elegant theory with musical practice. Tantalisingly dualism yields an ‘abstract’ mathematical solution for tonal music which doesn’t quite match physical reality and human experience. However, mathematical group theory and modern information theory may well provide a means whereby the essential insight of dualism could be integrated into the prevailing *monist* (fundamental bass) understanding of harmonic progression in tonal music. To reach the goal will also involve using modulating oscillatory systems and mutable base numbers; but first, before explaining how this might be achieved, it may be helpful if some of the musical meanings that are often attached to the word *inversion* are differentiated.

Both the concept of melodic inversion and its practical application are long established in the western musical tradition. Throughout the tonal era, technically inclined theorists and composers have gloried in the contrapuntal possibilities that polyphony allows to those with the skill and imagination to harness this resource. And by general agreement, the greatest exponent of them all was J.S. Bach. For many, myself included, the enriching play of contrapuntal voices is one of the chiefest of delights that music offers; notwithstanding, the MOS model teaches us that harmony is fundamental, not counterpoint.[1]

A central technical device evolved by contrapuntists writing music in the tonal tradition, is that of the *real* and *tonal answer*. A real answer involves the exact repetition of the intervallic relationships of a fugal theme, while a tonal answer ‘bends’ the occasional interval in order to better fit the melody into the prevailing harmony. More generally, composers have tended to use similar tactics in the elaboration and extension of their melodic themes, particularly where a theme is inverted within an unchanging key-center. This technical ‘sleight of hand’ is important on two counts: first, it confirms and charts the increasing dominance of the harmonic dimension over counterpoint through the tonal era (circa 1450–1900); and second, the technique of adjusting intervals in inverted or transposed melodic themes clearly distinguishes melodic inversion – and compositions built on melodic inversion – from harmonic *reflection*, which, in the meaning attached to the term in this book, demands rigid and rigorous inversion of all intervals, in all parts. Also, the term harmonic inversion has often been used with the general meaning of dyadic, triadic and chordal inversion. For example, the inversion of two notes a perfect fifth apart leaves them at the interval of a perfect fourth, a root position triad has a first and then a second inversion as the lower notes are transposed up an octave or more, and a seventh chord has three inversions. Again, the defining distinction between harmonic reflection and traditional harmonic inversion is that some intervals are adjusted in the latter process. In contrast to both melodic and traditional harmonic inversion, *harmonic reflection is absolutely interval invariant.*

To understand how this invariance arises, consider the following chords: x, y and z in Figure 14.1. Are they consonant or dissonant? By transposing chord x down an octave and chord y up an octave, the E-minor chord can be placed above the C-major chord, to form a single third chord, ‘z’. The constituent notes of chord z are all harmonics of its bottom note C. Chord z is composed of a selection of partials of the fundamental C-32Hz.

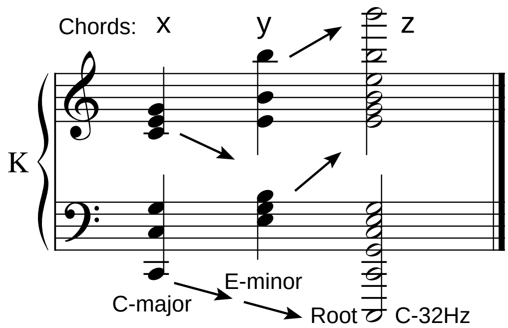

Figure 14.1 Two chords, C-major and E-minor, combine to form parts of one harmonic series.

Most people will agree that chords x (C-major) and y (E-minor) are consonant. But what about chord z, with B natural (h-15) sounding against the root note C? Notwithstanding this clash, it could be argued that chord z is a concord also – given the arrangement of notes presented in chord z. The reason for this perhaps somewhat theoretical opinion is that when the relationships of the notes are considered in terms of ratios of the harmonic series, a particular connection between the notes emerges. This connection is encapsulated in the lowest common multiple (LCM) of the whole number frequency ratios of all of the notes in the chord. The words make this simple concept appear rather more complicated than it is. Here it is in music, number patterns and figures – Figure 14.2.

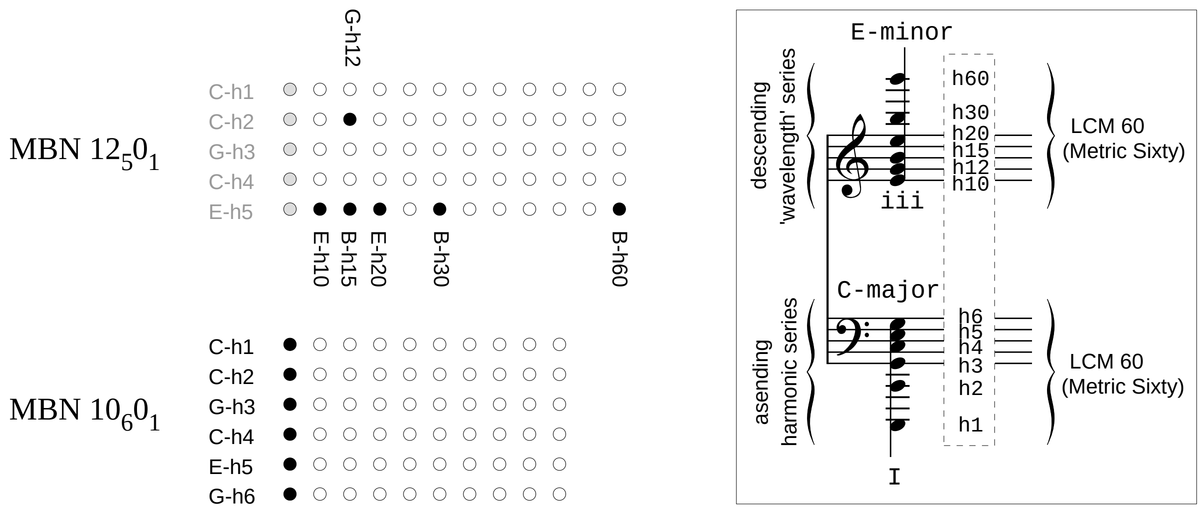

Figure 14.2 The consonant C-major and E-minor chords combine to produce ‘chord z’ (MBN $12_5 0_1 = 10_6 0_1 = 60_1$).

As can be seen in Figure 14.2, mysteriously, these very different chords, C-major, E-minor and chord z, all possess the same lowest common multiple of frequency ratios, based on the common foundation C-h1. That is to say that the lowest number into which all the numbers 1, 2, 3, 4, 5, 6 will divide perfectly (without remainder) is sixty – LCM 60. Likewise, 10, 12, 15, 20, 30, 60 will divide without remainder into sixty also – LCM 60. Thus chord z containing all twelve notes has an LCM of 60 as well. In earlier chapters these LCMs are termed the chord’s ‘Euler metric’ or simply ‘Metric’; written

Metric Sixty, or for shorthand M60. Now as all three chords have the same Metric, and chords x and y are recognisably consonant, should not chord z be viewed as consonant also? Or at least, a mathematical concord?

There is another connection between the chords x and y, but this is a rather subtle relationship which our ears cannot directly grasp, a harmony which a mathematician would appreciate: the reciprocal relationship of harmonic and arithmetic series. While chord x is built from the *frequency* relationships 1 : 2 : 3 : 4 : 5 : 6, chord y is built from the *wavelength* relationships 1 : 2 : 3 : 4 : 5 : 6. Musically, the wavelength relationships manifest themselves as a *descending or inverse 'harmonic' series*, with the descending intervals of chord y from the top note: an octave, fifth, fourth, major-third and minor-third. This is just the same pattern of (ascending) intervals found in chord x – the lowest six ratios of the harmonic series.

When dealing with patterns of relationship in the harmonic series, it is customary to use an 'h' prefix for the ratios (individual harmonics) of the harmonic series. Thus chord x would be h1, h2, h3, h4, h5 and h6. Extending this convention, as introduced in Chapter 7 and illustrated in Figure 7.6, it is also convenient to speak of 'arithmonics' for ratios of wavelength (an inverse series) and to label them: a1, a2, a3, a4, a5 and a6. This system of labels is illustrated in Figure 14.4 measure 1; and, from chord y in Figure 14.2, the arithmonics can be seen to have the relationship h60 = a1, h30 = a2, ..., h10 = a6. Let us explore a little further and take a look at a more complex chord, the dominant-seventh – Figure 14.3.

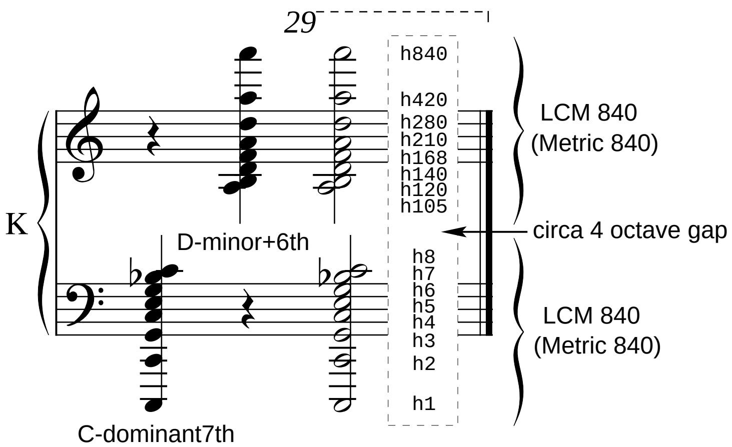

Figure 14.3 The dominant-seventh chord built on C with its inverse or 'reflection' – the D-minor chord with added sixth.

In this chord, though the Metric or LCM is much larger – M840 – the pattern is essentially the same. Notwithstanding whether the seventh and added sixth chords are considered to be consonant or dissonant, the connection between the two chords still holds true, though at a greater separation. Both chords have a Metric of 840 founded on C-h1 (32Hz). Thus we have so far examined two types of chord, the common major chord and the dominant-seventh chord and found that both have 'arithmetic' or inverse partners, the common minor chord and the minor chord with an added major-sixth interval, respectively. (Interestingly, for this latter pairing, Rameau described the dominant-seventh chord as the characteristic dissonance of the dominant scale degree – V – and the minor-added-sixth as the counterbalancing dissonance of the subdominant scale degree – IV). Generalizing these arithmetic relationships leads to the situation where *any chord/harmonic series will have a partner chord/arithmetic (wavelength) series of the same Metric founded on a common fundamental h1.*

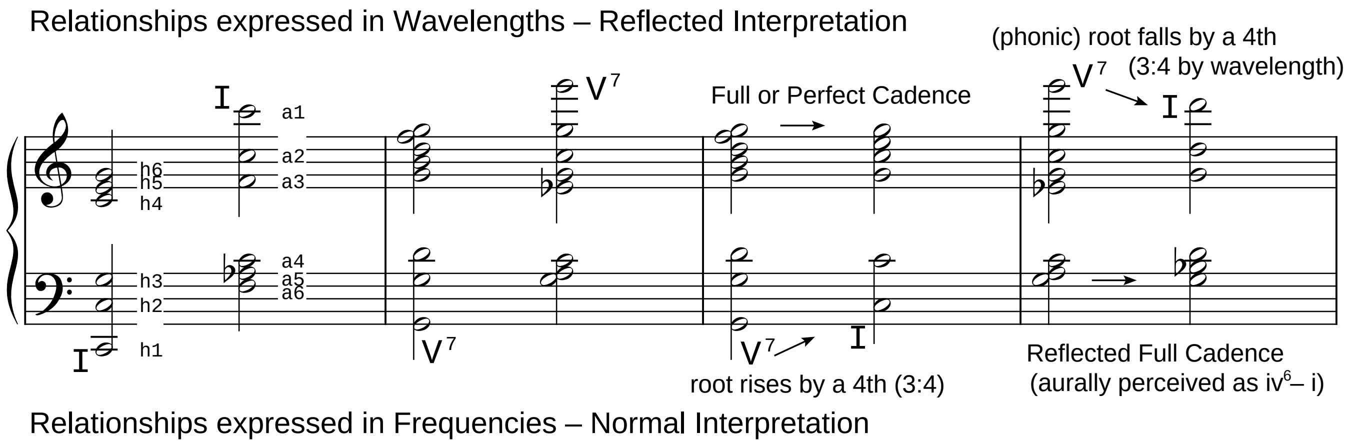

Figure 14.4a Tonic and dominant chords in the key of C major, expressed as ratios of frequency and ratios of wavelength. In the last two measures the exchange (computation) of the information implicit in the chord progression of a full or perfect cadence (V^7– I) is shown in 'normal' and 'inverse' form.

It might be useful, now that we have tonic and dominant chords in both normal and inverse forms, to briefly investigate the reflection of a chord progression, rather than just isolated chords. (The topic is further developed below in section two, *Integrating the Arithmetic*.) In Figure 14.4a above, the first measure shows the tonic chord in normal and reflected form and the second measure the dominant-seventh in like manner. The third measure contains the familiar full or perfect cadence chord progression (V^7–I). The fourth measure is probably less familiar, the reflection of the dominant-seventh to tonic progression, which could be interpreted as a minor chord of the added sixth resolving to a plain minor chord. That is, C^{+6}-minor in second inversion resolving to a root position G-minor (IV^6–I). A beautiful and gentle cadence that is worth savouring. Notice also that the note G forms a common point of stability within both cadences (measure 3 top note and measure 4 bottom note); and, that eight ratios (notes) of the dominant are exchanged for six ratios of tonic. That is, eight notes are exchanged for six notes, between fundamentals (tonic and phonic fundamentals in Oettingen's dualistic scheme) which have a mirrored frequency relationship of 3:4 (6:8) and 4:3 (8:6).

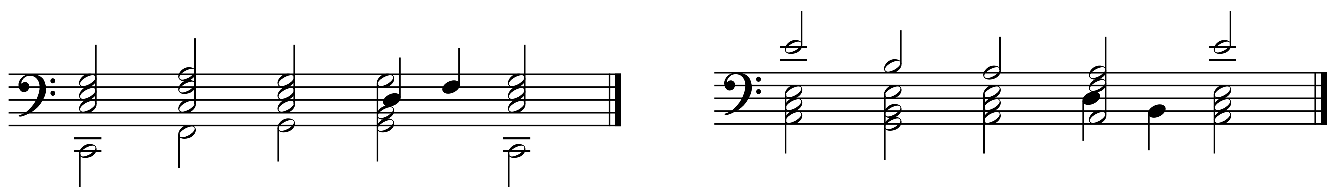

Figure 14.4b Oettingen's own illustration of the full cadence (left) and its aural and visual inversion on the right.

Beginning with the proposition that any chord in music could be viewed as a *configuration of partials* taken from some underlying harmonic series. For indeed our ears, by apprehending the harmonics of timbre and through manufacturing combination tones, tend to fill out the written notes/chords of music into more complete harmonic series. Then a further general conclusion, at least in theory, is that any chord (viewed as an incomplete harmonic series) will, as in the C-major/E-minor chord pairing of Figure 14.2, possess an inverse partner chord of identical Metric and perceived fundamental (h1).

The notion of the arithmetic or wavelength series having a low fundamental ‘h1’, as well as their ‘phonic fundamental a1’ – the top note – arises because our perception and aural interpretation of these inverse series is founded on an essentially ascending harmonic view of them. We simply cannot aurally ‘read’ these inverted series from top to bottom – the flaw dualism never solved. Though the basic information content in the relationships within harmonic and arithmetic series is the same, our perception of them differs as our ears ‘read’ the inverse series from the ‘wrong’ end – as configurations of partials – interpreting wavelength relationships as those of frequency. It therefore follows from this that if the information contained within the chords/series in a piece is changed to its equivalent inverted form – harmonic to arithmetic and vice versa – our perception of this same information content will be different: a different harmonic experience is recovered, a second alternative view of what is basically the same information. Thus, following from the above, each and every tonally organised piece of music possesses a *companion piece* of inversely arranged chords/series, which in informational terms is equivalent, but which we perceive as a different aural experience. That is, an exact upside-down reflection of the original, yielding different, sometimes strange, harmonic sequences.

Also for completeness, in addition to the symmetry between harmonic and arithmetic chords/series, there is a symmetry between forward and retrograde motion – temporal inversion – providing two further ‘views’ of the information contained within the original chord progressions: the retrograde and the retrograde reflection. Retrograde motion might be interpreted as a form of negation – discussed in section three, *Symmetry and Negative Numbers*.) Taking another step of exploration, consider the following (very) simple piece of music.

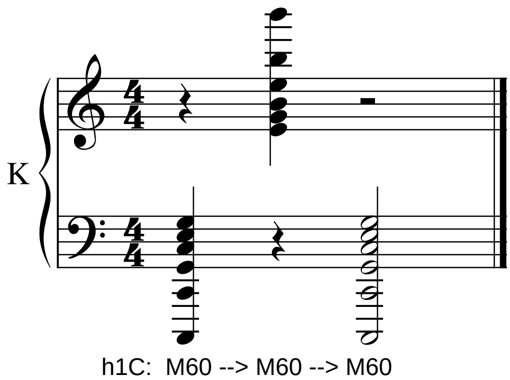

Figure 14.5 In principle, a piece of tonal music consisting of three chords: C-major, E-minor and C-major.

Given the connections that have been established between the various chords of identical Metric, founded on identical fundamental harmonic series, it would be possible to substitute one chord for another in a tonal composition, without altering the flow of Metrics and fundamentals. This substitution when applied to the ‘test’ composition shown in Figure 14.5, would produce the piece shown in Figure 14.6. The effect of the transformation is not unlike that of turning a garment inside-out: the end result may be rather odd looking but it is still a viable item of clothing. By expressing the information of the original chords (the ratios between notes) in inverse form, swapping frequency and wavelength relationships about, the effect is that of turning the music inside-out. For the most part these inverse companion pieces don’t make wonderful sense to our ears, though some do have a strange charm; however, all are theoretically, tonally organised systems – the tonal logic though inverted, still remains. The process of conversion between the two equivalent forms, with harmonic series becoming arithmetic configurations and vice versa, inverts treble and bass by placing the top notes at the bottom and the bottom notes at the top. Major chords are converted to minor chords, minor chords to major, the order of voices are inverted – potentially very good news for players of bass instruments, as the interesting bits tend to fall to the bottom – and melodic motion, up and down, is reversed.

One practical consideration to bear in mind is that in traditional tonal harmony narrower intervals occur in the treble and wider intervals in the bass, overall. (This is due to the spacing of resonances on the ear's detector membrane becoming more crowded at lower pitches.) Thus when a composition is inverted, narrower intervals tend to predominate in the bass which lead to a 'muddy' effect on the harmony. Therefore, arithmetic companion pieces are best heard at (higher) pitch levels that allow the bulk of narrow interval (now at the bottom) to be realised in the middle and treble ranges.

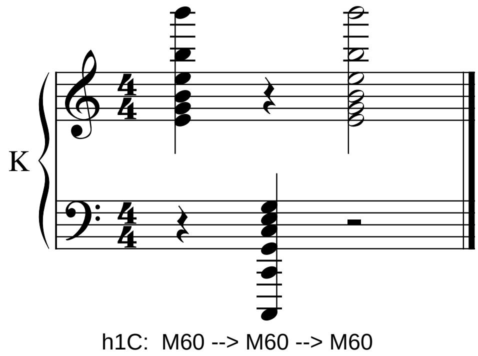

Figure 14.6 In principle, a second 'companion' piece which shares a common sequence of Metrics and fundamentals with the piece in Figure 14.5.

The music in Figures 14.5 and 14.6 is built from a complete series, to show the underlying principle; however most music is, of course, built from chords which form a limited configuration of partials and not complete series, but this doesn't undermine the logic of reflection for normal compositions as generally chords manifestly have roots. Below is a more conventional arrangement of the simple three-chord piece.

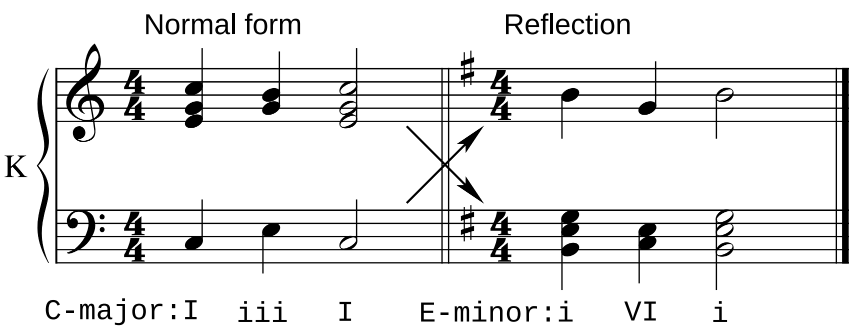

Figure 14.7 The test piece with C-major and E-minor chords in more familiar arrangements (incomplete series) and its reflected companion piece – in the key of E-minor.

Group Theory and Harmonic Inversion

In the discussion of group theory in Chapter 13, the rotation of an equilateral triangle in the plane, about its center of area, was used to illustrate the symmetries of the mutable number six. With the triangle limited to remaining in the plane, it was found to support three transformations: t_1, t_2 and t_0. However, if the triangle were allowed to leave the plane, it would be possible to rotate it about three other axes, drawn from each vertex through the center of the opposite sides – Figure 14.8. Each of the three flip-over operations produces one new transformation and these new transformations are different from the original exchanges where all vertices participated in the motion. In these new transformations two vertices are exchanged, while the third vertex remains untouched. As illustrated in Figure 14.8, the new transformations have the character of a mirror reflection which exchanges left for right. Just as when looking in a mirror, the reflection of your left side becomes the image's right side and vice versa.

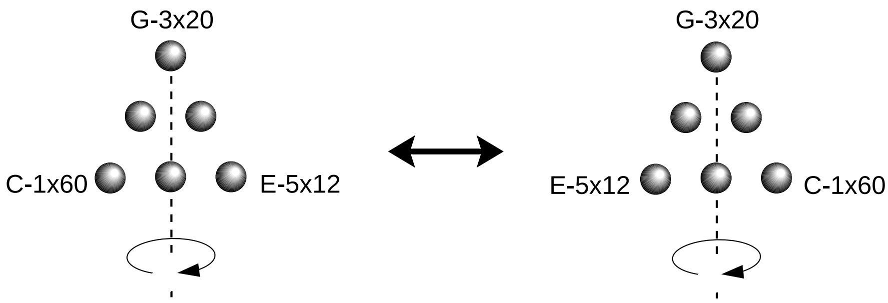

Figure 14.8 In terms of group theory, the flipping over from harmonic to arithmetic series is analogous to the 'mirror reflection' symmetry achieved by rotating a triangle 180 degrees about the axes running from the vertices through the centers of the opposite sides.

It is this quality of mirror reflection, the reversing of left and right or top and bottom, which is manifest in the inverse, reciprocal relationship between harmonic and arithmetic series. (For top to bottom reversal the mirror is placed underneath the object – see Note attached to Figure 14.9.) However, returning to the examples of reflection involving the chords of C-major and E-minor, which share a Metric of 60. Strictly, to apply the illustrative analogy of flipping over the triangle to these chords would require a plane regular polygon with 44 sides[2] (or 45 sides if the reverse of 1×60 is included); as that is the number of distinguishable digit arrangements that the mutable number sixty possesses. Notwithstanding this requirement, the triangle illustration will suffice, as here we are dealing with just two chords. Two chords united by the common mutable number value of sixty which is to say, in the terms of group theory, the chords of C-major and E-minor, viewed as mutable numbers, share a common symmetry or belong to the same symmetry group. Thus, it follows that the foundations of normal and inverse harmonic relationships as viewed from the perspective of the MOS model and mutable base numbers, are united through the principles and methods of the group theory of symmetry.

The connection between harmonic reflection, as outlined in this chapter, and the MOS model's interpretation of the minor mode is intimate. The seed of the idea of reflection can be traced back to *Le Istituzioni Harmoniche* (1558), where Gioseffo Zarlino began to investigate the puzzle of the minor mode in terms of arithmetic relationships; and overall, the process has a strong connection with Hauptmann and Oettingen's 'dualism' of a century ago – the proposition *that the minor mode is the equal inversional partner of the 'normal' major mode*. However, in contrast, the MOS model doesn't suggest that our ears and aural cognition can interpret arithmetic series 'natively', in the form of descending integer relationships, as a dualist might wish. Rather, through the serendipitous limits to our perception of the reciprocal form, we recover a different 'harmonic' view of the inversely arrayed information – the minor chord rather than an inverse major chord – and thereby gain extension of our musical experience. From this it follows, that for any tonal composition there exists a symmetric partner piece, a harmonically reflected companion composition consisting of precisely the same information (in inverse array), which the ear perceives as being different from its original. Figures 14.9–10 below provide two examples which demonstrate the process of reflection on a somewhat broader scale. More reflected compositions are listed in the Examples section.

Nun Komm' Der Heiden Heiland

Latin Hymn: Veni Redemptor Gentium, from Walter's Gesang Buch 1524
Set by J.S. Bach. Ratios of frequency reflected into ratios of wavelength.

Figure 14.9 Chorale preceded by its 'reflected partner', with harmonic analysis. Though the raw information content of the two pieces is the same, our perception of them differs as our ears 'read' the inverse series from the 'wrong' end – interpreting wavelength relationships in terms of frequency.

Note: Although the two mirror fugues (Contrapunctus 12 & 13) from the *Art of Fugue* by J.S. Bach are very close in appearance to these reflections, they are fundamentally different, in that their focus is essentially contrapuntal and centered on *invertible (tonal) counterpoint,* rather than *strict harmonic inversion:* as demonstrated in the first two bars of each piece, where the intervallic sequences are not exact reversals of the original themes.

Vater unser im Himmelreich

(Harmonic or frequency relationships of the lower staves, the chords, translated in the upper staves to arithmetic relationships, i.e. relationships of wavelength.)

Figure 14.10a Lower system: the normal harmonic form of Johann Krieger’s prelude, with its ‘inverse partner’ printed above to aid comparison.

Figure 14.10b Lower system: the normal harmonic form of Johann Krieger's prelude, with its 'inverse partner' printed above to aid comparison.

Briefly, for the sake of completeness, the other possible symmetry operations all involve the element of smooth transition (translation) rather than step-change, which might best be connected with the musical technique of glissando in the domain of pitch/timbre – gliding from one note to another. Translation also implies the possibility of arbitrary changes of pitch, a smooth continuum of choices, rather than the step-like quanta of musical scales. Thus although the examples shown in Figure 14.11 conform to scale steps, these have been used for convenience; at their core, translation symmetries are arbitrary in size and theoretically infinite. Though potentially of considerable application to contemporary genres and especially electronic music, translation symmetries, due to their their flowing character, are generally antipathetic to the discrete steps of harmonic progression and its underlying structure of whole mutable numbers. Therefore, where such glissando operations have been used in the tonal canon, they are most often limited in duration, structurally peripheral and disciplined by beginning and ending on scale degrees. In tonal music, glissando has been used more as embellishment than substance. However, in the domain of duration the smooth translation symmetry lies at the heart of expressive tempo variation, as discussed in Chapter 10 and illustrated in Example K.

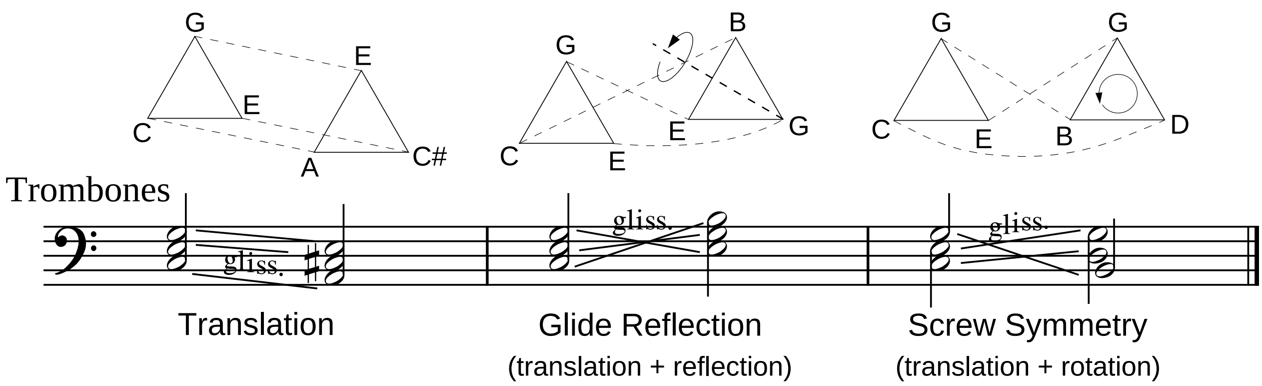

Figure 14.11 The translation symmetries. The smooth continuum implied by translation sits uncomfortably with the step-based organisation of tonal music and has generally played only a small role historically.

INTEGRATING THE ARITHMETIC INTO TRADITIONAL HARMONY

Introduced above in Figure 14.4, the reciprocal character of harmonic and 'arithmetic' chord progressions was briefly studied using the example of a full cadence in normal and inverse form. However, as the human ear does not register arithmetic/wavelength relationships to any meaningful degree, the question arises as to what relevance these mirroring arithmetic structures might have to our understanding and analysis of 'real' tonally organized music. After all they would appear, on face value, to be little more than theoretical constructs without direct or meaningful application: the flaw that dualism struggled to overcome. So, if the ear cannot *directly* grasp arithmetic organisation within music – as it does the harmonic series – might the ear in some manner, *indirectly* apprehend the arithmetic principle? In Figure 14.11, the reflected full cadence example is developed further, in order to investigate the manner and means by which the arithmetic principle might be incorporated into the 'normal' harmonic sphere of music. Though great strides have been made in areas such as neuroscience, psychology and physiology, a full understanding of the mental processes of aural cognition is still some distance away. Probably, if and when it does arrive it will not include 'arithmetic processing'. This being the case, the following example is offered principally as a method of drawing the inverse, arithmetic principle down into the 'real world' of harmonic organisation and perception.

As discussed above, the arithmetic or inverse full cadence is perceived as a minor chord of the added sixth on the subdominant, resolving to a minor tonic chord (iv^{+6} – i). Although the arithmetic or wavelength organisation of this inverse full cadence is the exact reciprocal of a normal 'harmonic' dominant-seventh to tonic progression – with an arithmetic series of eight ratios/oscillators exchanged for six ratios/oscillators, constructed on Oettingen's 'phonic fundamental' tones a perfect fourth apart – our perception is not that of a normal full close cadence, but of a beautifully delicate, minor, subdominant-tonic close. This perception – one might almost say mis-perception – arises from our inability to understand wavelength relationships, which pass unnoticed, leaving us with the relationships that can be grasped: those of frequency. In essence the ear is reading the arithmetic series from the wrong end, not from the top down but from the bottom up – as it would any harmonic series – resulting in the perception of a complex arrangement of harmonic frequencies, rather than the simple downward arrangement of wavelength relationships: the 'arithmonics' B-a1 through B-a8 and F#a1 through B-a6 (Figure 14.12).

The full cadence example given in Figure 14.4 separated the two domains of arithmetic processing (measures 2 and 4) and harmonic processing (measures 1 and 3); the two chord progressions were not integrated, except in their having an identical reciprocal structure; and in Figure 14.3, the integration pushed the two domains impracticably far apart. In contrast, Figure 14.12 illustrates the integration of the two domains, the arithmetic and harmonic principles acting in concert, and although the arrangement of oscillators is achieved at the cost of total adherence to whole number ratios (e.g. objective note C# in the first column ideally should be 274.3Hz rather than 274Hz), in practice, the accommodating tolerance of the ear can be relied upon to smooth over such slight deviations from integer values.

When the ear reads an inverse chord/series from the 'wrong' end, garnering as a result a rather complex arrangement of frequencies, it inevitably attempts to understand the relationships in terms of the normal, ascending, harmonic series. So it is into the arena of the *real* ascending overtone series that the *theoretical* relationships of the descending arithmetic series must be embedded. The example in Figure 14.12 is of the chord progression iv^{+6} to i – a minor imperfect plagal cadence, with the objective tones shown in black notes upon the treble clef. Do play them, it's a beautiful cadence.

Above the black objective tones in Figure 14.12 are white 'theoretical' tones completing the eight ratios of an inverse/arithmetic 'dominant-seventh' chord and the six ratios of the inverse/arithmetic 'major tonic' chord. Though chords are rarely arranged with such large interval at the top, adding the white notes on the same stem in with the objective (black) notes would not alter the harmonic sense. And, set above these, lie the fundamental arithmetic tones with upward pointed stems, terminating in the absolute fundamental wavelength F#, 'A1'. Interestingly, all of these 'white notes' have a real objective existence as partials of the sounding notes, it is just that the human ear and cognition cannot decipher the descending relationships they bear.

Below the objective chords, in gray notes, are the first five ratios of an aggregated/nested series combination, which encompass the 'real' harmonic gist of the two objective chords in whole number ratios. The '*background*' nested series are founded on C-16Hz and G-24Hz, with the objective (black) notes taking up the ratios B-h15, C#h17, E-20, G-h24, B-h30 for the first chord, and B-h10, D-h12, F#h15, B-h20 for the second. Placed within this background series is the '*foreground*' aggregated series, stepping in groups of five background ratios. It is this aggregated series which defines the roots of the chords and contains the structural intervals of fifth and octave that appear in the objective notes. Below this system, with downward pointing stems, lies the absolute fundamental nesting series, founded on C-8Hz, 'H1'.

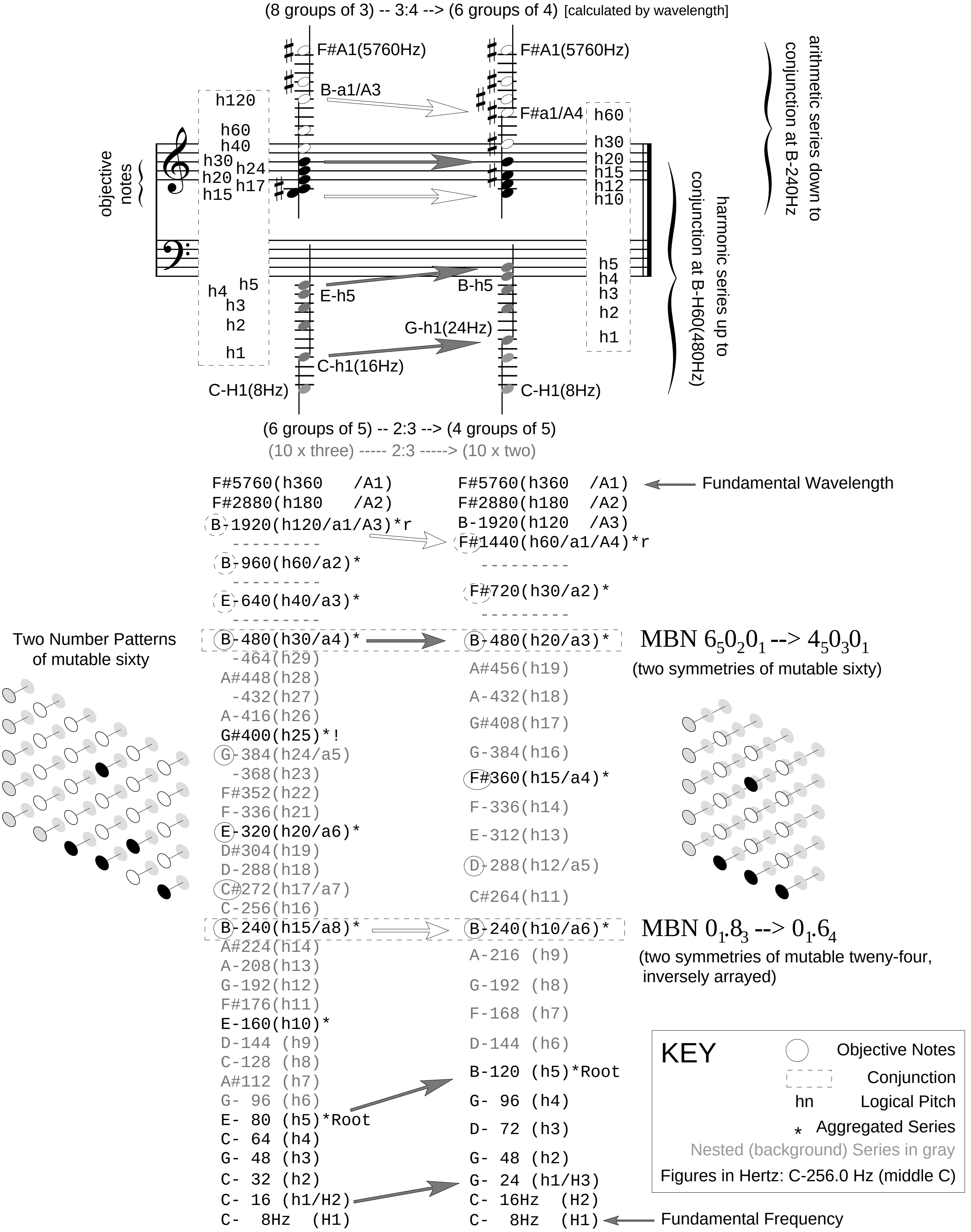

Figure 14.12 The minor chord progression iv^{+6} to i, arrayed in harmonic and arithmetic configurations.

So Figure 14.12 illustrates in some detail the MOS model's interpretation of the minor to minor chord progression iv^{+6} to i – a second inversion, minor chord of the added sixth on the subdominant, resolving to the minor tonic chord. The objective sounding notes are shown in black on the treble clef, above them are the theoretical 'arithmetic' tones in white, and below them, the remaining harmonic tones of the MOS analysis in grey.

Below the staff are two columns listing the ratios of the fundamental, nested and aggregated harmonic series, which encompass all the objective notes in integer values. The text and arrows around the staves indicate the modulation exchanges taking place. Gray arrows show the perceived harmonic exchange in the aggregated series, a sesquialtera 2:3 modulation of six groups of five built on E-80Hz for four groups of five built on B-120Hz. This foreground aggregated exchange is set within the broader context of a similar exchange of thirty ratios built on C-16Hz for twenty ratios built on G-24Hz in the underlying nested series. Below these two layers of transformation lies the stability of the fundamental nesting series founded on the frequency C-8Hz. Ultimately, the two sesquialtera 2:3 exchanges in the aggregated and nested series are components nested within this fundamental series, executing what is at bottom a reconfiguration between two digit sequences of the mutable base number sixty.

MBN $6_5\,0_2\,0_1$ –> $4_5\,0_3\,0_1$

In total MBN sixty has 44 distinguishable arrangements or symmetries to choose among.[3] This is the MOS model's account of the real musical experience, based on the ear's apprehension of objective frequency relationships. However, there is another way of approaching the chord progression, an approach which depends more on the mathematical 'ear' and sensibilities. This involves looking for the simplest relational explanation in terms of the algorithm of symmetrical exchange; that is, the analysis to be found in the 'arithmetic' domain of wavelength relationships. The white arrows in Figure 14.12 outline the arithmetic exchange of a sesquitertia 3:4 modulation of eight ratios for six, descending from B-1920Hz and F#1440Hz, respectively (4:3 by frequency), and set within the stability of the unchanging fundamental wavelength series built on F#5760Hz 'A1'. The ratios/notes of this descending nested system are set above the objective notes, sweeping downward to their conjunction at B-240Hz, so eventually including the objective chords also. This arithmetic system is the familiar full or perfect cadence exchange upside down! That is to say, the sesquitertia 3:4 modulation performed with relationships of wavelength. However, for consistency and to maintain the distinction between the harmonic and arithmetic spheres, a fractional mutable number value should, perhaps, be used to describe the rearrangement of digits for the mutable number value of one twenty-fourth.

MBN $8_3\,0_1$ –> $6_4\,0_1$	harmonic/frequency notation of sesquitertia 3:4 exchange
MBN $0_1.8_3$ –> $0_1.6_4$	arithmetic/wavelength notation of sesquitertia 3:4 exchange

The expression of fractional values in mutable numbers is covered in some detail in Appendix B (Chapter 17) and so after this brief mention they will be left till then. Basically the logic in using fractional extensions to encapsulate arithmetic/wavelength series rests on the fact that our normal understanding of music is essentially that of the harmonic domain. Therefore it seems both right and convenient to recognise the natural ascending ordering of the harmonic series by way of assigning whole number values to it and to leave the fractional values for the more theoretical arithmetic domain. However, in the spirit of group theory, as the two domains bear a symmetrical relationship to each other, the choice is ultimately an arbitrary one. As can be seen from the comparison between arithmetic and harmonic values, basically, and quite logically, the columns are reversed: twenty-four is transformed into one twenty-fourth. In the expression given above, one small point of detail to notice, literally, is the ‘fractional point’ added after the units column. Strictly, this is redundant as the units column itself defines the beginning of the fractional expansion, but a point may be added for familiarity and similitude with the decimal number system.

Now, returning to the two column-lists of the background harmonic series founded upon C-16Hz and G-24Hz, these are the series that contain every objective note as integer relationships of frequency. The lists contain entries for all the harmonic ratios up to B-480Hz – the top objective note in each chord. Above this shared *conjunction* at B-480Hz, only the ratios of the descending arithmetic series are shown for brevity, and interestingly they are all divisible by five. In the harmonic domain the grouping of ratios by five is further underlined through the use of black type, with all the other ratios in gray, excepting the fundamental grouping of the first five ratios of each background nested series. This grouping of h1 to h5 forms a base layer of nesting, upon which the foreground objective notes build their own, aggregated series (i.e. aggregated into groups of five). It is these foreground or aggregated series which we recognise as the ‘harmonic gist’ encapsulating the chords, with the aggregated series’ fundamental tone (the combination of h1 through h5 of the background series) perceived as the root harmonic tone. Crucially, the aggregated foreground series do not contain every objective tone; only the background series encompasses all objective tones. One might characterise the background series as the ‘parent’ and the foreground as the ‘child’ series; or in group theory terminology, group and subgroup.

By setting the objective tones within the broader context of the background nested series in this way, quite extended and therefore complex series are created, which, under the influence of the second law of thermodynamics, will have a tendency to break down into less complex structures – the aggregated harmonic series. The particular form that these foreground structures (aggregated series) take, is strongly influenced by the pattern of the objective notes either considered separately and/or in combination. When the configuration of objective notes, for the most part, falls in line with a particular numerical grouping of the ratios of the underlying/background series, it will encourage that particular aggregated series to emerge from the shadow of the more complex parent series – as a lower energy ‘shorthand’ alternative. (Basically, the process amounts to a physical form of mathematical factorization, which maintains the information content – i.e. the value being represented – as discussed in Chapters 1 and 9.) However, in the background, the parent series is essential and cannot fade from view entirely, as it, and it alone, contains a number of the objective tones – C#-h17, G-h24 in chord iv^{+6} and D-h12 in chord i. These stray objective tones lodged in the background series, interject themselves within the foreground structures as *coloring tones*, lending their character to the ‘plain vanilla’ aggregated series. The prime example of this coloring or shading process is found in the minor triad, where the ‘major-third’ (h5) of the aggregated series – G#h25 in the iv^{+6} chord/series – is displaced by the objective tone G-h24 from the background series. Similarly, the interval of a major-sixth, C#h17 again drawn from the background series, further alters the hue of this already minor-shaded chord.

Figures 14.13, 14.14 and 14.15 step through the exchanges described above. The foreground and background series are labelled, and the range of the graphs is limited so as to include only the ‘moving parts’. Notice how much more efficiently the nested structures (aggregated series) reach the conjunction at B-480Hz – in the graphs, energy and complexity increase from left to right. While the background single series structure requires thirty oscillators, the foreground nested structure, only requires ten.

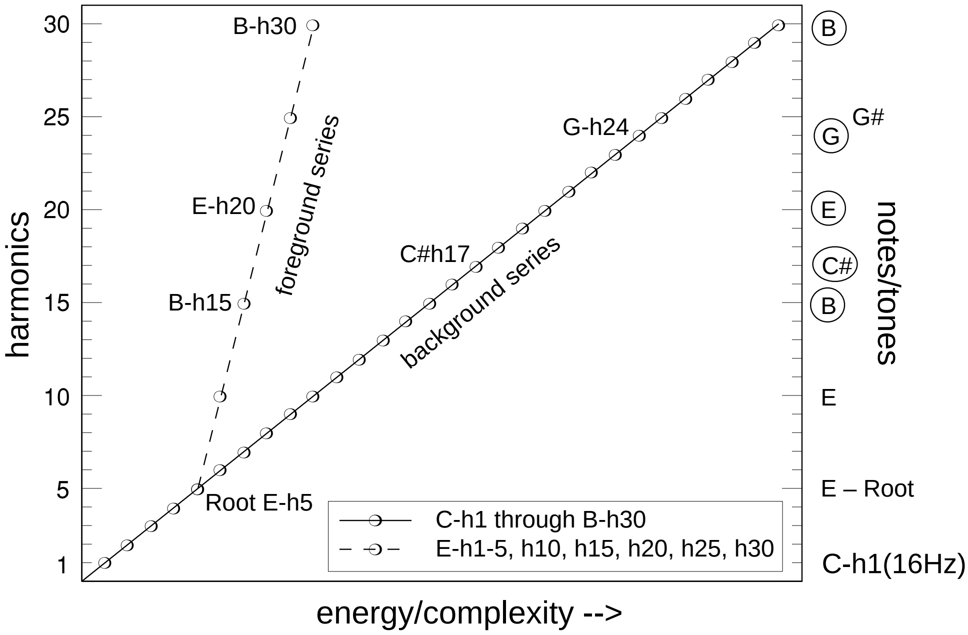

Figure 14.13 A graph display of the foreground and background series of the first chord – iv^{+6}. Energy and complexity increase from left to right in arbitrary units. The objective tones are circled on the right-hand side and labelled on the graphs. The root tone is also labelled, marking the fundamental group of the aggregated series (E-h1–5).

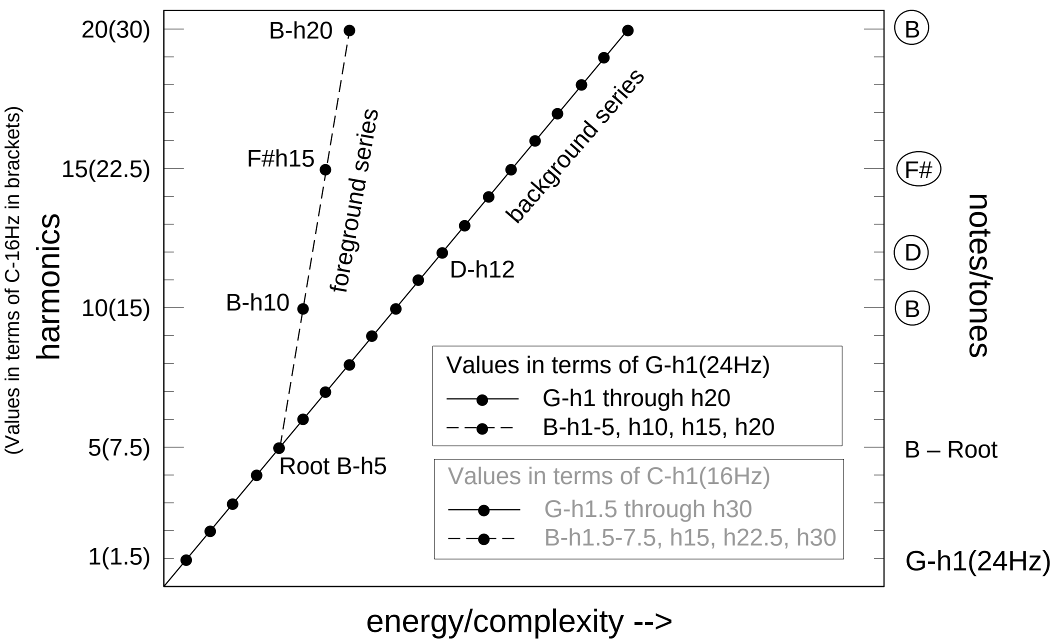

Figure 14.14 A graph display of the foreground and background series of the second chord, the tonic minor triad. The graduations in harmonics of G-24Hz are also shown in gray in terms of C-16Hz.

By superimposing Figures 14.13 and 14.14, it can be seen that the cadence (iv^{+6} – i) leads to a relaxation of stress, as both the foreground and background series move toward the left-hand side.

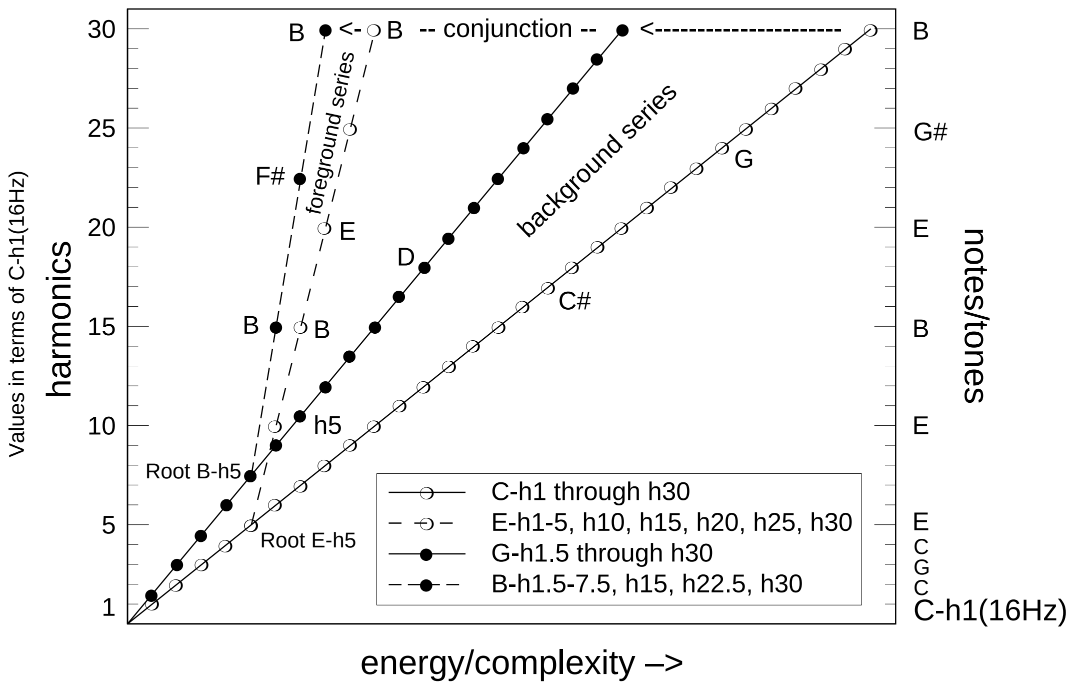

Figure 14.15 The superimposition of Figures 14.13 and 14.14 reveals the change between the two foreground and background series.

Some Extra Detail

There will be no harm in skipping over this short section which attempts to dig a little deeper into the mechanics of this cadence. Indeed, one could argue that the ear and processes of aural cognition simply leap from one configuration to the next as the objective stimuli of the notes sweeps onward, without seeking a logical path of transition for the lower levels of structure. However, what might be the fuller story of this exchange is shown in Figures 14.16 and 14.17. The exchange as drawn in Figure 14.15, has a problem. While the conjunction at B-480Hz allows for a smooth transition between the two background nested series in full and the upper portions of the two foreground aggregated series, there is no conjunction for the lower portion of the aggregated structure (i.e. C-h1-5 –> G-h1-5). How does the system get from root E-h5 to root B-h5? The answer is, perhaps, by small incremental steps.

The first chord in the cadence (iv^{+6}) has an alternative arrangement at its disposal, a configuration that places three of its notes in the background series and adds one of the notes of the second chord (D) to its foreground aggregated series. This is illustrated in Figure 14.16. Incidentally, this new configuration is slightly more efficient: though it too requires ten oscillators, given that all other factors are equal, overall the frequencies are lower, and so the system would be proportionally less energetic.

In the transition from one chord to the next, as the objective tones change, so too might the balance between which grouping of ratios (in the underlying series) best fits the objective pattern: inducing a shift in favor of a different number of aggregations, as the processes of aural cognition first try to fit the new stimuli (i.e. the second chord) into the existing pattern. In Figure 14.16 this effect is shown – drawing up the aggregated series to the root G-h6. In so doing G-h24 is promoted to the foreground and E-h20 and B-h15 are relegated to the background. The tone D-h18/h12 is also introduced, because the crossover point between chords is the moment under consideration.

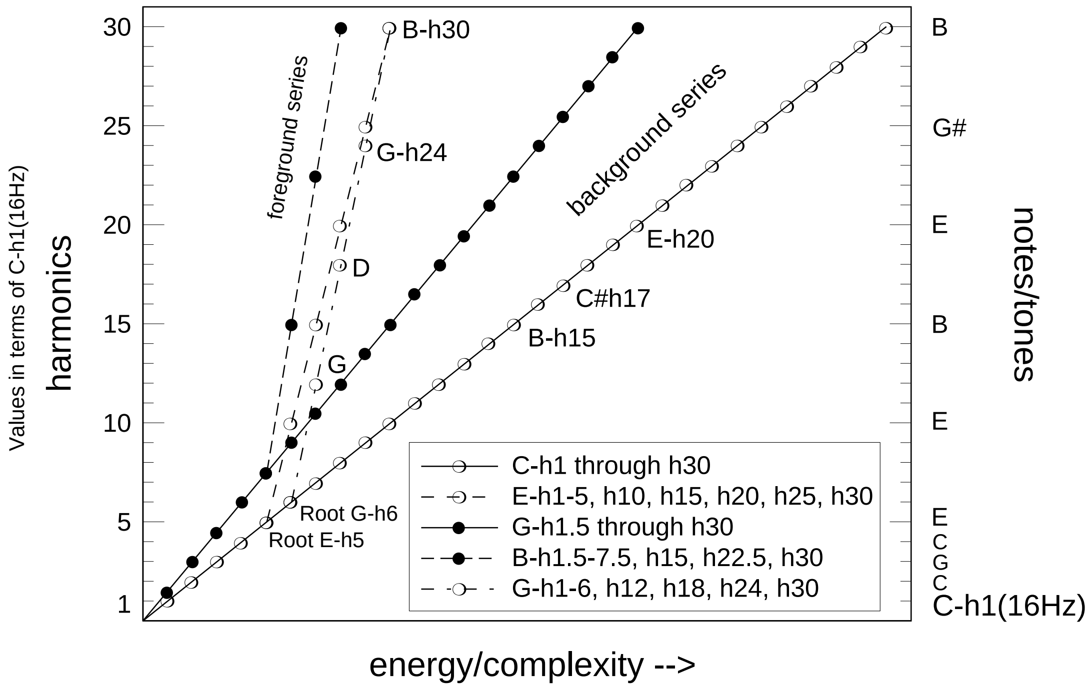

Figure 14.16 A slightly more efficient configuration of the first chord's internal arrangement is obtained by means of a sesquiquinta 5:6 modulation exchange, from six groups of five underlying harmonics, to five groups of six.

Figure 14.17 completes the story; the left hand area of the graphs has been enlarged and the right-hand omitted. Once the foreground structure has executed a sesquiquinta 5:6 modulation exchange to arrive at the root G-h6, the road is then open for the lower portion of the system to make a sesquialtera 2:3 exchange, of six ratio (h1-6) based on C-16Hz for four ratios built on G-24Hz (h1-4). The upper portion of the structure remains unchanged, during this stage. However, with the lower portion of the nested structure now consisting of four oscillators based on G-24Hz, the upper section is again free to lose a little more energy by undertaking a sesquiquarta 4:5 modulation, exchanging five ratios founded on G-h4 for four ratios built on B-h5. Again both structures (before and after the 4:5 exchange) require the same number of oscillators, eight, but the latter structure is slightly less energetic. Also this closing modulation mimics the overall change within the nested structure, from start to finish: ten harmonics are exchanged for eight. Finally, it should be noted that the movement of the background series, from fundamental C-16Hz to fundamental G-24Hz, a ratio of 2:3, implies the existence of the even broader and more fundamental series founded on C-8Hz.

The whole sequence is traced by black arrows in Figure 14.17. Each step of the way the system is losing energy and complexity; first the upper portion finds a modulation exchange, which in turn opens up the possibility for the lower section to release a larger amount of energy/complexity through the modulation algorithm, which, in its own turn, allows the upper portion to again make an energy-saving exchange. A summary of this process might be: a chain reaction of steps pursued by a 'system' toward higher entropy, spurred on by the ever watchful eye of the second law of thermodynamics. It is a process remarkably reminiscent of a particle's decay path as described in nuclear physics.

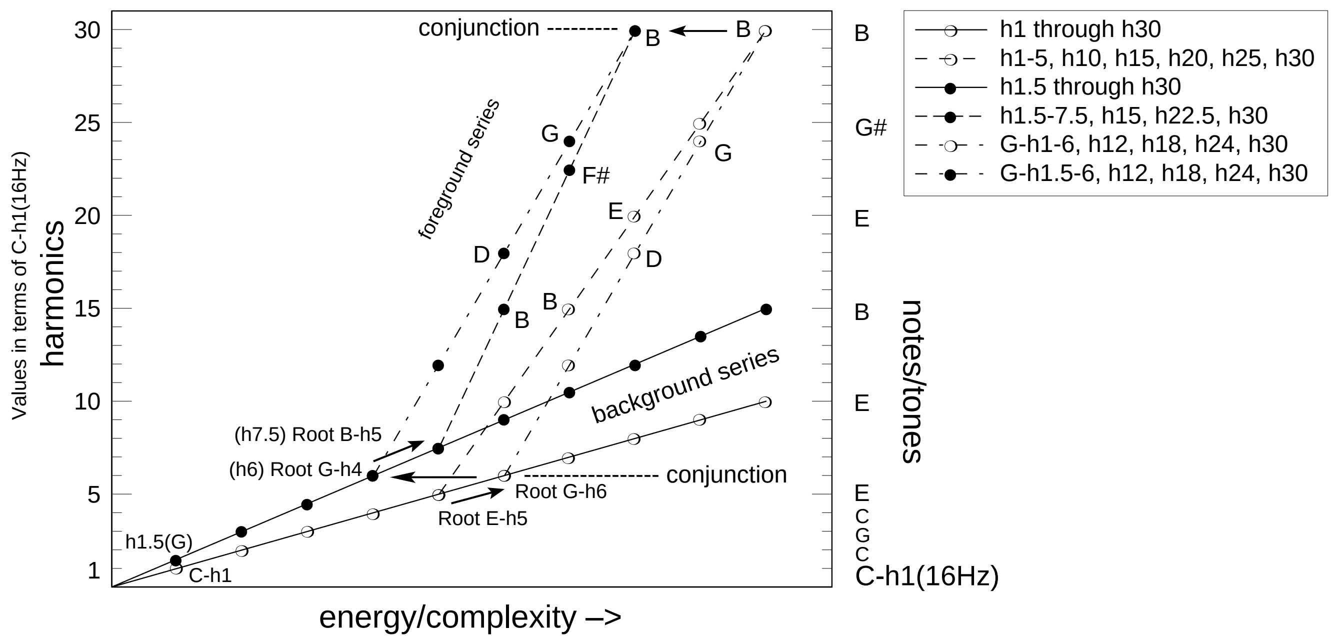

Figure 14.17 The modulation exchanges which allow for a smooth transition between the chords (enlarged view). The small exchange in the upper section of the aggregated series, illustrated in Figure 14.16, opens the way for a larger exchange in the fundamental grouping, across the conjunction at G-96Hz; after which another small adjustment in the upper section takes the system smoothly from chord iv^{+6} to i, each exchange powered by a loss of energy.

Reflection of Tonal Compositions

Although the arithmetic principle remains a somewhat theoretical and ethereal concept, in the example presented in Figure 14.12, it can be seen that it is possible to incorporate aspects of its nature into a thoroughly harmonic analysis, based on the ascending overtone series of real experience, combined with the the modulation algorithm of symmetrical exchange. The analysis offered here does appear to sustain, at least in this example, a certain reciprocal character in the way that the arithmetic, or inversely arrayed, full close cadence is mirrored in the harmonic domain by a minor plagal cadence (vi–i). Also one might note that despite the delicate character of the cadence – formed from objective notes set at a considerable distance from their defining fundamental periods – the progression manages to transmit a sense of gentle extended leverage, exercised from afar, to bringing about the considerable step change of 2:3 in their respective roots.

However, overall, the main interest in ‘drawing-down’ the arithmetic dimension into the harmonic domain is that it enables the construction of complete, inverse, companion compositions from existing works. There remains a great deal of work to do in further developing the line of thought set forth here; hopefully, perhaps in doing so, roles may be be found for other ‘arithmetic’ chords/progressions to play, in extending our understanding of the familiar harmony of the ascending overtone series. For, beside the reciprocal partnering of the dominant-seventh chord with the chord of the added sixth, and the major triad with minor triad (examined in Chapter 11), with little difficulty and with the aid of the accommodating tolerance of the ear, the principle of inversion can be extended to include the diminished and augmented triads, which by virtue of their (nearly) symmetrical structure partner themselves (Figure 14.19). While additionally, the other shade of seventh chord (i.e. major triad plus major-seventh, C–E–G–B) would appear to translate, by inversion, into a minor triad plus minor-seventh (i.e. E–G–B–D).

```
                                          (h80)   E- a1
  Diminished Triad                        (h40)   E- a2
  partners itself                                 A- a3
                                          (h20)   E- a4
                                          (h16)   C- a5
                                                  A- a6
(h36)  D-a1                              F#h11----F# a7
(h18)  D-a2                               -h10     - a8   Augmented
(h12)  G-a3                              D- h9----D- a9   Triad: A#-D-F
 (h9)  D-a4                               - h8     -a10
A#h7---A#a5                              A# h7----A#a11
G-h6---G-a6   Diminished                 G- h6
E-h5---E-a7   Triad: E-G-A#              E- h5
C-h4                                     C- h4
G-h3                                     G- h3      Augmented Triad
C-h2                                     C- h2      partners itself
C-h1                                     C- h1
```

Figure 14.18 Although the diminished and augmented triads, taken as *configurations of partials*, are not composed of identically proportioned minor-third intervals (i.e. 5:6 and 6:7 for the diminished triad), the tolerance of the ear combined with the adjustments of temperament produces an impression of symmetry. Interestingly, the 'arithmonics' correctly invert the subtly different intervals, a refinement perhaps lost on all but the most sensitive of ears.

SYMMETRY AND NEGATIVE NUMBERS

The mathematical theory of groups provides a wide-ranging technique for the development of axiom systems from collections of related 'objects', thereby creating the basis upon which whole mathematical systems may be built. Although for thousands of years an implicit group theory process of extracting (traditional) mathematics – geometry, arithmetic, etc. – from the realm of daily experience, alluded to at the beginning of Chapter 13, proceeded unrecognised. The slow motion shattering of the assumed direct link between traditional mathematics and physical reality, which followed the discovery of new, non-Euclidian geometries in the nineteenth century, eventually forced mathematicians to confront and investigate the nature of this process head on, leading them to formally describe the principles upon which an axiom system can be developed from a scheme of related entities. In Chapter 13, and above, something of these group theory principles have been applied to the MOS model of harmony in particular, portraying tonal music both in terms of a dynamical system in the material world and as formal mutable numbers.

As briefly mentioned towards the end of the first section in this chapter, retrograde harmonic progression could be viewed as a form of negation, and because mutable numbers necessarily reduce to physical oscillatory systems, a minus sign appended to a mutable number could be taken to represent phase opposition. On the small scale this form of negation can be illustrated by the time reversal of a single wavelength, for example, reversal of the mutable number two as shown in Figure 14.19, where the original wave in graph A, is reversed in graph B (representing minus two) and then the two waves are combined in graph C to form zero. From this opposition or anti-phase comes an anonymous, symmetric, two-valued quality, the negative gaining meaning only in relation to the positive and vice versa. Which of these states we choose to name positive and which negative is rather arbitrary – as indeed with the assigning of whole or fractional values to the harmonic and arithmetic spheres.

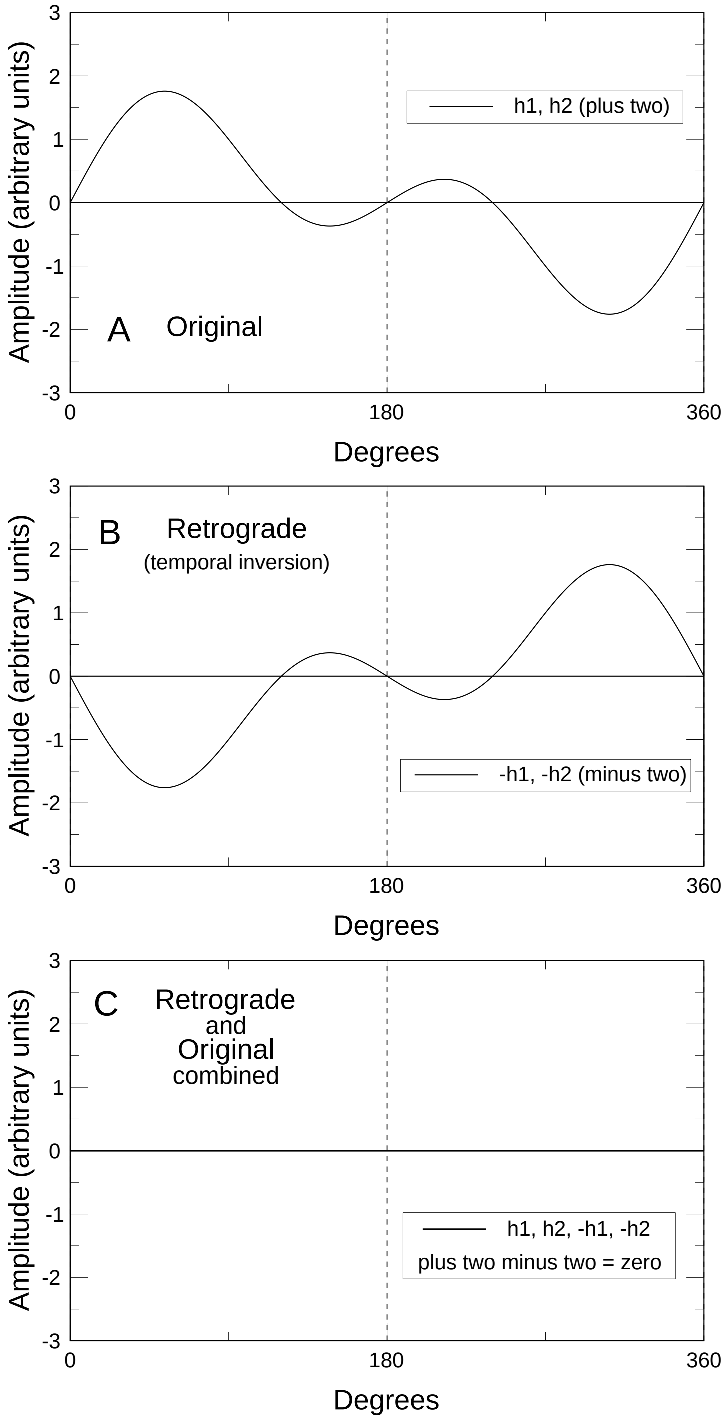

Figure 14.19 Graphs of the mutable number two (A), its negation by retrograde motion (B) and their annihilation (C).

In more extended form, negation might additionally be interpreted as the reversal of a sequence of modulation exchanges. In this form the temporal inversion of a composition (playing the harmonic progressions backwards) would equate to the reversal of the original flow of mutable digit sequence exchanges. Thus for example, in the last two bars of Prelude No.1 (Example S, page 35) the original (forward) computation would produce in reversal the negation of the values 192, 160 and 32:

Original: 384G + 32 = 416A + 160 = 576D + 192 = 768G

Negation: 768G – 192 = 576D – 160 = 416A – 32 = 384G

Following through the logic of associating phase opposition with negation in physical mutable numbers produces some number operations that contrast with traditional practices. Multiplying together two negative mutable numbers, that is nesting one number within the other (e.g. MBN $-2_1 \times -2_1 = -4_1$) produces another negative quantity. Similarly, dividing one negative number by another, which amounts to removing a tier of nesting, results in a negative outcome. Allowing contrasts of this type would be equivalent to making changes to the basic axioms of traditional mathematics, where minus times minus yields a positive result. Although this might at first seem a radical proposal, traditional mathematics has long recognised the possibility of many, indeed an infinity, of different axiom systems. This recognition of other mathematical structures, built on separate choices of axioms, came about with the parting of the ways that was marked by the discovery in the early nineteenth century of non-Euclidian geometry by J.C.F. Gauss and others. Gradually through the nineteenth and twentieth centuries other geometries, arithmetics, algebras and logics were found. However, for any of these mathematical structures to be meaningful in the way that traditional Euclidian geometry and arithmetic are – that is to say they yield sensible answers to appropriate questions – the choice of axioms or basic rules needs to provide consistency and a reasonable level of completeness. Consistency in this context implies a lack of contradiction: the axioms must not allow truth and falsehood to coexist, in the sense that a statement can be proved both true and false by tracing different arguments from the same starting point. Completeness and range are desirable, in that for practical use a mathematical construction limited to only half a dozen 'bricks' won't get you very far. Generally, simple small mathematical structures can be entirely complete, but complex systems, like traditional arithmetics, are not complete, but they *are* usefully extensive in the range of procedures that can be demonstrated to be valid within them. It was Kurt Godel (1906–1978) who initiated a mathematical earthquake when, in 1931, he proved arithmetic's incompleteness. For years afterwards many traditional mathematicians remained in denial, some in despair.

From ancient times, and particularly from the time of the Greek geometers, one method of establishing the validity of mathematical processes was to use physical congruence to demonstrate equality. For example, if one triangle could be placed on top of another, fitting perfectly, then the triangles could be considered equal. And this concept of congruence could be taken further by asking, for example, how many ways may an equilateral triangle be placed over itself and fit perfectly? The answer as we have seen above is three, or six if the inversion is allowed. Although these rotational and inversional symmetries have of course been intuitively recognised from time immemorial, it was again only in the nineteenth century that serious attention was paid to such symmetry groupings. With the general study of symmetry – *group theory* – being developed initially by Niels Henrik Abel (1802–1829), Evariste Galois (1811–1832) and Arthur Cayley (1821–1895), as they investigated not only physical symmetry, but abstract symmetries, such as the symmetry of numbers.

Just as the rotational symmetry of shapes is intuitively understood, so musicians have intuitively recognised rotational symmetry in music. For the most part in western music, the explicit recognition of this feature has been confined to melody, with J.S. Bach, amongst many others, making extensive use of the technique of rotational symmetry in thematic manipulation – in the forms of original, retrograde, inversion, and retrograde inversion. However, as discussed above, the concept of rotational symmetry can also be applied to harmony; and in a similar way to mathematics, harmonic symmetry has been present in western tonal music implicitly, to a considerable degree, from the outset. The principal example of this harmonic symmetry is of course the duality of the major and minor triads – the basic building blocks of tonal music – the major triad being formed from major-third and minor-third ascending and the minor triad from major-third and minor-third descending. Some other harmonic symmetries have also been touched upon in this chapter, for example, that of the dominant-seventh/chord of the added sixth. However, rather more fundamentally, rotational and reflective symmetries of the harmonic series – the harmonic number line – which are illustrated in Figure 14.20, could be viewed in a similar light to the encompassing symmetries of the plane from which geometric figures, like for example an equilateral triangle, selects a subset. Thus the individual symmetries embodied by pairs of chords, such as the major and minor triads, could be viewed as being particular expressions of the broader symmetry of the harmonic/arithmetic series.

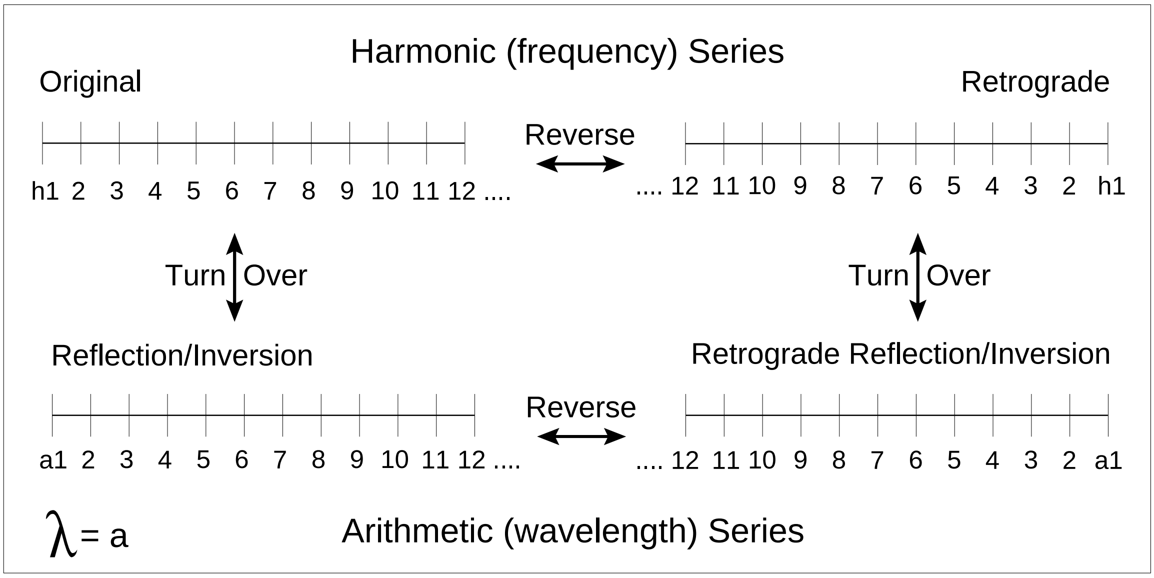

Figure 14.20 The four elements of the symmetry group formed by the valid rotations of the harmonic number line. (The established symbol for wavelength, lambda, is a perfectly acceptable alternative to ‘a’, arithmonic.)

Essentially, over many generations, through the era of tonal harmony, musicians were intuitively exploring a range of valid relationships within a set of symmetry groups – ‘little worlds’. Through trial and error, and guided by performance validation, composers and musicians discovered the rich and varied language of tonal harmony which could be built up from the simple group axioms provided by these symmetries. In a way, each successful composition – pieces that have passed the performance test – represent valid theorems of harmony in the mathematical construction, which is, traditional tonal music.

However, due to the characteristics of the human ear and aural cognition, some regions of tonal relationship in the musical set of symmetry groups, lie beyond the reach of performance validation – ‘musical proof’. The reflection of complex chords into even more complex entities in particular, stretches the limits of tonal perception. The ear loses track and cannot extract tonally meaningful information from many of these exchanges. This is analogous to incompleteness in the mathematical sense, where some true statements cannot be proved from the group axioms – as Godel demonstrated for traditional abstract arithmetic. Equally, in music, some compositions will fail performance validation, though they represent correct constructions of the system’s axioms. Though such pieces may sound ugly and unnatural, knowing that the information they contain is as valid as a Mozart symphony, and indeed under reflection might actually *be* a Mozart symphony, perhaps points to new areas for composers of tonal music to explore. As for the consistency of tonally organised music, there would appear to be little to doubt, due to the relentless and ordered succession of natural number relationships computed by the sequence of primary modulation exchanges: 1:2, 2:3, 3:4, 4:5,... n:(n + 1). All that would be required to disprove music's consistency, is one primary modulation that runs counter to this pattern; however, I do not believe that such an exchange exists.

Notes

1. It was with some surprise and reluctance that I was compelled by the MOS model to accept the primacy of harmonic progression as the central dynamic organisational force in tonal music. Though equally the MOS model encompasses the metrical dimension, meter plays the largely static and subservient role of a temporal container measuring out the ground upon which the 'game of harmony' takes place; normally, only becoming an active participant through occasional metrical change (e.g. a hemiola). Counterpoint essentially animates and fills-out the meter (rather analogous to the way the pitch element of arpeggios fills-out the harmony), but also, it vastly entertains and delights aural cognition's propensity to track individual sound sources, and thereby greatly enriches the musical experience.

2. Flipping over regular polygons with odd and even numbers of sides produces somewhat different results. For odd-sided polygons the exchanges are in the vein of the triangle illustration with one vertex remaining unchanged, but for even-sided polygons the axes of rotation pass through the centers of each pair of opposite sides, and so all vertices are involved in exchanges.

3. As both aggregated and nested series move in step by the proportion 2:3, another equally valid interpretation could be a two-series system: MBN $6_{10}0_1$ –> $4_{10}0_1$ – a nested/nesting pair.