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Mathematical Miscellany

GEOMETRY, SYMMETRY AND COMPUTATION

To begin with, mathematics grew out of the practical needs of human beings as they went about their daily lives. Down to earth questions concerning number and geometry would inevitably emerge for our distant ancestors: How many sources of that particularly tasty berry and in what direction from home? How many beasts in that wild herd and what route will they follow across our territory? What distance and direction to travel to intercept the herd and how much time will it take? These are all practical mathematical problems of number and geometry clothed in the concerns of ordinary life. Today the same questions might be re-phrased: When does the local supermarket close and if I cut across town town can I also make it to the butcher's shop? In this way mathematics was with humankind from the start. However, it was to take a considerable time for mathematics to separate completely from the human sense of place and purpose, and to finally emerge as a totally abstract discipline. Certainly in ancient times the process of abstracting a math from the everyday world had begun, with the concept of number being recognised in its own right, as distinct from a number of pebbles, sheep or soldiers.

Probably the first steps towards a formal system of mathematics, separate from the practical math of daily living, were to come with the increasing social organisation of agricultural communities and their vital concerns with the cycle of planting and harvesting – reflected in the regular patterns and positions of the Sun, stars, and Moon. With the increased productivity of settled societies came the possibilities of elites, organised religious practices, large-scale warfare and taxation. Taxation, with its necessity for surveying land and assessing production, particularly required the development of an abstract math, as also did the calculation of a calendar to some degree. Early civilisations in Egypt, Mesopotamia and elsewhere began this process with the development of written number systems and procedures for land survey and measurement. Often these early steps involved ad hoc solutions to practical problems like calculating the area of a circle or the relationships of right-angled triangles. In the ancient western world, it was the Greek peoples above all others who were responsible for the development an abstract mathematical system, or rather an intellectual tradition of abstract mathematical thought, from Thales of Miletus and Pythagoras of Samos in the fifth century BC, onward. Both Thales and Pythagoras were acquainted with Egyptian and Babylonian knowledge, upon which they built, and crucially, they sought to extract general principles and procedures from the particular and concrete. Geometric shapes were understood by the Greeks as representations or approximations of ideal Platonic forms, and physics and mathematics became named categories of inquiry in natural philosophy. Even so, subtle connections with the human perception of the physical world were to continue, going unrecognised for centuries and even millennia. It was to be a slow, unplanned untangling of mathematics from physics. This was the Greek genius, to begin in earnest the abstraction of mathematics, taking the individual problems and solutions involving practical objects and magnitudes in the everyday world, to the level of abstract generality, logical principles and notions of mathematical proof.

As noted by the great early nineteenth century mathematician J.C.F. Gauss (1777–1855), Greek mathematical thought had a geometric bias to it. This is not surprising: as mathematics emerged from the concerns of daily life, so it was natural for the first developments to reflect that arena of natural life – the place or space in which human beings find themselves. When we measure, we usually measure physical objects and distances, geometric parameters for the most part, the objects, shapes, volumes, planes and dimensions of our familiar 'human' space. Mathematics is intimately and intuitively linked with our sense of place. And so it was that the first great expansion of the mathematical horizon undertaken by the ancient Greeks principally focused on geometric aspects of the physical world. The Greek achievements were summed up in Euclid's *Elements*, a definitive compilation of existing mathematical knowledge – a collection of geometric definitions and theorems codified in Alexandria around the beginning of the third century BC. In the *Elements* Euclid had essentially produced an abstraction of the familiar 'human' space of everyday life, a generalization of practical intuitive geometry; significantly, he classifies some of the definitions as 'common notions', self-evident truths based on the 'real' experience of life in the material world.

This dependence on the physical world for the ultimate validation of formal mathematical principles and processes would continue throughout ancient times and up to the nineteenth century AD. Perhaps the foremost example of this approach is the *assumption* that three-dimensional space, and time for that matter – the arena of human experience – is 'flat', like some inflexible cubic grid, with an external immutable clock counting off the seconds, minutes and hours. The validity of this assumption, which certainly appears correct at the human scale, would not be seriously questioned until Carl Friedrich Gauss began to wonder about one of Euclid's postulates – that parallel lines never intersect. Gradually, Gauss found that by relaxing this common-sense notion, he could construct a new 'geometry', another internally consistent geometric world, separate from that described by Euclid.

By the end of the nineteenth century the effects of the work of Gauss, his student Bernard Riemann and others finally broke the last nexus between intuitive 'human' experience and geometry. In the future, mathematics would be entirely distinct from the study of the physical world – that is physics. Mathematics had finally shaken off the last vestiges of reliance on the 'great world' and come of age, as a fully abstract formal discipline: a 'little world' built on a small set of assumptions (axioms) and developed through theorems into an extensive logical structure; indeed, so extensive it probably has no end.

Once mathematics had broken free from the assumption that its ultimate validation derived from the self-evidence of the physical world, mathematicians were free to roam a truly vast landscape, including the exploration of unfamiliar spaces and geometries of unlimited dimensions; and to ask, why do we experience just three physical dimensions in this world? Why not four or ten? For example, a line has one dimension, length; while a square has length and breadth and a cube length, breadth and depth – three dimensions. But can an object be constructed, if not visualised, with four spatial dimensions – a 'hypercube'?

In a world of any given number of 'flat' (normal) physical dimensions it should be possible to draw a straight line which partakes or passes through all dimensions; and by making this line the hypotenuse of a right-angled triangle its length can be determined from other lengths partaking of fewer dimensions. Each new dimensional step is of unit length. In one dimension, using a 'squashed' right-angled triangle with one angle of zero degrees at point 'A' and two of ninety degrees, the hypotenuse partaking of only one dimension equals the square root of one, which is one. An odd special case. In two dimensions, the familiar right-angled triangle of Pythagoras' theorem yields a diagonal hypotenuse of root two length (1.414...). This has created a new unit of length which can never coincide with the first dimension's unit of length – truly a new dimension. In three dimensions, by drawing the dotted line across the cube to meet the unit step of depth, and inclining the right-angled triangle to one side, the hypotenuse can be calculated to be root three length (1.732...). Again a new unit length is created which can never coincide with the previous two units (1 and 1.414...). So far so good: three dimensions we intuitively understand, but next if the same procedure is used again we must trust the mathematics to guide us where our intuition cannot. In four dimensions, although we can't visualise it, a unit length step will extend at right-angles from each vertex of the cube, into the (flat) fourth physical dimension. Again using the same procedure, by joining up point 'A' to the further end of the next unit step, the hypotenuse is calculated to be the square root of four, that is *two first dimension unit steps*.

$$\sqrt{a^2 + b^2} = h$$

$$\sqrt{1 + 0} = \sqrt{1} = 1$$

$$\sqrt{1 + 1} = \sqrt{2} = 1.414...$$

$$\sqrt{2 + 1} = \sqrt{3} = 1.732...$$

$$\sqrt{3 + 1} = \sqrt{4} = 2$$

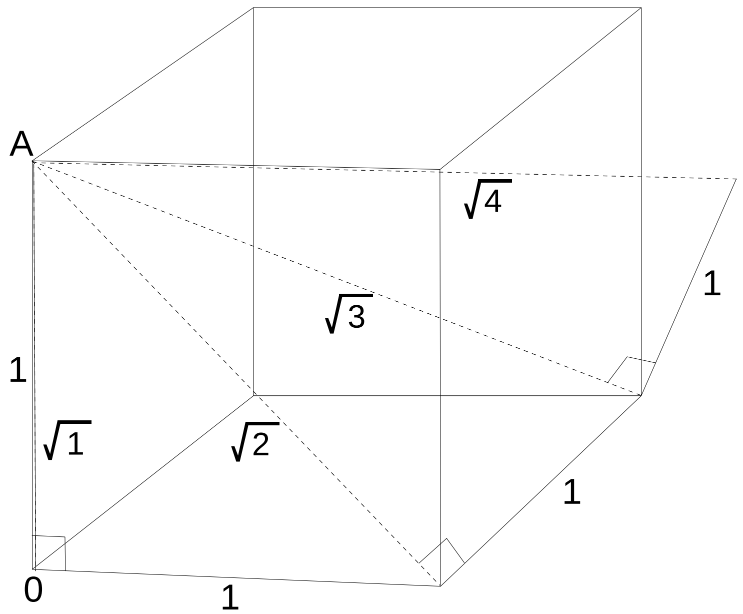


Figure 13.1 Calculating the diagonal length of a four-dimensional 'hypercube' using Pythagoras' theorem.

These calculations illustrate the proposition: that by stepping out into a fourth physical dimension we might not find a new, unique, fourth unit of length, but might actually find that we return to the whole-numbered units of the first dimension. Also, this example shows that the awkward irrational unit lengths of root two and root three (developed by moving into the second and third spatial dimensions) may be accessed by multiple unit one steps. Whether or not this approach yields any clue as to why, in the real material world, we only experience three dimensions of space – length, breadth and depth – the reason for including such a speculative mathematical digression is that it echoes and materially illustrates the recursive pattern of twelve tonal-centers generated by the spiral of fifths/twelfths (Figure 9.20). By the repeated application of a simple rule, as in Pythagoras's theorem above, cellular automata in Chapter 5 or the modulation algorithm of symmetrical exchange in tonal music[1], systems may develop a structural variety, held within an overarching unity, by periodically 'return to themselves' – a concept central to the study of symmetry, discussed below.

Picture courtesy Wikipedia

Johann Carl Friedrich Gauss (1777–1855), the son of a laborer was born in Brunswick in 1777, and in his youth, was a mathematical child prodigy of Mozartian proportions. Luckily, Gauss came to the attention of the Duke of Brunswick, a man with progressive ideas, who encouraged and supported his education at first in Brunswick and later at the University in Gottingen. As a young mathematician, Gauss made his name by calculating the orbit of the newly discovered ‘planet’ Ceres (a large asteroid) thus allowing astronomers to reliably locate it in the night sky. And at the early age of twenty-one he put together many of his mathematical discoveries in the large volume *Disquisitions Arithmeticae* (published 1801). In 1807 he was appointed Professor of Astronomy at Gottingen, a position he held until his death in 1855. Gauss’s father had seen little value in education and it was his mother Dorothea and her brother who supported and encouraged his early schooling. Like many men of genius, Gauss was not of a particularly easy or social character, resentful toward his father and in his time quarrelling with his own son. However, for a short while, under the optimistic and uncomplicated influence of his first wife Johanna (1780–1809), his rather pessimistic inclinations were counterbalanced. Sadly she died young in child birth and though Gauss later remarried and fathered another three children he never again experienced a period of such domestic happiness. In addition to family tragedy, the criticism his book received for its difficult and opaque style, combined with the death (in battle) of his patron and by then friend, the Duke, cast further shadows. In the first quarter of the nineteenth century, Gauss began to wonder if it might be possible, at least in abstract mathematical terms, for space to be curved or distorted. This insight was perhaps stimulated by his involvement in surveying the state of Hanover which was carried out to a level of accuracy that introduced the effects of the curvature of the earth into his calculations. Although Gauss had worked out a reasonably complete formulation of curved (hyperbolic) space by 1825, he never published it, only sharing his ideas with trusted friends and mathematical acquaintances. It was to fall to two of these, Johann Bolyai and Nikolay Ivanovich Lobachevsky to bring these developments into the open. Both men believed they had discovered ‘curved space’, but from the examination of Gauss’s earlier notebooks and the connections between the three individuals, it is fairly clear that the underlying insight came from Gauss – one of the greatest mathematicians of all time. Only rarely did Gauss find students of a calibre to match his own ability (Riemann, Dedekind and Bessel) and yet he left Gottingen the undisputed leading center for mathematics – his presence and reputation a sufficient reason for Napoleon to spare the city. Many discoveries and advances lay buried in this notebooks until after his death; always sensitive to criticism and naturally secretive (he usually declined to explain the source of his insights and discoveries) in later years Gauss became an increasingly a reclusive figure. He died haunted by his own agnosticism and thereby doubting the value of what he had achieved.

Unlike the development of mathematics, the path of separation or abstraction from the physical world would be impossible for the discipline of music, unless musicians were prepared to relinquish the option of meaningful performance: by making music, as well as writing and thinking about it, musicians are constantly confronted by physics, the constraints, imperatives and opportunities of material existence. In a sense, performance in tonal music plays a similar role to proof in mathematics – it is the means by which the validity of particular constructions can be assessed. As musicians set about exploring the range of possible tonal relationships, they were like their brother mathematicians, mapping out a coherent ‘little world’ – a mathematics. However, the difference between the two maths is that a mathematics built on the axioms of the harmonic series would forever remain intimately entwined with its physical background, whereas a mathematics built originally on axioms of common sense and observation could, and ultimately did, pull itself clear of the material world, through the application of logical thought. (Perhaps it could be argued, that a similar path of abstraction has been taken by atonal music?)

Picture courtesy Wikipedia

Georg Friedrich Bernhard Riemann (1826–1866) was born at Breselenz in Hanover, the son of a Lutheran minister. Though a shy and self-conscious child, Riemann's undoubted abilities led his father to foster his education in the hope that his son might find, in time, a position within the church. Bernhard Riemann studied first at Hanover, where he lived with his grandmother, and then after her death in 1842, at the Gymnasium Johanneum in Luneberg. At Luneberg, with the encouragement of its director, the rather solitary and serious Riemann developed his taste for mathematics. In 1846 Riemann went to Gottingen intending to obey his father's wishes by studying theology at the university. However, he soon found the attractions of the mathematics lectures too great to resist, and begged his father's permission to be allowed to change his course. Wisely, his father bowed to the inevitable. Within a year of beginning his mathematical studies in earnest at Gottingen, Riemann move to Berlin, and for two years soaked up the more up-to-date lectures available there. In 1849 Riemann returned to Gottingen, where he would remain for the rest of his career, rising to become the senior professor of mathematics by 1859. The same year saw his publication of a ground-breaking paper on prime numbers. In 1862 he married Elise Koch, with whom he had a daughter. Riemann was another mathematician of the highest order following in Gauss' footsteps at the University at Gottingen. Bernhard Riemann, who was to die at the age of thirty-nine from tuberculosis, made contributions to many fields of mathematics, discovering amongst other things, another form of curved mathematical space – elliptic space. It was Riemann's formulation of curved space that was to be used by Albert Einstein in the twentieth century to articulate his General Theory of Relativity – the geometric theory of gravity as the curvature or distortion of both space and time.

THE UNION OF PITCH AND DURATION

One hundred miles away from Gauss and Riemann in Gottingen, and at about the same time, Mortiz Hauptmann in Leipzig produced his treatise on *The Nature of Harmony and Metre*, in which he sought to derive all the elements of music using one basic process – Hegelian dialectics. And somewhat similarly, throughout these documents, it is argued that tonal music can be understood in terms of modulating oscillatory systems and their analog, *mutable numbers* – below this idea is given a geometric perspective.

Stepping back nearly two hundred years before Gauss, to the time of Rene Descartes (1596–1650), brings us to another outstanding mathematician whose work was intimately connected with the concept of 'abstract' mathematical space. Descartes' great achievement was to unite arithmetic and algebra with geometry, through the invention of coordinate (analytical) geometry, a form of 'visual arithmetic' conducted in an abstract mathematical plane or space, which could be thought of as a higher or multi-dimensional form of the number line. His great contemporary, Pierre Fermat (1601–1665) also developed a similar system at about the same time, but didn't publish it!

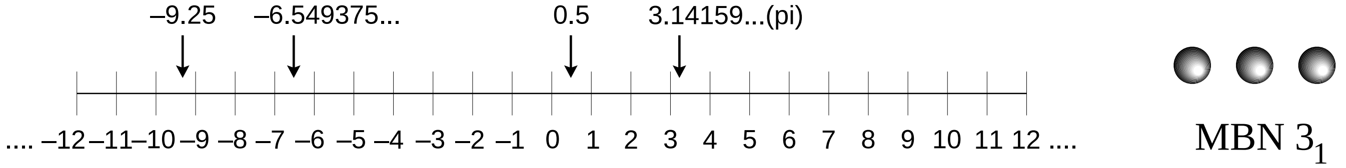


Figure 13.2 The one-dimensional number line, a most useful visual representation of the range of positive and negative numbers. The space between the vertical divisions is taken to represent every possible value lying between the whole numbers.

By placing two number lines perpendicular to one another, a plane or two-dimensional grid can be created, producing effectively a map of an abstract mathematical plane, which may for convenience be divided into four quadrants. Points in the plane are referenced by a pair of numbers described as ‘x,y’ coordinates. Conventionally, the horizontal number line delineates the x-axis/units and the vertical number line the y-axis/units. The position ‘0,0’, where the number lines usually intersect, is referred to as the ‘origin’. This ‘Cartesian plane’ could be viewed as a development of physical number patterns, abstracted and extended to embrace the smooth continuum of real numbers (Figures 13.2–4), and through this connection a linkage might be drawn between mutable numbers (i.e. tonal music) and geometry.

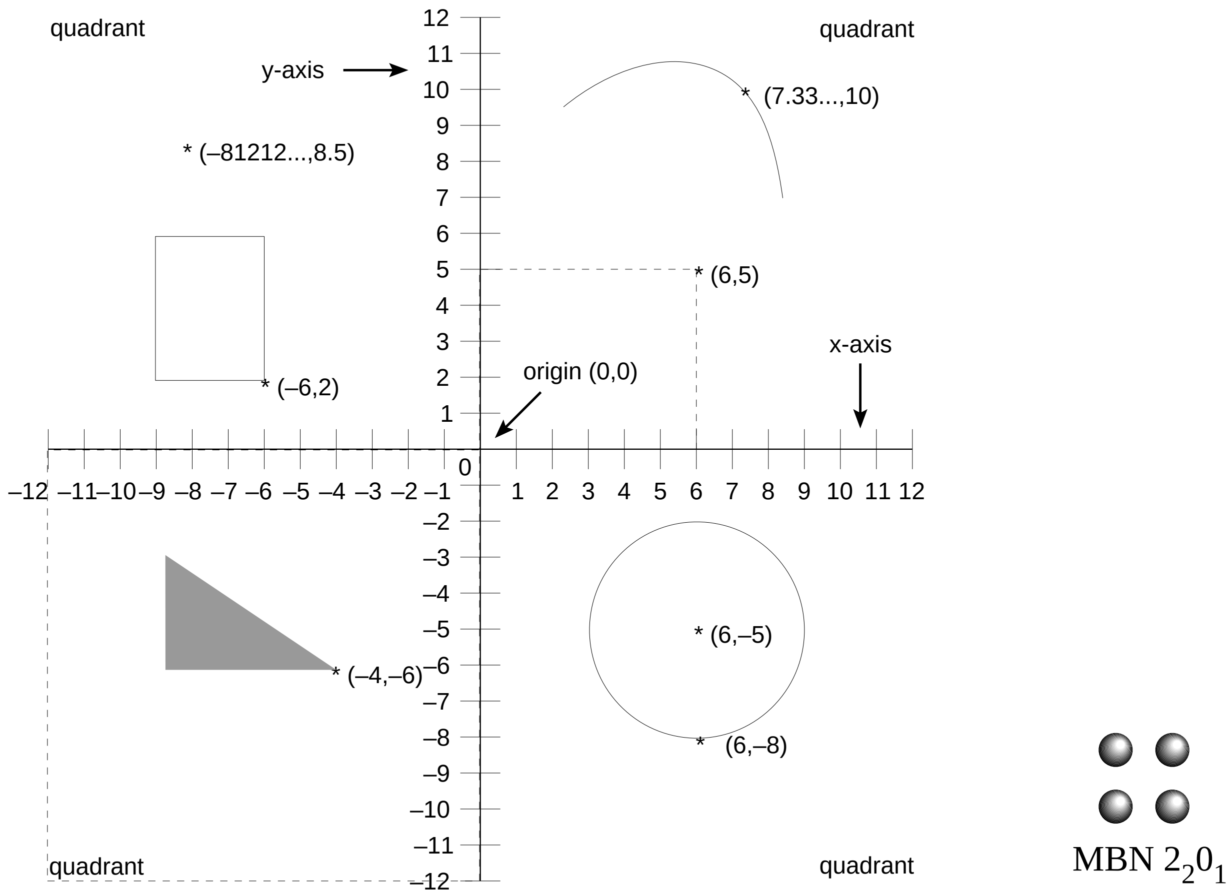


Figure 13.3 A two-dimensional coordinate grid of x and y values. By specifying sufficient coordinate pairs, any plane shape, figure or area can be described in numbers – i.e. by arithmetic.

Plotting points in the plane is only the beginning, by plotting many points and joining them up, straight lines, curves, shapes and areas can be described by coordinate numbers. Geometry and arithmetic are united by an equivalence between the arithmetic of numbers (coordinates) and the geometry of an abstract plane. By adding one further number line, a ‘z-axis’ the two-dimensional plane of Figure 13.3 can be expanded into a three-dimensional space (Figure 13.4), with each point now referenced by ‘x, y, z’ coordinates, bringing solid forms such as cubes, cylinders, cones and spheres within the reach of arithmetic. These axes do not necessarily have to be drawn so as to cross at their zero points of origin, though most often they are. Depending on the values of the coordinates being drawn, it can be convenient to show axes crossing at points other than zero. As described above, until well into the nineteenth century, it was taken as read that this ‘flat’ three-dimensional mathematical grid was a true representation of the self-evident physical space of daily experience, and indeed, the ‘absolute’ space within which Newtonian mechanics was described.

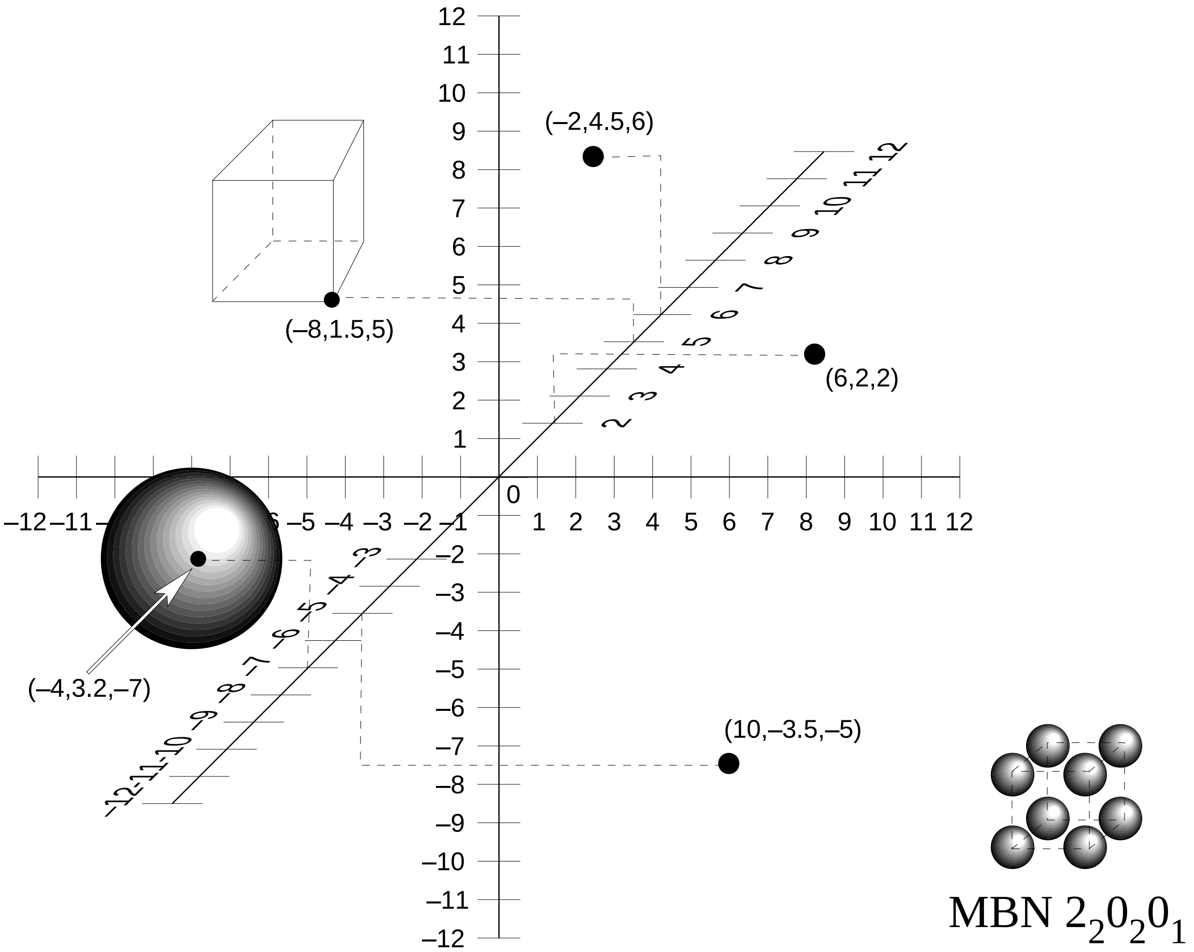
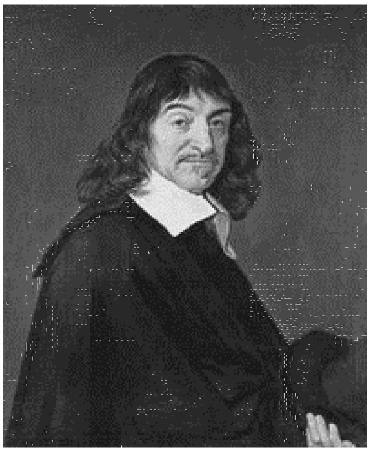


Figure 13.4 A three-dimensional grid of x, y and z ‘number line’ coordinates, with a few example points and figures drawn within the volume. Bottom right the cubic number pattern of the mutable number eight.

Essentially Descartes’ invention of coordinate geometry allows positions in space, and objects described by multiple points within that space – lines, curves, vertices, etc. in a coordinate space – to be handled as arithmetic (i.e. operations on coordinates), and manipulated in a general form, as algebra, thus bringing geometry within the orbit of arithmetic. Although some mathematicians disliked the idea of their abstract world of numbers and equations being dragged into the ‘daylight’ of visual Cartesian geometry, others, like for example David Hilbert (1862–1943), saw an opportunity to verify the geometric insights of the ancients through the rigour of numbers.



Picture courtesy Wikipedia

Rene Descartes (1596–1650) was born into an aristocratic French family. His father was a lawyer and judge. Sadly, his mother, a woman of noble birth, died before he was old enough to know of her. Like his mother, Descartes did not enjoy robust good health: during his schooling by the Jesuits, at university and in later life he rarely left his bed before midday. He studied law to please his father; however, highly intelligent, sceptical and of an enquiring cast of mind, he chose to further his knowledge of the world through travel and a little amateur soldiering. During his sojourns around Europe, Descartes met and was much influenced by the Dutch mathematician Isaac Beeckman, whose interest in mathematical physics left an indelible mark, and they remained friends for life. While considering the new ideas about motion and mass brought into focus by the work of Galileo, Descartes discovered his graphical (Cartesian) approach to these problems, now known as coordinate or analytical geometry. He claimed that the concept came to him in a dream... in bed again, of course. Descartes long delayed the publication of his discoveries and ideas on a wide range of scientific and philosophical topics, fearing the same fate that befell Galileo – condemnation and punishment. Eventually, without the author's name, *Discourse on the Method* was published in 1637. Little is known of Descartes' personal life: he did not marry and had few close friends; however, he did briefly set up family with a mysterious woman named Helen in the 1630s and had a daughter by her, who died in infancy. Rene Descartes maintained a long correspondence with Princess Elizabeth of Bohemia, and accepted, after much prevarication, an invitation to become the private tutor of Queen Christina of Sweden. Unfortunately Descartes died shortly after arriving in Sweden, having endured the long journey only to be met by one of the cruelest winters ever to strike Sweden. A principal intellectual figure of seventeenth century, making wide contributions across philosophy and the emerging new science, he is particularly remembered for the epithet "I think, therefore I am", and in mathematics, coordinate geometry, which Newton and subsequent scientists including Einstein would use to map out the arena of space and time in which their theories operate.

The conduct of arithmetic implies naturally enough a number system, and generally the decimal system is used to performed coordinate geometry. However, any number system is equally able to perform the function, and the mutable number system, in particular, is well qualified for this purpose, in that it can combine and express in one digit sequence the perhaps many coordinates of a Cartesian point. For example, the mutable number twenty-four expressed as MBN $8_3 0_1$ might be taken to represent the coordinates (8,3) or, as MBN $6_4 0_1$ would translate to coordinates (6,4). Now, as we have seen in previous chapters (Figures 6.2–4 and 13.5), these two mutable numbers also represent the full or perfect cadence, taken in isolation. Thus it is a short step to turn these two digit sequences/coordinate pairs into positions in an abstract space delineated by a coordinate grid, and by this means chart the dominant–tonic chord progression (V^7–I) as a transition from one position to another within that space. The idea is somewhat analogous to the concept of *configuration spaces* used in physics and other sciences, where every possible structural arrangement that a system can take (i.e the full cadence) is allotted a unique position within an abstract space. Then the evolution of the system can be charted in terms of positional changes within that space.

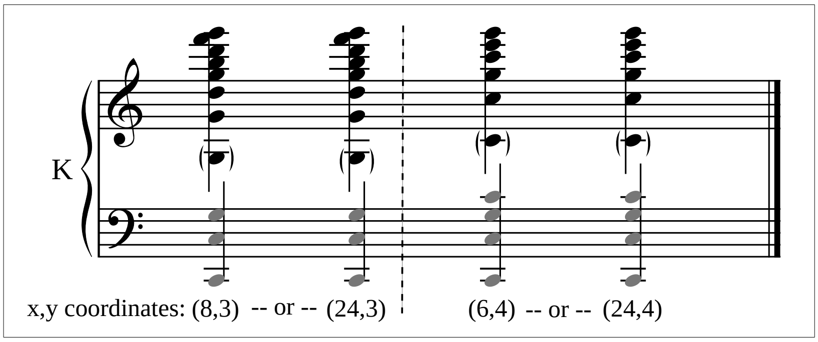


Figure 13.5 The coordinates of an abstract space expressed in harmonic/musical form (chords V^7 and I). The gray bass notes representing values on the y-axis and the black treble clef notes represents x-axis values.

In Figure 13.6 the full cadence is charted in two ways, using the the ‘raw’ digits as coordinates and by multiplying out the individual columns so as to express all the coordinates in units of the fundamental series. Using the raw digit values (8,3 and 6,4) produces a dynamic space in which the ‘coordinate metrics’ change from plot to plot, as the units of the x-axis are in multiples of the y-axis units. Thus somewhat confusingly, each x-axis unit is worth three y-axis units for the dominant chord (MBN 8_3 0_1) and four y-axis units for the tonic (MBN 6_4 0_1). This dynamic rendering of the coordinates is interesting in that the space is changed or distorted by the points occupying it, but is notionally ‘flat’ when empty of plots.

The second method of plotting points by ‘multiplying out’ the columns in mutable numbers produces what is essentially the standard Cartesian absolute plane or space, where plotted points take up positions in a uniform flat grid, which remains unaffected by what it contains. This method also places the conjunctions between nested series/mutable digit sequences/chord progressions in a ‘space’ consisting of one less dimension. In the example illustrated in Figure 13.6, the conjunction lies along the line of value twenty-four, on the x-axis; one dimension less than the plane of the x,y axes.

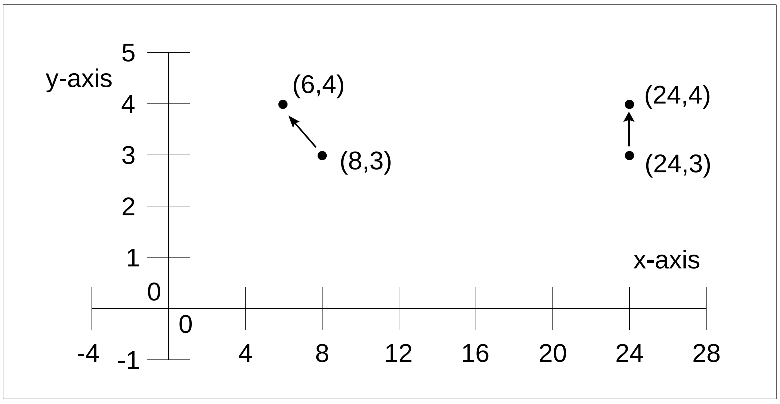


Figure 13.6 The dominant-tonic chord progression (V^7 – I) in isolation, with mutable number digit sequences for twenty-four used as x,y coordinates (MBN 8_3 0_1 and 6_4 0_1) yielding a graphical display of the cadence. There are two possible interpretations, either the raw digit sequence (8,3 --> 6,4) or the multiplied-out coordinates (24,3 --> 24,4).

Figure 13.6 is just a simple example of what could be a most useful way of illustrating and visualising the evolution of mutable number digit sequences – that is harmonic progression in tonal music. Indeed, one can imagine using far larger numbers than twenty-four to perhaps model complex, dynamical, multi-point systems. The full cadence V^7–I, is a simple example to introduce the principle. From studying the Bach Prelude in Chapter 12 and Example S, it is apparent that in most non-trivial tonal compositions at least three dimensions will be required, and this is not so straightforward to represent on a flat page for more than a few positions on three axes. In Figure 13.7 the first four-bar phrase from the prelude is plotted in three dimensions. Overall, the positions plotted move from front to back and bottom to top, with the phrase progressively losing complexity and energy as it proceeds towards the cadence. The recalibration at the end of measure 4 (applied in Chapter 12) then draws the prelude back to the lower front position of the space, in preparation for the start of the next phrase.

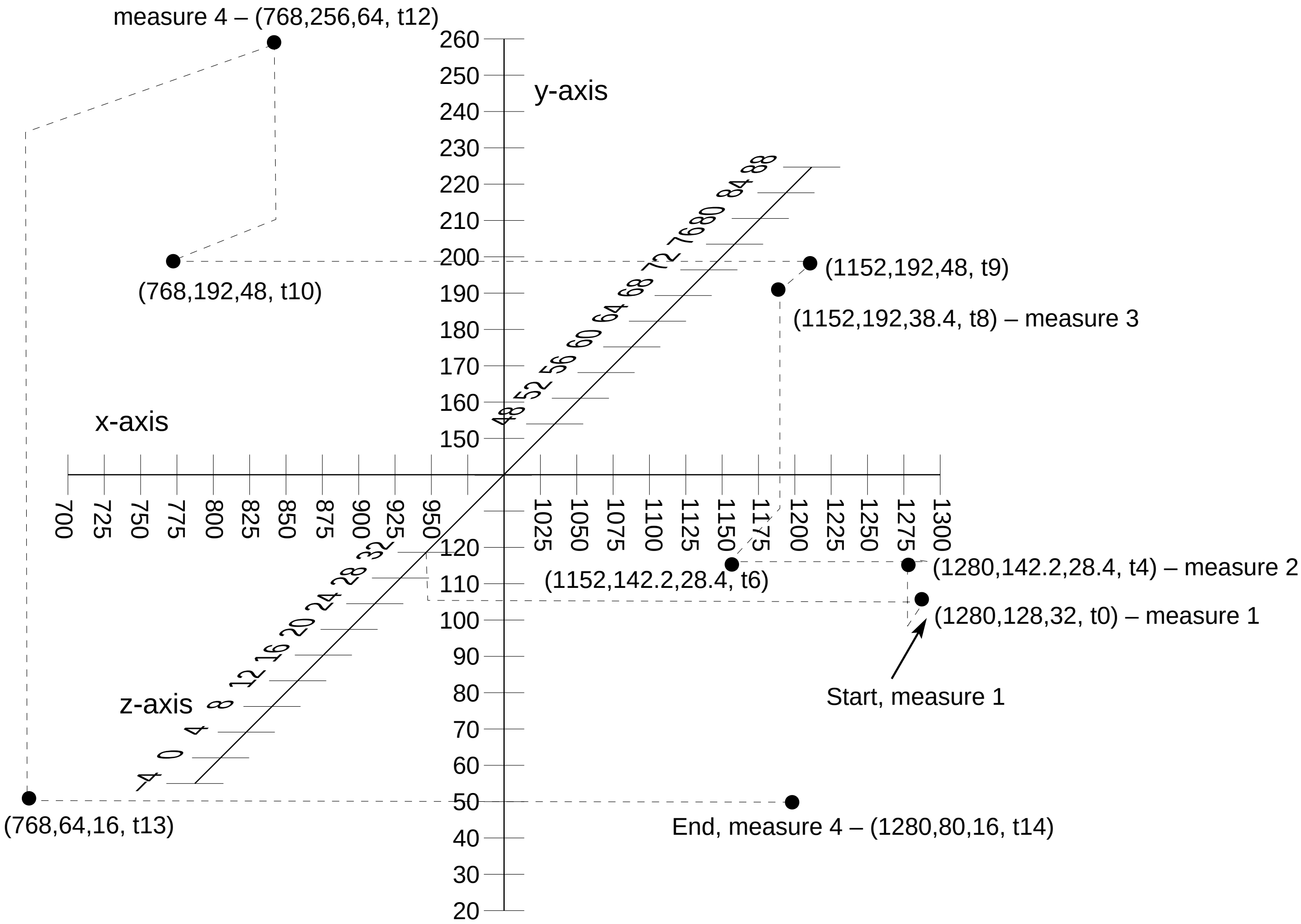


Figure 13.7 The first four bars of Prelude No.1 from the Well-tempered Clavier – J.S. Bach, with the mutable base digit sequences plotted as coordinates in an abstract mathematical space.

Finally, having reached the point of describing a musical system moving in a three-dimensional space in Figure 13.7, it only remains to add the forth dimension of time to the picture. The dimension of time enters at the lowest level, as a harmonic series set at the foundations of the structure, and as described in Chapter 5 (Figure 5.13) it is built from units of duration, the pulse (or subdivisions or multiples thereof). Thus each plot in Figure 13.7 can be assigned a position in time, a position on a fourth ‘t-axis’. Although it is difficult to imagine or represent yet another axis (as with the hypercube), in practice this is not such a problem because motion in the ‘t’ dimension in mono-directional. As a consequence of this irrevocable directionality, by linking the three-dimensional plots of Figure 13.7 in ascending t-axis units (the dotted line), beginning from the origin t = zero, a sequence of time-ordered events is obtained without explicitly drawing a t-axis on the page. The t-axis units are quarternotes in Figure 13.7.

In the study based on Bach's prelude (Example O and Figure 13.8 below), when played at concert pitch (A = 440Hz) and the ridiculously precise tempo of 61.318... MM, there will be an exact correspondence between the period of the fundamental harmonic series describing the harmony of the piece (C-H1) and the period of the beat in performance. That is, C-H1 = 1.022Hz = quarternote at 61.318...Malzel Metronome.

Mutable Base Numbers: 1280-E 1280-E 1152-D

	Mutable	10	9	8	
Pitch: middle C = 261.626 Hz	Digit	4	5	5	etc...
		32	28	28	
C-H1 = 1.022 hertz	Sequences:	1	1.016	1.029	

Figure 13.8 The first two measures of a study for alto recorder or flute, based on Prelude No.1 from The Well-tempered Clavier, where the stately tempo of MM 61.318... and pitch of A = 440Hz would combine the fundamental harmonic period (C-H1) with that of the beat.

In Bach's prelude, the shortest effective unit of duration in regard to harmonic progression, registered as changes in mutable number digit sequences, is the quarternote. Though the figuration is set in sixteenth notes, this subdivision of the pulse can be regarded as extended chordal elaboration. Indeed, the repeated pattern in the Prelude illustrates well how rhythmic articulation and chordal figuration are often used to 'fill out' and support both the metrical and harmonic structure. However, where the rhythm and/or figuration occasionally run counter to the prevailing order, as they sometimes do, an equivalence can be perceived between the process of rhythmic syncopation in the domain of duration and suspended dissonance in the realm of pitch – both being disturbances in the normal nested structure of the MOS. Whilst the overall union of the metrical and harmonic aspects of a composition is rather theoretical, and in practice tonal music is probably best modelled as two separate modulating oscillatory systems running in parallel – a durational MOS underlying a pitch MOS. It is nevertheless remarkable that a single coherent notional structure, built from nested harmonic series and articulated by the algorithm of symmetrical exchange, stretching from the period of the whole piece (or movement) through to the high frequencies of timbre, can be constructed at all.

SETS, GROUPS AND MUTABLE NUMBERS

Early in the twentieth century Bertrand Russell (1872–1970) devised a mathematical scheme that generated the natural counting numbers out of 'thin air' through a process of nesting. His technique relied on the *theory of sets* which had been developing over the preceding fifty years and was famously used by Georg Cantor (1845–1918) in the nineteenth century to investigate the nature of infinity. A 'set' is simply a collection of 'things'. The collection is enclosed within curly brackets and the elements are separated by commas. For example, a set of numbers {1, 2, 3}, a set of birds {sparrow, pigeon, starling, blackbird}, a set of unrelated objects {pencil, house, ship, planet}, or crucially for Bertrand Russell an empty set. The null or empty set has no elements, and the 'Ø' symbol is used to represent it. Russell's idea was basically that by defining zero to be the empty set 'Ø' and the number one to be the set containing the empty set '{Ø}', then, through a process of nesting sets within sets, all the other counting numbers may be produced.

0 = Ø – the empty set
1 = {Ø} – { the set containing the empty set }
2 = {Ø, {Ø}} – { the set containing the empty set, { and another set containing the empty set } }
3 = {Ø, {Ø}, {Ø, {Ø} } } – { the set containing the empty set, { and another set containing the empty set }, { yet another set containing the empty set , { and another set containing the empty set } } }

Since the 1960s, set theory has been extensively applied to the analysis of atonal music. Atonal set theory focuses upon the equal-tempered scale of twelve chromatic tones and seven diatonic notes interpreted in terms of the set of integers modulo 12 and modulo 7. That is, division by 12 and 7 of absolute values so as to reduce the positive and negative integers to the range {C0, C#1, D2, D#3, E4, F5, F#6, G7, G#8, A9, A#10, B11} and {C0, D1, E2, F3, G4, A5, B6} respectively. Conventionally these sets are based on C but any scale could be used, and considerable work has been done in exploring the range of transformations that map one collection (e.g. scale, chord or note) to another. Rather surprisingly far less attention has been devoted to the set of harmonic partials {h1, h2, h3, ... hn}.

Set notation within curly brackets is quite well suited to expressing mutable number digit sequences in the form of harmonics, including the feature of nested structure. Indeed, a more fundamental nesting set takes on the form of a single element within the scope of its nested series. For example, the primary sesquitertia 3:4 modulation exchange as introduced in Chapter 9 page 4.

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{{h1C,h2C,h3G}, h6G, +h9D, +h12G} -3:4-> {{h1C,h2,h3G,h4C}, h8C, h12G}
```

Here four elements of a nested series {h3, h6, h9, h12} are exchanged for three elements {h4, h8, h12} within the wider context of the nesting series {h1, h2, h3, h4, h5, h6, h7, h8, h9, h10, h11, h12}, with the first element of each nested series subsets of the underlying nesting series, e.g. {h1, h2, h3} and {h1, h2, h3, h4}.

Borrowing a little from the notational style of atonal/neo-Riemann set theory, the modulation algorithm of symmetrical exchange could be expressed in terms of a function that carries one harmonic series/mutable digit sequence to another:

$$\mathrm{MBN}^{a}_{m:n} \quad an_{H(f)} \;=\; am_{H\left(\frac{n\times f}{m}\right)}$$

for series of 'a × n' and 'a × m' harmonics, with positive integer values of 'a' representing primary, secondary, tertiary, etc. exchanges and 'H(...)' signifying the fundamental frequencies of the respective nested series. Thus for the dominant–tonic secondary sesquitertia 3:4 exchange:

```
{{H1C,H2C,H3G}, h6G, h9D, h12G, h15B, h18D, h21F, h24G} -->
                    --> {{H1C,H2C,H3G,H4C}, h8C, h12G, h16C, h20E, h24G}
```

the values would be:

$$\mathrm{MBN}^{2}_{3:4} \quad 2\times4_{\mathrm{H}(3)} \;=\; 2\times3_{\mathrm{H}(\frac{3\times4}{3})}$$

As well as mapping a given nested harmonic series from its 'domain' to its 'image' or 'range', the function also calculates the corresponding adjustment in the fundamental nesting series by producing the new value for the nested fundamental, (for the above example 3×4/3 = H4).

So far so good, but can this function be applied to more levels of nesting? As found in the analyses given in Chapter 12 and the Examples. The answer is yes and no! Intrinsically, the algorithm of symmetrical exchange operates on a nesting and nested pair of harmonic series – a fundamental nesting series within which a 'child' series is nested. But luckily a parent can have more than one child, and this is the case for fundamental harmonic series also. You will perhaps recall that aggregated series were named as such for the convenience of avoiding the ambiguity of referring to two different nested series, and that ultimately all levels of nested series can be viewed as nesting within the one absolutely fundamental series. With this perspective it becomes possible to deal with unlimited levels of nesting by separating out each individual level of nesting (i.e. aggregated or nested series) and treating it as being a 'child' of the system's fundamental nesting series. Take for example the first chordal exchange between measures 1 and 2 of the Prelude in C by J.S. Bach, Chapter 12 page 10 and Example S.

fundamental, nested, aggregated series

measure 1: { { { H1, through H32 }, h2, h3, h4 }, h8, h12, h16, h20, h24, h28, h32, h36, h40 } -->

measure 2: --> { { { H1 through H28 }, h2, h3, h4, h5 }, h10, h15, h20, h25, h30, h35, h40, h45 }

The complex arrangement above of fundamental, nested and aggregated series may be separated into two simpler pairs of series, 1) the fundamental and aggregated pair with ratio of exchange (below),

fundamental, aggregated series

{ { H1, through H128 }, h2, h3, h4, h5, h6, h7, h8, h9, h10 } --9:10-->

--9:10--> { { H1 through H140 }, h2, h3, h4, h5, h6, h7, h8, h9 }

and 2) the fundamental and nested pair with ratio of exchange (below).

fundamental, nested series

{ { H1, through H32 }, h2, h3, h4, ..., through h40 } --9:8-->

--9:8--> { { H1 through H28 }, h2, h3, h4, h5, ..., through h45 }

Notice, however, that although the two separated pairs of series have different ratios of exchange (9:8 and 9:10) they both share the same conjunction frequency 1280 Hz. This is somewhat obscured by the frequency of H28 actually being 28.444... Hz and H140 being 142.222... Hz. As mentioned in earlier chapters a degree of flexing of relationships must be allowed for in the dynamical approach taken in the MOS model. In practice these variations tend to mostly cancel each other out over an extended piece.

Now that we have simplified the exchange by separating out the nested and aggregated series, it is just a matter of inserting the appropriate values into the transformation function, which are respectively:

$$\mathrm{MBN}^{1}_{9:10}\ 1\times10_{\mathrm{H}(128)} = 1\times9_{\mathrm{H}(\frac{10\times128}{9})} = \mathrm{h}9_{\mathrm{H}(142.2...)} = \mathrm{H}1280_{\mathrm{H}1}$$

$$\mathrm{MBN}^{5}_{9:8}\ 5\times8_{\mathrm{H}(32)} = 5\times9_{\mathrm{H}(\frac{8\times32}{9})} = \mathrm{h}45_{\mathrm{H}(28.4...)} = \mathrm{H}1280_{\mathrm{H}1}$$

Although no difference will arise from the calculation of one series before another, it is preferable to start at the top with the aggregated series and work downward from there, as this sequence mirrors the MOS model's view of the system being 'driven' from the top by the objective sound. The music – the chords of a composition – primarily impose their relationships upon the aggregated series, with a few stray notes falling through this uppermost level to be caught by the finer mesh of harmonics cast by the middle-level nested series, and ultimately, in very extraordinary circumstances, a note could theoretically even fall right through to the fundamental series.

The flexibility afforded by using three levels of nesting, or three column mutable numbers, is about the basic requirement needed to accommodate the flow of normal harmonic progression in the music of the tonal era. This 'three-dimensional' quality to the MOS model of tonal harmony is interesting, both for its reflection of the three spatial dimensions of the physical world (which likewise afford a particularly rich variety of relationship), and its underlying two-dimensional foundation, which in turn is reflected in the newly discovered holographic principle briefly introduced in Chapter 3 page 9.

George Boole whose early work on symbolic logic would a century later bear fruit in underpinning the techniques of the modern electronic computer, was also a foundational figure in set theory. Boole was noted for the confidence and daring of his highly original deductions and calculations; indeed, Boole placed a deep trust in the outcomes of mathematics' abstract, visually based symbolic procedures. One can only speculate as to what effect a more materially orientated 'aural' math might have given rise. However, when mathematics did eventually step meaningfully into the physical world, through the employment of binary logic in the circuitry of computers, the results were to be far-reaching both for mathematics and for mankind.

George Boole (1815–1864) was born in Lincoln, England, the son of a tradesman. Boole was industrious and largely self-taught, and, against the odds, acquired a position as a teacher. By the time he was twenty, Boole had founded his own school, yet despite the workload maintained his life-long habit of constant study. It was only at this point in his life that he began the serious study of mathematics, yet after only a few years he published many mathematical memoirs and two treatises. His development of a highly original *algebra of classes* (i.e. sets), along with *Boolean logic* were perhaps his most enduring achievement. At the age of thirty-four he was appointed Professor of Mathematics at Queen's College, Cork, Ireland. In 1855 Boole married Mary Everest, the niece of a famous Indian surveyor. George Boole died unexpectedly after a short fever on the 8th December 1864, aged 49 years.

Group Theory

Group theory is the field of mathematics concerned with the study of symmetry. It creates a rigorous language in which the nature of symmetry may be described, and the operational principles of symmetry understood. The mathematical theory of groups provides a foundational conceptual environment in which the structures and processes of the MOS model may be securely built. Broadly similar to set theory, it is a math that deals with collections of (mathematical) objects that share some common feature, for example, the group of whole numbers (... −3, −2, −1, 0, 1, 2, 3, ...) or the rotational symmetries of regular plane or solid geometric figures such as a square or cube or twelve-faced dodecahedron. The theory of groups provides a remarkably general approach to the study of collections of related items, which can be applied in a wide variety of different contexts. It underpins much of particle physics and cosmology, where group theoretic principles of symmetry transformations provide a basic framework within which theories can be worked out.

Though many ideas and concepts concerning symmetry had implicitly seeped into mathematics during its long history, a formal theory of groups was not developed until the nineteenth century – initially emerging from the inspired work of Niels Henrik Abel (1802–1829), Evariste Galois (1811–1832) and somewhat later Arthur Cayley (1821–1895). In a manner rather analogous to thermodynamics in physics, the principles of group theory were slow to emerge, and the broad scope of their application took even longer to be fully appreciated. Indeed, after a false dawn in the writings of Paolo Ruffini (1765–1822) at the turn of the nineteenth century, it would be nearly a hundred years before definitive texts were to be produced. Throughout the twentieth century the theory of groups was to be continuously further developed and refined by mathematicians, with a notable landmark, after thirty years' concerted effort, the classification of all finite simple groups in the 1980s. This feat is often cited as perhaps the greatest mathematical achievement of the century. Overall the development of group theory has been democratic, in the sense that many mathematicians have contributed to the field over a long period of time. However, if a founder were to be identified it would probably be Evariste Galois, a young Frenchman who discerned a subtle connection: a symmetry between the solutions of certain equations, the nature of which can be gathered from the equation $x^2 = 4$, which has two solutions: x can be 2 or −2. Two and its negative hint at a symmetry: they are different, and yet also, the same.

The 'objects' or elements or units of a *group* form a mathematical structure, a 'little world' of relationships. The number of elements in a group may be limited, as in the four rotations of a square which bring it into coincidence with itself without leaving the plane, or unlimited as in the group of all integers. For a group to be mathematically valid, it needs to satisfy some or all of the following criteria:

1) Operations must produce the same result despite the order in which they are applied to a fixed sequence of the group's elements (associativity);
2) Operations upon elements in the group must always produce other valid elements of the group (closure);
3) One special 'identity' element within the group will produce no change when combined with elements by the group algorithm – a null operation (identity);
4) Every transformation can be undone by some process of inversion or reversal (inverses);
5) Operations must produce the same result despite the order of elements to which they are applied (commutativity).

In the above example of the group of integers – the positive and negative whole numbers – the algorithm would be *addition/subtraction* and the identity element *zero*. Addition of any two or more integers will always produce another integer (i.e. 3 + 6 = 9 or 10 − 7 = 3), and zero when added to any element produces no change of value (i.e. 7 + 0 = 7). And every transformation by addition for the whole numbers can be reversed by subtraction – the addition of a negative integer. In essence, the group approach extracts a 'little world' of mathematics, an 'algebra' out of the valid manipulations of the group's elements, by the group's algorithm. (As did humankind, similarly, over millennia, deduce traditional mathematics from their experiences of the physical world.)

Symmetry Groups

Up to this rather late point in the story there has been no direct application for the triangular number patterns introduced in Figure 1.7, but here at last they have a purpose to serve in illustrating the rotational symmetry of a plane figure – an equilateral triangle. For the Pythagorians of ancient times the triangle held a special place in their philosophy: they proved the relationship between the sides of right-angled triangles and used the first four counting numbers (the tetrakyts) to construct a sacred triangular number pattern of ten.

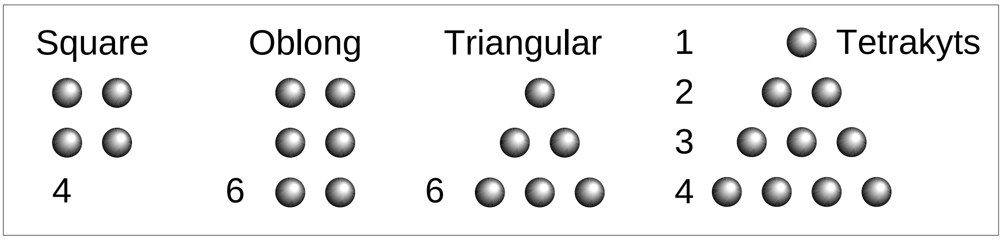


Figure 13.9 Sample square, oblong and triangular number patterns.

If an equilateral triangle is rotated in its plane, about an axis running through the central point of its area, it will come into coincidence with itself, its original position, three times in each complete rotation. Vertex A goes into B, B into C and C into A with 120 degrees of rotation, followed by A into C, B into A and C into B at 240 degrees of rotation and finally completing one cycle, A into A, B into B and C into C. (Further rotation beyond 360 degrees is not considered here, excepting those rotations arising from multiple applications of the three basic ones, which are taken to cycle around as the figure 'returns to itself'.)

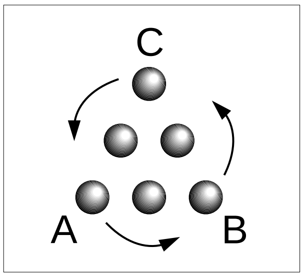


Figure 13.10 The rotation in the plane of an equilateral triangle about its central point.

Now, if the product of any two transformations or rotations is taken to be their successive application, and if each transformation given above is labelled respectively t_1, t_2 and t_0 (the latter transformation taking the triangle back to its original position, the same outcome as rotation by zero degrees), then a few examples selected from amongst all possible combinations, and order of combinations, are:

$$t_0 \bullet t_0 = t_0 \qquad t_0 \bullet t_1 = t_1 \qquad t_0 \bullet t_2 = t_2 \qquad t_1 \bullet t_1 = t_2 \qquad t_1 \bullet t_2 = t_0 \qquad t_2 \bullet t_2 = t_1$$

Also quite naturally, by reversing the rotary motion, the reversal or undoing of each transformation is achieved, and plainly more than two transformations may be applied successively, in any arbitrary order. Reverse motion could be denoted by a minus sign. Note that though the process of transformation is represented by '•', this does not denote addition or multiplication but merely *transformation*. Here are some more successive 'algebraic' transformations of the elements t_1, t_2 and t_0:

$$t_1 \bullet -t_2 = t_2 \qquad t_1 \bullet t_2 \bullet t_1 = t_1 \qquad -t_1 \bullet -t_1 = t_1 \qquad t_2 \bullet t_2 \bullet t_2 = t_0 \qquad -t_0 \bullet t_1 \bullet t_0 = t_1$$

Thus from amongst the rotational symmetries of an equilateral triangle – the positions of coincidence – there has emerged a scheme of relationship and manipulation. A system of mathematics, albeit a rather limited one.

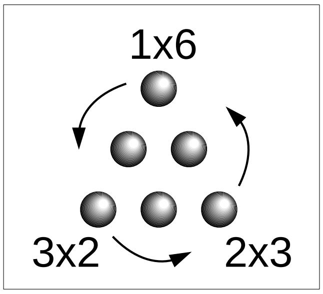


Figure 13.11 A representation of the ‘rotational’ symmetries of the physical number six: MBN 6_1, $2_3\,0_1$ and $3_2\,0_1$.

Next, if the vertices of the triangle were relabelled to the different configurations of partials that the physical number six can sustain, the result is as shown in Figure 13.11. This relabelling would not alter the little scheme of transformations open to the system as such, but it does cast the manipulations in a new light. The triangle now represents the aurally distinguishable transformations[2] of the mutable number six obtained by the modulation algorithm. Each rotation of 120 degrees carries the vertices from one digit sequence to another. In all cases the ‘rotations’ yield the number six by different means (1×6, 2×3 and 3×2), and therefore they are symmetries or ‘samenessess’ of the object. What this example illustrates is that within most mutable numbers there resides a ‘little world’ of relationships, a set of configurations (i.e. various digit sequences) each of which is a symmetry of the particular single value being represented. That is to say, each mutable base number forms a *symmetry group* consisting of the range of valid digit sequences available to it. Some of these mutable numbers will possess a wide variety of internal arrangements, some will have only a small arsenal of configurations to choose from, and the primes will have but one.[3] For example, MBN twenty-four has 20 distinguishable internal arrangements, while MBN six has only 3 configurations to choose among (Figure 13.11), as also do MBN twenty-one and twenty-two, but MBN twenty-three, a prime, has only one viable internal arrangement. Twenty-four, twenty-three and twenty-two are as close as whole numbers can be in value, yet one is fecund, one barren and one relatively stinted in the ground that they offer for variety to emerge.

This is not all that must be considered, because taken at face value MBN sixteen is also a rich number, having 35 different internal arrangements in total, though only eight of these are separately distinguishable. They are for the record: 24 arrangements of 2×2×2×2, six of 2×2×4, two of 4×4, two of 2×8 and one of 1×16 (discounting 16×1).[4] Thus if MBN sixteen were used as the basis for a ‘little world’ of transformations, so long as all the arrangements were clearly labelled, as in Figure 13.10/11, it would produce a relatively rich array of states (though MBN twenty-four if treated similarly would be richer still with 43 arrangements). What this application of group theory to mutable numbers is also illuminating is a dichotomy between the face value fecundity of different numbers. From the point of view of a group theoretic axiom-system approach, MBN sixteen is relatively highly symmetric: the mutable number sixteen is rich in ‘rotational’ symmetries and therefore can provide a wealth of elements or viable transformational states. Similarly, MBN twenty-four is relatively highly symmetric and likewise can host a wealth of transformational states. The difference between MBN sixteen and MBN twenty-four is that in the former, less than one quarter of these states, are individually distinguishable while in the latter almost half are. Therefore in terms of information and entropy these two highly symmetric numbers are significantly different. MBN sixteen is highly symmetric *but relatively information poor*, in that most of its internal arrangements look the same – like the random arrangements of a gas at equilibrium. MBN sixteen has relatively high entropy. In contrast, MBN twenty-four is highly symmetric *and information rich*, in that nearly half of all its possible internal arrangements are distinguishable and could be used to encode twenty bits of information, more than twice the information density of MBN sixteen. MBN twenty-four has relatively low entropy.

Picture courtesy Wikipedia

Evariste Galois (1811–1832) was born at Bourg-la-Reine near Paris, the son of Nicolas-Gabriel Galois a school-master and Adelaide Marie Galois (nee Demante) the daughter of a senior jurist. The family were literary, and liberal in politics, with Galois' father becoming Mayor of Bourg-la-Reine in Napoleon's time. Evariste was educated at home until the age of twelve (his father ran a local boys school) and his strong republican ideals probably date from this turbulent time after the restoration of Louis XVIII. In 1823, Evariste was sent to the Lycee Louis-le-Grand in Paris as a boarding pupil, were he did well academically, but also became embroiled in the student politics, reflecting the wider tensions between liberalism and conservative royalism in French society. Partly responding to a new and enthusiastic teacher of mathematics, and perhaps partly looking for escape from his dreary surroundings at the school, by the end of 1827, Evariste had become engrossed in mathematics. Remarkably, by April 1829 his first mathematics paper had been published and such was the rapidity of his mathematical development, that by June of the same year, the outlines of his seminal idea concerning symmetry groups had been written up and submitted to the French Academy of Sciences. Frustratingly, even the foremost mathematicians of the academy, Cauchy, Fourier and Legendre, dallied and failed to appreciate the significance of Galois' visionary leap, as they had also with similar work by the young Norwegian, Niels Henrik Abel, a few years earlier. Indeed, it would take all the century for the fuller implications of Galois' and Abel's work to be digested. For quite some time after Napoleon's fall Nicolas-Gabriel managed to hang on as Mayor of Bourg-la-Reine, but after an increasingly vicious and personal campaign, he committed suicide in July 1829. This must have been a devastating event for Evariste and no doubt confirmed his firebrand republican views. From this time Galois' political activities intensified, which led to his imprisonment 1831. Interestingly, he found the prison environment conducive to mathematical research. Upon release, the impetuous young Evariste became involved in a romantic affair which culminated in a duel which left him fatally wounded. Galois died in hospital on 31st May, 1832, just 21 years old.

Generating the Natural (Mutable) Numbers

We now turn our attention to mutable numbers as a *group*-like structure, that is the set of counting numbers 1, 2, 3, 4, etc., but unlike Bertrand Russell's scheme here not including zero. Originally the set of all integers (... −3, −2, −1, 0, 1, 2, 3, ...) was proposed as an example of a group that, taken with the algorithm of addition, satisfies the criteria for a group given above. The addition of any and all integers results in another integer, subtraction provides the inverse operation and zero plays the role of identity element in this scheme. However, in applying group theory in the round, to modulating oscillatory systems, the example of the natural counting numbers – the positive whole numbers – is more appropriate. And though '•' was used above to represent the mode of transformation now the group operation may be signified by '×', because the structure of nesting one series within another used within the group of positive whole mutable numbers is clearly multiplication.

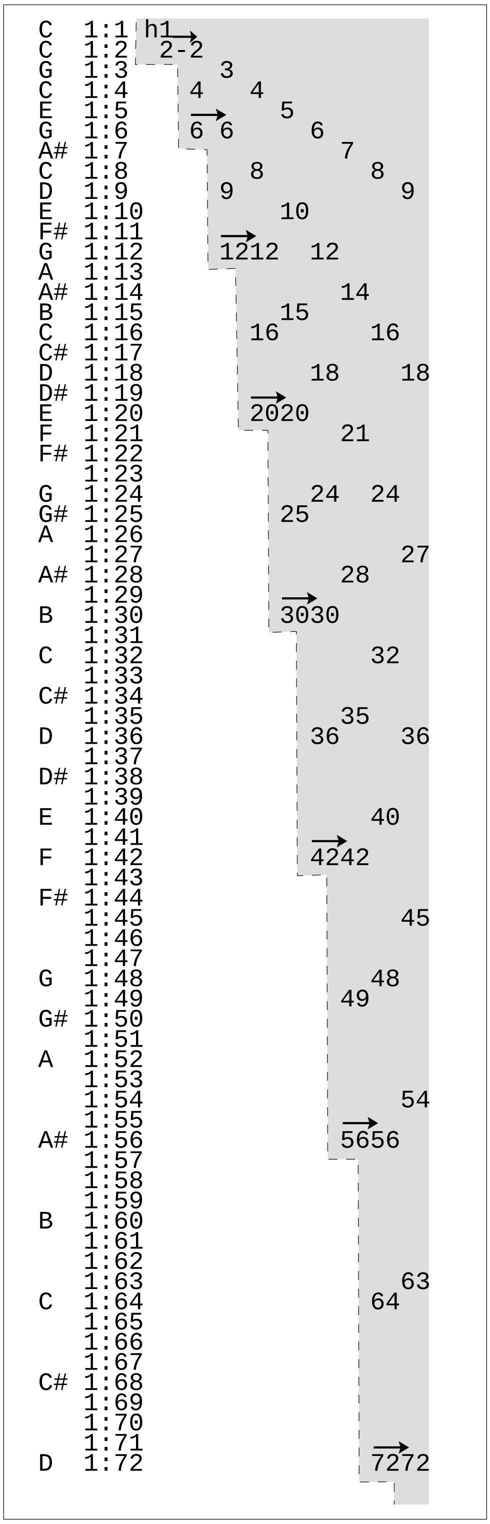


Figure 13.12 The first eight primary modulations – the means by which the algorithm of symmetrical exchange generates the natural numbers out of unity. In order, the modulation exchanges are: dupla 1:2, sesquialtera 2:3, sesquitertia 3:4, sesquiquarta 4:5, sesquiquinta 5:6, sesquisexta 6:7, sequiseptima 7:8 and sequioctava 8:9.

In such a multiplicative scheme the identity element that leaves other elements of the group unchanged under transformation is the number one (h1), instead of zero. So rather than $t_n \bullet 0 = t_n$ we now have $t_n \times 1 = t_n$ (*identity*), while the nested structure of mutable numbers naturally supplies the operation of multiplication between two group elements in their prime states. For example, h1 is one, h1+h2 is two, three is h1+h2+h3 (also nested as h2+h4+h6 in Figure 13.12) and four is h1+h2+h3+h4 (nested as h3+h6+h9+h12 in Figure 13.12) and so forth. Thus by nesting one (or more) harmonic series within another, the outcome is always a valid harmonic in the underlying series (*closure*), and no matter what order the series are nested in, the valid (higher) harmonic is unchanged (*associativity* and *commutativity*). Rather neatly, this valid upper harmonic is the conjunction value (i.e. frequency) used for exchanges, and because of this each nested series forms a ‘regular’ or *normal subgroup*[5] of the fundamental nesting series. Which is to say that the nested series divides the fundamental series without remainder, which it must do given that it shares a common conjunction with the fundamental series – the highest harmonic in the nested system. However, the natural numbers do not have *inverses* and so they are not called a group but a *monoid*. A monoid is just a weaker form of group structure which doesn’t include inverses. Though if the counting numbers (e.g. 1, 2, 3, ... n) were expanded to include all the fractions that could be made from them, then each number would have an inverse (e.g. two’s inverse is one-half: 2 × 1/2 = 1), and so they would form a group – the group of *rational numbers*. (The rational numbers, discussed below, play a significant role in articulating the exchanges of the modulation algorithm.)

Figure 13.12 illustrates the first eight *primary* modulation exchanges in the table of nested harmonic series (or Sieve of Eratosthenes). These exchanges are also illustrated in their musical guise in Figure 9.3 and stepped through as ratios in Figure 9.4. The set of natural numbers in this scheme are formed by the diagonal line of column-heads – h1, 2, 3, 4, 5, 6, 7, 8, 9, etc. – together with their accompanying nested harmonic series (which are given a gray backdrop). Thus, as the system steps through the sequence of primary modulations (i.e. una, dupla, sesquialtera, etc.) – powered by the repeated procedure of adding two ratios/oscillators to each nested series followed by the relaxation of a modulation exchange – it generates the set of natural numbers in the fundamental series. For example, the fundamental series moving from natural number 3 to 4:

```
   t₃     ×      t′₄      -->       t₄     ×    t′₃     (3 × 4) --> (4 × 3)
H1,H2,H3 n~ h6,h9,h12  -->  H1,H2,H3,H4 n~ h8,h12   Sesquitertia 3:4 exchange
```

One might envision the nested series as being the argument taken by the transformation algorithm and the fundamental series as the result yielded. And for consistency the presentation is, as usual, in the form of harmonic rather than arithmetic series. Taking the numbers and transformations in the ascending order of Figure 13.12 yields a sequence commencing:

Table 13.12a

	Ratios Exchanged	Transformation		
|	h1 --> <u>h1</u>	una	1:1	
|	h1+h2 --> <u>h2</u>	dupla	1:2	
|	h2+h4+h6 --> <u>h3</u>+h6	sesquialtera	2:3	
|	h3+h6+h9+h12 --> <u>h4</u>+h8+h12	sesquitertia	3:4	
V	h4+h8+h12+h16+h20 --> <u>h5</u>+h10+h15+h20	sesquiquarta	4:5	etc.

The underlined ratios are the column-heads of Figure 13.12 and represent the creation of the succeeding natural numbers as they accumulate within the fundamental series through the application of the transformation algorithm of symmetrical exchange. As the group of natural numbers is infinite, the process of transformation is potentially unending. However, the modulation algorithm is not restricted to this outward stepwise development. The algorithm works equally well in reverse,

Table 13.12b

	Ratios Exchanged	Transformation	
|	h5+h10+h15+h20 --> h4+h8+h12+h16+h20	sesquiquarta	5:4
|	h4+h8+h12 --> h3+h6+h9+h12	sesquitertia	4:3
|	h3+h6 --> h2+h4+h6	sesquialtera	3:2
|	h2 --> h1+h2	dupla	2:1
V	h1 --> h1	una	1:1

as well as for arbitrarily wide steps ascending and descending.

Table 13.12c

Ratios Exchanged	Transformation	
h6+h12+h18 --> h9+h18	sesquialtera	2:3
h7+h14+h21+h28+h35 --> h5+h10+h15+h20+h25+h30+h35		7:5
h2+h4+h6+h8+h10+h12+h14+h16+h18 --> h9+h18		2:9
h8 --> h2+h4+h6+h8	quadruple	4:1
h9 --> h9	una	1:1

Thus the modulation algorithm of symmetrical exchange, acting from one, or h1, or MBN 1_1, both generates the sequence of natural mutable numbers and allows an arbitrary freedom of navigation amongst them via the general relationship introduced in Chapter 9:

$$n_{H(\frac{m}{a})} \longleftrightarrow m_{H(\frac{n}{a})}$$

For integer m and n representing series of ‘m’ and ‘n’ harmonics, integral values of ‘a’ representing primary, secondary, tertiary, etc. exchanges and ‘H()’ signifying the relative fundamental frequencies of the respective series. However, here for the generation of the natural numbers only primary exchanges have been required which simplifies the relationship to:

n harmonics of relative frequency m <---> m harmonics of relative frequency n,

which yields straightforward mutable digit sequence exchanges like MBN 3_20_1 --> 2_30_1 [H=3] when applied to the rows of Table 13.12c above, thus:

Table 13.12d

Ratios Exchanged	Transformation		
h6+h12+h18 --> h9+h18	2:3	$3_{H(m)}$ --> $2_{H(n)}$	n = 3, m = 2
h7+h14+h21+h28+h35 --> h5+h10+h15+h20+h25+h30+h35	7:5	$5_{H(m)}$ --> $7_{H(n)}$	n = 5 , m = 7
h2+h4+h6+h8+h10+h12+h14+h16+h18 --> h9+h18	2:9	$9_{H(m)}$ --> $2_{H(n)}$	n = 9 , m = 2
h8 --> h2+h4+h6+h8	4:1	$1_{H(m)}$ --> $4_{H(n)}$	n = 1 , m = 4
h9 --> h9	1:1	$1_{H(m)}$ --> $1_{H(n)}$	n = 1 , m = 1

Symmetry within Mutable Numbers

Earlier, the symmetries of a triangle in a plane were examined; these symmetries arose through the rotation of an equilateral triangle by 120, 240 and 360/0 degrees. However, in a more general view, the equilateral triangle could be seen as ‘selecting’ three specific rotations from amongst the unlimited range of rotational symmetries of the plane itself. If a circle, rather than a triangle, were chosen for the figure in the plane, it would share more fully in the plane’s symmetries, in that every infinitesimal rotation of a circle about its center would be a symmetry of the figure. Looked at from this more general perspective, the full group of symmetries of the plane could be viewed as *acting* upon figures in the plane; with the symmetries supported by various figures in the plane selecting subgroups of symmetries from within the broader complete group of symmetries of the plane itself. So now the rotations of an equilateral triangle in a plane, might be considered in terms of the *action* of the full group of symmetries of the plane, upon the triangular figure. Of all the possible symmetries of the plane, only three produce valid invariant symmetries of the triangle, the rotations of 120, 240 and 360 degrees. (That is without the triangle leaving the plane – i.e. turned over.) All the other limitless symmetries of the plane, lateral displacements and other angles of rotation about the central point of the figure, don’t take the figure into itself but leave the triangle at variance with respect to its starting position. The smooth rotation of the triangle might be described as an orbit of the vertices of the triangle, under the action of the rotational symmetries of the plane – an orbit that contains just three invariant positions. In group theory terms, these rotations that give rise to points of invariance, are termed the stabilizer subgroup[6] of the complete group acting upon the particular set: the group of symmetries of the plane acting upon the set of vertices of the triangle.

Moving on from the three vertices of a triangle to the three mutable number digit sequences of value six, a similar description to that above could be applied. However, instead of the group of symmetries of the plane acting upon a triangle, the group of positive rational numbers acting upon the set of the harmonic series, of extent or cardinality ‘n’ –i.e. {h1, ..., hn}[7] is substituted. Staying with our example number six, this would be the set {h1, h2, h3, h4, h5, h6} or MBN 6_1, with the group of positive rational numbers acting by multiplication upon them. That is, the group of rational numbers in the form of all the positive whole number ratios, often written in fractional form, for example: ..., 1/3, 1/2, 1/1, 2/1, 3/1, ...; and all other combinations of the positive whole numbers such as 2/3, 5/4, 27/589 or 1025/308.

Taking the identity operation first, which is the action that leaves elements unchanged, for the rational numbers, this is multiplication by the number one (1/1), the una modulation. Multiplying the six harmonics of the set by one leaves the set as it was, still six harmonics. And this multiplication would similarly leave any other set of harmonics unchanged also.

In contrast, if the set of six harmonics is multiplied by one-half, this does change the set – only three harmonics remain {h2, h4, h6}. However, as the operation is seeking to describe the modulation algorithm of *symmetrical* exchange, when the upper nested series is multiplied by a rational number, the lower series must also be multiplied symmetrically – by its *reciprocal or inverse*. Thus, if the original set of six harmonics is taken to be the nested system of series (H1 nesting h1 through h6), when h1 through h6 are multiplied by 1/2, H1 the proto-fundamental series must be multiplied, symmetrically, by 2/1. Yielding the exchange:

{ {H1}, h2, h3, h4, h5, h6 } --> { {H1, H2}, h4, h6 }

MBN 6_1 --> MBN $3_2\,0_1$

In regard to this operation the rational numbers 1/2 and 2/1 might be considered as applying a mirror-reflected symmetry. Alternatively, the original set of six harmonics could have been multiplied by one-third and the proto-fundamental series by three, to yield the exchange:

{ {H1}, h2, h3, h4, h5, h6 } --> { {H1, H2, H3}, h6 }
MBN 6_1 --> MBN $2_3 0_1$

Or, as illustrated in Figure 13.15, once the number system has arrived at MBN $3_2 0_1$, a further multiplication by 2/3 and 3/2 can be used to step from this arrangement of harmonics to the ground-state configuration, MBN $2_3 0_1$. These are the three symmetries of the mutable number six: MBN 6_1 MBN $3_2 0_1$ and MBN $2_3 0_1$. There are no other viable configurations of harmonics that can be reached by the rational numbers acting upon this set – excepting that is the trivial multiples such as 2/4 and 4/2, and the aurally indistinguishable inverse of MBN 6_1 which is written as MBN $1_6 0_1$. And continuing from the ground state arrangement of {{H1, H2, H3}, h6} on the right of Figure 13.15, a further multiplication by 3/1 of the upper nested series and of 1/3 for lower nesting series, would return the system to the left-hand starting configuration {{H1}, h2, h3, h4, h5, h6}.

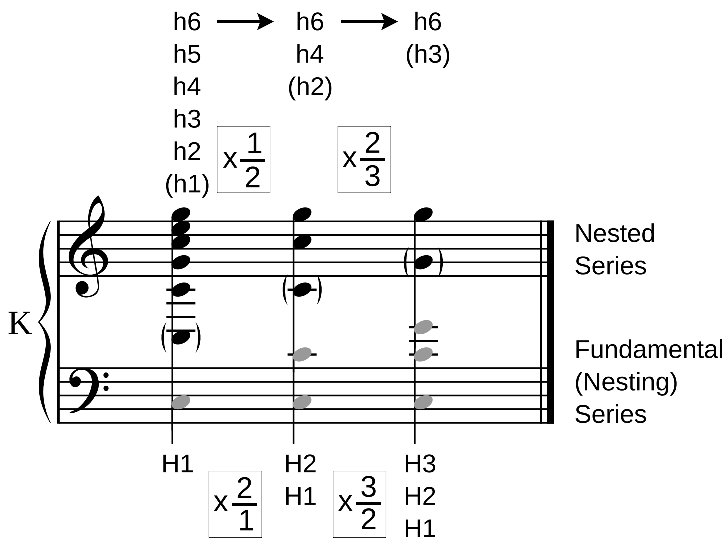


Figure 13.15 Two examples of the action of the group of rational numbers upon a set of six harmonics. From left to right: MBN 6_1 is transformed into MBN $3_2 0_1$ and then into MBN $2_3 0_1$ – these are the three aurally distinguishable symmetries of the mutable number six. (All other non-trivial rational numbers will produce invalid results.)

The first example of a group cited above was that of the integers, the positive and negative whole numbers, in which zero plays the role of identity element and the operation of transformation is addition/ subtraction. The group of integers, just like the group of rationals, can also be interpreted as *acting* upon the set of the harmonic series {h1, ..., hn}. Here again only a subgroup of the integers induces valid transformations in any particular set of the harmonic series (i.e. mutable number digit sequence), and in contrast to the rationals, only the identity operation of the addition of zero produces a symmetry – the symmetry that leaves the set unchanged. All other valid operations of addition/subtraction leave the set at variance with itself. However, these valid non-identity operations do provide a mechanism by which chords, rendered as extended harmonic series, may seek out new conjunctions to match up with the next succeeding chord in a harmonic progression. These valid *actions* of addition (or subtraction) operate upon the sum or ‘outer surface’ of the target set of harmonics, and so must step in the same aggregate multiples as the upper level of the nested system, thus constraining the choice of possible additions or subtractions.

For example:

MBN $6_1 + 1_1 = 7_1$ { {H1}, h2, h3, h4, h5, h6 } --> { {H1}, h2, h3, h4, h5, h6, h7 }
(Decimal: 6 + 1 = 7)

MBN $5_3\,0_1 - 1_3\,0_1 = 4_3\,0_1$ { {H1, H2, H3}, h6, h9, h12, h15 } --> { {H1, H2, H3}, h6, h9, h12 }
(Decimal: 15 − 3 = 12)

MBN $2_2\,0_1 + 6_2\,0_1 = 8_2\,0_1$ { {H1, H2}, h4} --> { {H1, H2}, h4, h6, h8, h10, h12, h14, h16 }
(Decimal: 4 + 12 = 16)

Invalid Addition: MBN $6_1 - 7_1$ (6 − 7) Number ceases to exist at zero
Invalid Addition: MBN $5_3\,0_1 + 2_2\,0_1$ (15 + 4) Addition only in multiples of three, the aggregate

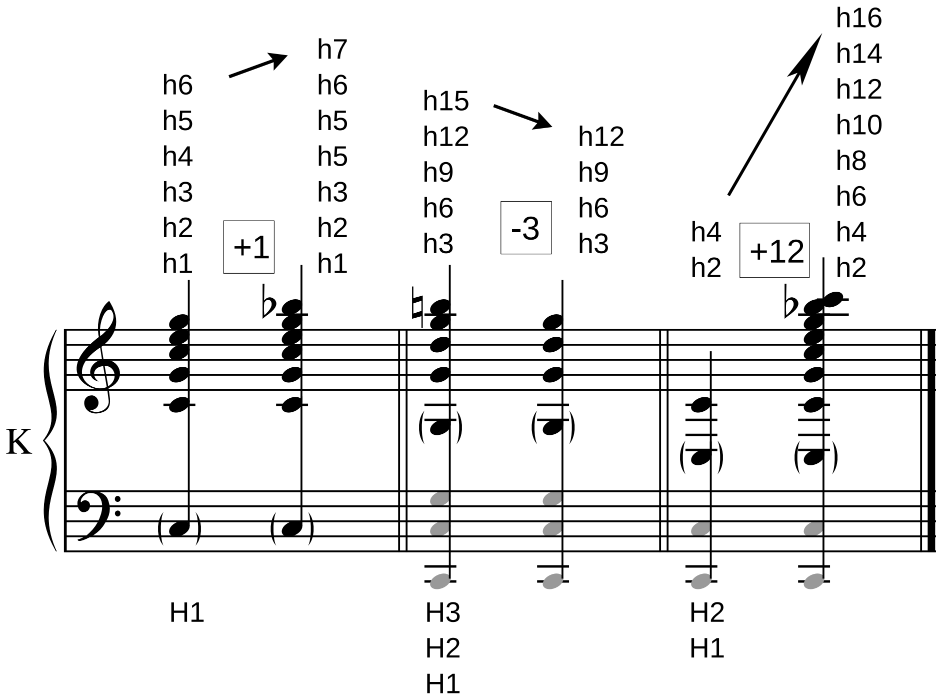


Figure 13.16 Three examples of the action of the group of integers upon sets of harmonic series. From left to right: one is added to MBN 6_1 transforming it into MBN 7_1 then three is subtracted from MBN $5_3\,0_1$ (Decimal 15) and twelve added to MBN $2_2\,0_1$ (Decimal 4).

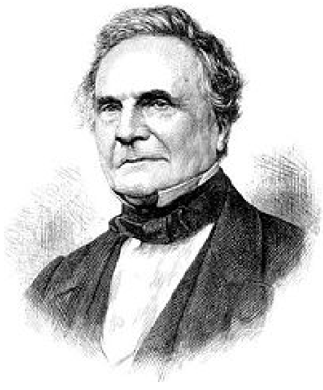
Physical numbers (and the music they represent) are much more awkward to work with than abstract numbers: physical systems impose severe limits upon what is, and is not, possible. As musicians are well aware, arbitrary chord sequences usually result in tonal chaos – only a limited subset of all chord sequences prove to be aurally intelligible and satisfying, that is, represent valid exchanges and manipulations of digit sequences in the mutable base position-value number system.

The above, though far from an exhaustive application of set and group theory to the principles and mechanisms of modulating oscillatory systems, is at least a first step along a path of integration. These branches of mathematics, and perhaps related fields such as category theory, appear to provide established areas of recognised practice capable of accommodating and underpinning the conception of tonal music as ‘physical’ mathematics and computation, as presented in *Journey to the Heart of Music*. And, perhaps in the future, these domains might provide the route by which a more fully developed music theory built upon the MOS model and mutable numbers, could be securely grounded within the broader structures and rigorous procedures of traditional mathematics.

COMPUTATION WITH MUTABLE NUMBERS

Chapter 10 is titled ‘Chord Types: *Numbers in Music*’, and more generally throughout this document, the argument is advanced that tonal music, stripped down to an elemental structure of nested harmonic series, forms essentially a sequence of values, *mutable base numbers*, written in an acoustic physical notation. In support of this contention – that chords in tonal music are in essence positional numbers written in sound, (i.e. parts of harmonic series h1 through hn) – a demonstration ‘composition’ or ‘tonal procedure’ is presented below, which performs the overtly numerical task of finding the divisors of any given whole number. Judged on its merits, in purely musical terms, the piece is trivial, consisting as it does of repetitive arpeggios and scale passages. However, this procedure does result in a ‘composition’ of sorts which is both recognisably tonal music and recognisably practical mathematics. The example given below seeks out the divisors of seventy-two, though equally the procedure could be applied to any number that lies within the range of musical instruments, and in theory could be applied to any positive whole number.

The procedure is also given below in the parallel form of a computer program. This program, as in Chapter 3, is written in the almost readable prose of the BASIC programming language (additional versions in AWK and Perl can be found in the SCRPT.ZIP directory). All these versions of the procedure/ program require a digital electronic computer to function. That is, a physical device capable of handling positional binary numbers by means of representing the digits zero and one as the absence or presence of a defined level of electrical potential within the computer’s circuitry. Going back in time to the nineteenth century, the mathematician Charles Babbage designed similar devices: the mechanical difference and analytical engines. Though never finished in their own day, these were likewise physical devices, but ones that used cog wheels and cylinders to represent the digits of positional decimal numbers – with which we are all familiar. Theoretically, there is little to distinguish between the modern computer and Babbage’s engines, beside the technicalities of operation (and of course a huge speed differential). Interestingly, Ada Lovelace (the daughter of Lord Byron, who collaborated with Babbage on the project, and is the author of the fullest account of the analytical engine’s true potential) suggested that among other things the device might: “compose elaborate and scientific pieces of music”.



Picture courtesy Wikipedia

Charles Babbage (1791–1871) was born in London into a prosperous banking family with land holdings at Teignmouth in Devon. Babbage was educated at home with tutors and at a variety of schools and private academies, and in 1810 went up to Trinity College, Cambridge. At Cambridge he meet and became friends with many of the coming generation of scholars and scientists such as John Herschel, the son of the great astronomer. With others of this circle he founded the Analytical Society in 1812 and in the same year transferred to Peterhouse College were he felt the teaching of mathematics was superior. Though a good scholar himself, he was to leave Cambridge with only an honorary degree conferred without examination. His somewhat awkward academic career was to be a story oft repeated throughout his life. Babbage was a man of proud character with a talent for alienating friends and colleagues; he held on to, not to say nourished, differences and arguments with a rare passion. Ultimately, this trait would lead to him failing to realise the great potential of his principal contribution: the development of serious mechanical computation. The stimulus for this invention came from the difficulties encountered in producing reliably accurate astronomical, navigational and mathematical tables by human hand. Babbage saw that the basic but mind-numbingly repetitive arithmetic could be better done by machine, and he set about designing a device capable of performing such a task: the difference engine. The history of mechanical calculation is long, going back to ancient methods of reckoning involving fingers, pebbles, etc., devices such as the abacus, and many later inventions, through to work by Leibniz on automated arithmetic – which Babbage had read at Cambridge. Building on this knowledge, Babbage came up with the design of his difference engine, for which he was able to secure government funding. All looked well for the project at first and much was made of his plans at the intellectual gatherings he regularly hosted at his London house. One visitor was Augusta Ada Byron, daughter of the poet Lord Byron, a keen amateur mathematician. Indeed, Ada's mother, also a mathematician (a "Princess of Parallelograms" in Byron's cutting put-down) encouraged her study in the hope that it might counteract any inherited traits of her father's character! Ada was fascinated by Babbage's invention and requested the plans for closer study. Though their relationship was never more than platonic, Babbage liked Ada, helped her to further her mathematical studies and respected her for perceiving the true nature of his invention. Perhaps crucially, being a woman she didn't provoke or challenge his pride. Remarkably Ada probably grasped the wider implications of Babbage's engine more fully than he did himself, and she is now considered to be the first ever computer programmer. Later, Ada, Countess of Lovelace was to render Babbage a great service by translating the account of his more advanced analytical engine, which he gave to Italian mathematicians (having alienated most English colleagues), adding substantial material of her own. It is this document above all which preserved Babbage's work, and nearly a century later, communicated it to Alan Turing the father of the modern computer. Despite all Ada's enthusiastic interest and support, there were delays and eventually total failure in the construction of the difference engine. This was partly due to domestic tragedies in Babbage's life, ill health, a lack of focus on one single design, and as always, personal animosities. However, Babbage did make contributions over a wide area of engineering, science and mathematics; he held the post of Lucasian Professor of Mathematics at Cambridge from 1828 to 1839. Charles Babbage died in London on the 18th October, 1871.

Similarly, in seeking to use positional mutable numbers as the basis of operation for computation, an appropriate physical device is required, that is, a physical device specifically designed to match the particular characteristics of its operational number system. For mutable numbers (i.e. chords in tonal music) an appropriate physical device is a musical instrument, though one could imagine far more powerful oscillatory processors, with frequency ranges and sensitivities far in excess of that required for the pursuit of music. However, the instruments we have and use to make music are entirely adequate for the demonstration of tonal computation in sound.

First the mathematics. An underlying mechanism for finding the divisors of a given number is described in Chapter 7, in the section on *bow waves* in the Table of Harmonic Series (or Sieve of Eratosthenes). There the formula $N = n^2 - s^2$ was derived and described, for any odd number N. A slightly more complicated formula was found for where N is an even number. However, a further development of the simpler odd-number relationship, allows the inclusion of even numbers within a single algorithm. The best way of seeing how this algorithm works is visually, by picturing the numbers as areas – as squares and rectangles. Figures 13.17 and 13.18 illustrate this extension of the $N = n^2 - s^2$ relationship from odd to even numbers with two examples. The odd number $N = 133$ is examined first and then the even number $N = 82$.

Because the same 'remainder' relationship holds true for both odd and even numbers, with the two gray leftover rectangles and one square (Figures 13.17–18) necessarily taking integer values when whole numbers divide N (despite n and s themselves not always being whole numbers), this characteristic allows a simple algorithm to be devised: whenever a perfect whole numbered square (i.e. gray square below), equal to or less than the given number 'N', is subtracted from area N, leaving over two rectangular areas;

then, two whole numbered divisors of N will be: 1) the root of that perfect (gray) square, and 2) the sum of that root, plus the area of the two rectangles divided by that root (i.e. the sum of the non-root sides of the rectangles).

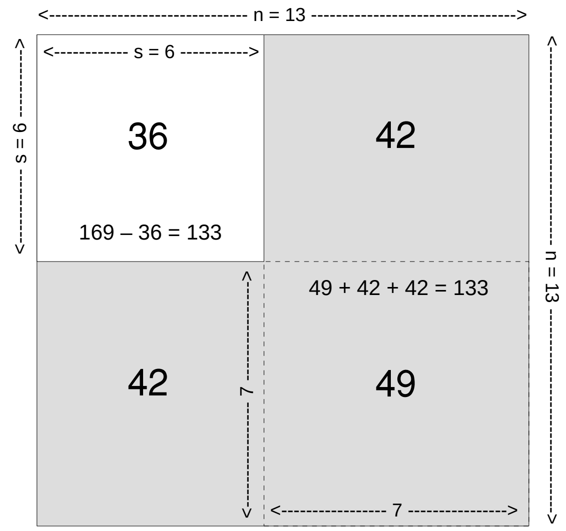


Figure 13.17 Number = 133. By subtracting s^2 from n^2 the number N is deduced (i.e. 13 x 13 minus 6 x 6 equals 133). Looking at the squares of n and s, superimposed, reveals that N 133 (the gray area) is composed of another square, plus two identical rectangles: (7 x 7) + 42 + 42 = 133 which may be combined into one rectangle 7 × 19. Therefore divisors are 7 and 19 (i.e. 7 + (42 + 42)/7).

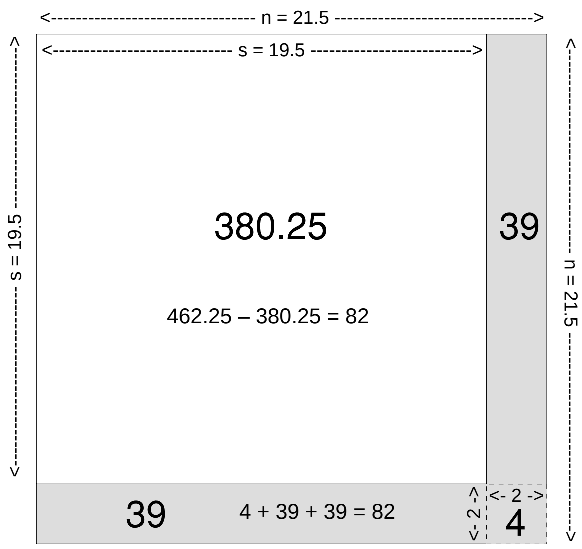


Figure 13.18 Number = 82. The same relationship as illustrated in Figure 13.17, holds true for even numbers too, in spite of n and s being fractional: 21.5 x 21.5 minus 19.5 x 19.5 equals 82. The imposition of s^2 upon n^2 again reveals another square and two identical rectangles: 2 x 2 + 39 + 39 = 82 which translates into the rectangle shown in Figure 13.19. Therefore divisors are 2 and 41 (i.e. 2 + (39 + 39)/2).

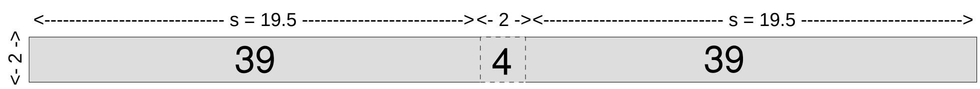


Figure 13.19 Rotating one leftover rectangle perhaps makes things clearer: area N = 82 = 2 x (19.5 + 2 + 19.5)

Thus when presented with any number ‘N’ for which one wishes to find the divisors, first calculate the largest perfect square equal to or less than N and then proceed in whole numbered steps downward from this square, testing each descending square in turn against the algorithm. Whenever the procedure produces an integer result for the ‘leftover’ rectangles, two divisors of N have been found. Essentially, the algorithm anchors the largest (gray) perfect square that will fit within ‘area N’, in N’s bottom right corner, and sequentially compresses this (gray) square, in whole number steps, to one.

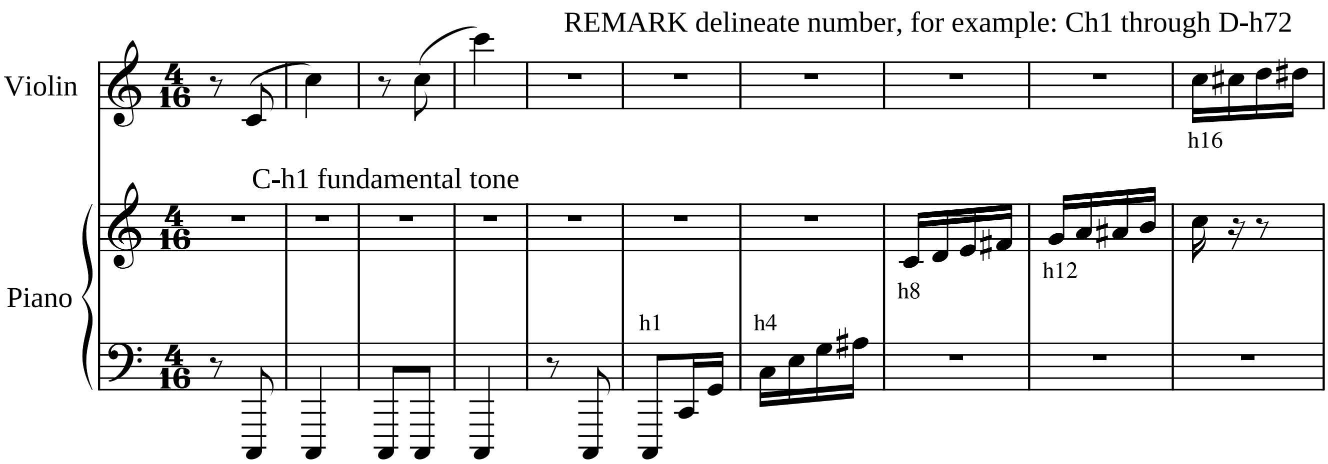
In the BASIC programming language (BBC Basic V) this procedure could be written out as three steps: First, acquire the number to be divided; second, find the largest perfect square that is less than or equal to it; and third, check each perfect square from this largest down to the unit square, in integer steps, for leftover rectangles with whole number areas. Whenever the result meets this criterion, print out the whole numbered divisors found.

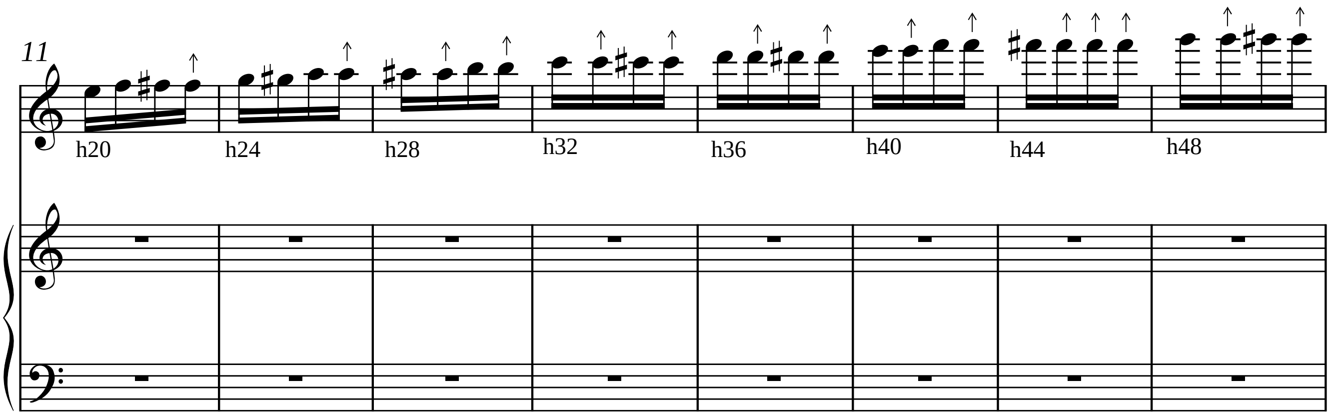
```
REMARK delineate number, for example 72.
INPUT "Please specify whole number to be divided", note_number
REMARK Loop 1. find largest perfect square equal to or less than note_number.
sqrt = 0
REPEAT
sqrt = sqrt + 1
square = sqrt * sqrt
UNTIL square >= note_number
IF square > note_number THEN sqrt = sqrt - 1
REMARK Loop 2. work down from value of sqrt to 1 in whole steps.
WHILE sqrt >= 1
  square = sqrt * sqrt
  difference = note_number - square
  result = difference / sqrt
  REM test if result is a whole number.
  IF result = INT(result) THEN
    divisor_1 = sqrt
    divisor_2 = sqrt + result
    PRINT "Divisors: "; divisor_1; " x "; divisor_2
  ENDIF
  sqrt = sqrt - 1
ENDWHILE
END
```

Applying the selfsame procedure as given in the above BASIC program, but using mutable base numbers operating upon the ‘physical devices’ that we call musical instruments (and writing out the progress through each loop exhaustively), produces the score given in Example R for the input number seventy-two. Below the first three pages are repreoduced for quick reference. The score requires microtonal notes to be played in the upper part (violin), indicated by small arrows above the notes where one staff note covers a range of two or four harmonics. For example, the written top C may stand for four frequency inflections: h64, h65, h66 and h67.

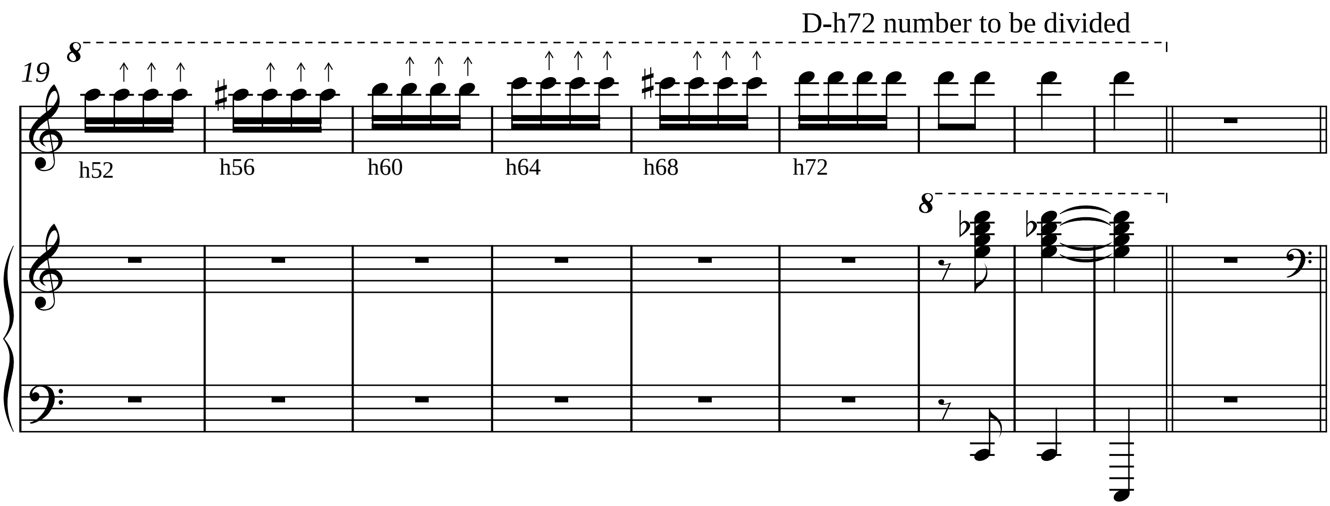
Study: The Divisors of Seventy-Two

Example of Tonal Computation using Mutable Base Numbers





[Where more than one harmonic of the fundamental tone C-h1 is represented by a single note,
e.g. F♯h22 and F♯h23 above, arrows (↑↓) are used to distinguish between them.
The arrow symbol indicates roughly an eighth-tone, quarter-tone or three eighth-tones as appropriate.]



Loop 1. Find largest square number equal to or less than D-h72


66
Nine squared
(Loop 1.9)
8
64 < 72
9x9 = E-h81
79
8
Therefore eight squared is largest
square number equal to or
less than D-h72
Nine squared, E-h81 is greater than D-h72
8
81 > 72
8
81 > 72
88
8
64 < 72
8
64 < 72

<u>Notes</u>

1. Strictly the cycle of fifths (the twelve tonal-center relationships) only approximately 'returns to itself' – as marked by Pythagoras' comma – and no matter how many fifths are taken (e.g. 12, 41, 53, 306, etc.) no power of two (octaves) will ever exactly equal a power of three (fifths/twelfths).

2. As human ears cannot distinguish between the structure of MBN 6_1 and MBN $1_6\ 0_1$ they are here treated as being synonymous (1 × 6)**.**

3. In rigorous group theoretical terms, prime state mutable numbers (i.e. digit sequences such as 1 x 23 and 23 x 1) probably should be counted as two different states: one group of twenty-three (MBN 23_1) and twenty-three groups of one (MBN $1_{23}\ 0_1$). Though hearing can not distinguish between the two digit sequences, mathematically there is a difference, as demonstrated by stepwise addition MBN 23_1 grows to MBN 24_1 (twenty-four) while MBN $1_{23}\ 0_1$ leaps forward to MBN $2_{23}\ 0_1$ (forty-six).

4. A perl script 'digseq' which calculates the total number of digit sequences (i.e. internal arrangements) for a given value, and the number that are distinguishable, can be found in the folder SCRPT/ZIP, in CHPT19.

5. By viewing nested harmonic series as automorphisms of their underlying fundamental nesting series it is possible to further construe the whole system of fundamental and nested harmonic series:

{{H1 through Hn}, h2 through hm}

as an automorphism group of cardinality 'n × m' and thereby devolve to it the attributes of group structure.

6. Also the terms stationary subgroup, isotropy group, or little group are used.

7. More precisely, the set of harmonics being acted upon – {h1, h2, h3, ..., hn} – is typically the upper aggregated series in a nested arrangement of harmonic series representing a particular chord in a particular key, and in traditional mathematical group theory such a nested series might be construed as an *automorphism* of the fundamental nesting series within which it is encompassed. Equally, the conjunction frequency linking two nested series engaged in a modulation exchange could also be considered a member of an automorphism of the fundamental nesting series (i.e. another nested series) delineating the complete collection of frequencies (theoretically of unlimited extent) connecting the preceding nested series to its successor. Often the conjunction frequency in an exchange will be the 'h1' fundamental frequency of this 'conjunction automorphism' though not exclusively, as seen in the common V^7–I secondary sesquitertia 3:4 exchange where the lowest conjoined frequency lies an octave below the explicitly identified conjunction. Indeed, all non-primary modulation exchanges exhibit this feature. Thus a rigorous mathematical description of the MOS model would probably identify all the multiple links in the conjunction automorphism/series – rather than the single frequency lying next above the highest written note which in these documents is customarily assign a unique value.