12

Prelude by J.S. Bach

MOS/MBN EXAMPLES

In the following pages the principles and techniques outlined in the previous chapters are applied to Prelude No.1 in C major from the *Well-tempered Clavier* by J.S. Bach, after which a passage from Mozart's *Piano Sonata K545* is also examined. This chapter is best read in conjunction with the full analysis presented Example S.

MOS – Modulating Oscillatory System
MBN – Mutable Base Position-value Number System

R.F. Goldman[1] describes the first prelude from the *Well-tempered Clavier* as "a model of what C major, or key in general, means". Effectively the piece provides the classic description of the extent of a key or tonal area. In Figure 12.1 and subsequent examples this *key area* – C major – is defined by the fundamental nesting series built on a frequency of 1.0 hertz, with its ratios marked in upper case –i.e. H1 to Hn. Notionally all pitches are drawn from harmonics of this series. However, in practice the computations/modulations of nested and aggregated series introduce a degree of *flexing* into the tonal fabric. This arises because the model is seeking to accommodate the dynamic motion of the objective notes, in its upper level, while also holding to a fixed fundamental series below, representing the key center. For example, the sesquitertia 3:4 modulation exchange from C (16.0 Hz) to F, yields F- 21.33... Hz rather than the whole ratio H21 (21.0 Hz). The tolerance of the ear and aural processing can be counted on to smooth over such small deviations, and this flexing is registered as small changes in the period of H1, when applying the formalism of mutable numbers to the prelude (pages 9 and Example S 19–23). But in the dynamical MOS analyses (pages 10–13 and Example S 2–18) this flexing of relationships is expressed as decimal fractions of frequency, measured in hertz, in the upper layers. Effectively in the analyses on pages 10–13 the upper level aggregated series (represented in black type), which mirrors the objective notes, including their partials and summation tones, is allowed free reign to 'drive' the system wherever it will. Over the course of the whole composition this might very well mean that the pitch drifts some distance from its starting level in terms of *external* measurement. However, from the *internal* point of view of the MOS analysis this is of no consequence, since the composition is being considered solely as the sum of its internal relationships: that is as a separate relational 'universe'. In the analysis of Prelude No.1 given in Example S, this tendency of pitch to drift is dealt with by occasional 'recalibration' back to the standard of middle C = 256 Hz. (In contrast, for the other analyses provided in the EXAMPLES folder, the pitch calculations are allowed to run their course from start to finish.)

In essence, one might imagine the representation of a tonal composition provided by the MOS/MBN model, as being like a lake of ratios, through which the music navigates. The lake bed lies at 1.0 hertz, or even some fraction of this, at a depth far beyond the effective range of musical sound, in fact, well within the domain of rhythm and duration. It is only in the upper level of the lake, closer to the surface, that the objective notes of music stir the ratios into aurally discernible patterns – these are the aggregated series described in black italic type: *(8 groups of 4) --3:4--> (6 groups of 4)*, that is, a secondary sesquitertia transformation consisting of eight units exchanged for six units. (Here the units are bundles of four ratios taken from the nested series). It is from the fundamentals of these top level aggregated series that the 'rootedness' of chords emerges. However, not all the objective notes can be accounted for by these short foreground aggregated series; this leaves scope for a middle level nested series to take on a shadowy background function. The nested 'background' series are shown in examples in gray type and are described in terms: (3 × five) --4:5--> (3 × four). That is, a tertiary sesquiquarta modulation exchange of fifteen ratios for twelve. These middle-level nested series are picking up the odd objective notes not covered by the aggregated series, in particular the minor third in minor chords ('h12'), thus allowing intelligible patterns to emerge, by placing most of the information (objective notes/ratios) in the simpler, topmost, aggregated series, while still accounting for the awkward ratios in the broader nested series. The foreground (aggregated) series nests within the broader background; sometimes the two series move in parallel steps and at other times their motion is disjunct. Occasionally chords, such as the diminished-seventh, lack a pattern of objective frequencies that can support a relatively simple top level aggregated series. When such an awkward pattern of notes occurs, the background nested series scoops up all the ratios in its far wider net (Example S, pages S.8, S.12, S.13, S.15). However, the complexity of this extended middle-level series is beyond the ear's ability to fathom, and the chord emerges as 'rootless'. In theory the chord will have a root – the fundamental tone of the extended nested series – but it lies beyond the range of human aural cognition to unravel. Finally, in principle at least, the foreground and background series are themselves nested within the absolute fundamental series, built on H1 (1.0 hertz) – the 'lake bed'. The notion of key could perhaps be associated with some sense of this most fundamental series. All levels of nesting, top-level aggregated series, middle-level nested series and bottom-level fundamental series share the same common conjunction, even though the proportions of their individual modulation exchanges may differ. Each level is nested within those below: the aggregated series within the nested series and the nested series (which includes the aggregated) within the fundamental series. Their mutually shared common conjunctions might equally be thought of as ripples and waves upon the lake's surface.

A brief glance at the first two measures of the Prelude reveals that an upward and downward extension of the objective notes occurs when they are embedded within top-level aggregated series (written in black type). Indeed, the C-major and D-minor chords have an octave worth of ratios added above their highest notes and three or more octaves below the lowest! What justification can be made for this arrangement? On the purely mathematical and formal level, the justification is simply that a structure of mutable number digit sequences can be made to fit any collection of notes simply by extending and nesting harmonic series to the point where all the notes in the score may be accommodated. Of course whether or not applying such a formalism to scores is judged to be useful is another matter. For the historic canon of western tonal music from the period of common practice there is, I believe, a good case for the techniques of MOS analysis, which generally produces relatively short upper level series intelligible to the ear. And indeed as mutable number analyses essentially operate at the elemental level of the relationship between two adjacent chords, the model might well be integrated into established longer range dynamic theories of tonal music. For other broadly tonal western music from before and after the period of common practice the case becomes progressively less convincing as the traditional harmonic content declines. By the end of the nineteenth century, in progressive music of the late romantic school – Wagner, Mahler and Richard Strauss – the dearth of clearly tonal chord progressions becomes quite problematical for the model to handle. Little by little tonality was being squeezed out of avant-garde art music. And for strictly atonal music there would appear to be little point in pursuing an approach which can only misunderstand the music.

Beyond the formalism of mutable numbers lies the far more intractable question of aural cognition, which at present remains an unconquered mountain shrouded in mist. But a little of its topography can, perhaps, be gleaned from a close study of the tonal canon, in that the body of music in existence forms an external manifestation of the, as yet only partially understood, mechanisms of hearing and aural cognition. Composers, musicians and their audiences have together, over time, selected musical structures that are meaningful, pleasurable and give satisfaction when apprehended by the ear. It is reasonable to assume that secreted within this corpus lie hints and clues to the inner logic of music, and thereby also aural cognition. It is to be hoped that at some time in the future, neuroscience will be able to dispel the swirling clouds completely, allowing the light of knowledge to illuminate the summit, and then, perhaps, a fully objective understanding of music might lie within our grasp. However, until that day comes to pass we must make do with speculative theory, guided by what is already reasonably well established.

Looking at the upward extension of the top level aggregated series first, and basing the discussion on the model of the human hearing mechanism given by James Beament;[2] the most significant fact to bear in mind is that a note only very rarely consists of one single frequency. The note middle C (256Hz) when played on almost any instrument, actually consists of multiple frequencies, for example 256Hz, 512Hz, 768Hz, etc.; that is, a harmonic series h1, h2, h3, etc. And the face value note-frequency in a score, here middle C, 256Hz, may not even be the strongest harmonic in the series. (Most non-musical sounds, noises in other words, also consist of many frequencies but they generally don’t have the stable, ordered arrangements typical of musical sound. Indeed, the satisfactory conduct of tonal music is only possible with note-sounds that have a good approximation to ideal harmonic series – or where this is not so, the vast bulk of the energy in the sound is deposited in the fundamental, rendering inharmonic higher frequencies insignificant.) For musical sound and noises alike, the ear harvests all the frequencies within the range of hearing, ordered harmonics of timbre or disordered frequencies of noise, which it separates as far as the detector membrane allows and processes so as to find the direction, distance and character of the sound. From the evolutionary point of view this was, and is, vital. For example: is that roar a sabre-toothed tiger or a stag in rut? Our ancestors, by definition the ones that survived to reproduce, needed ears that could tell the difference between a roaring sound that meant lunch or that they might be lunch! Vast amounts of processing are carried out unconsciously by the brain. When was the last time you considered consciously word order in an animated conversation? Yet the flowing words come out in the right order and that is not to mention the awesome feats of coordination involved in controlling the muscles used in speech. Similarly, we are not consciously aware of the many stages of lower-level processing undertaken by the mechanisms of hearing. The fundamental and harmonics of timbre are acquired and separated by the ear, processed along the auditory pathway to extract information about direction, distance, character plus much else; and then relevant parts of this information, packaged into sensations, seamlessly, effortlessly, become available to the conscious mind. One small part of that which is conveyed to the mind, when listening to tonal music, is the commensurability of chord progressions. Without conscious effort (though with considerable low-level unconscious processing) the one ‘wrong’ note or infelicitous chordal succession is instantly identified for us by the hearing mechanism. The ‘laws of tonality’, if they exist, most probably reside in the mechanics of the human hearing system.

Virtually all natural sounds are noises, that is disordered unstable collections of frequencies unrelated by any simple pattern. Long before language or music, the human (mammalian) ear evolved to analyse this form of sound which lacks, for the most part, steady fixed frequencies. Thus the character of the sound, the type of noise, was principally the impression created by an amalgam of a *range of varying frequencies* of *varying relative strengths* – a spectrum of frequencies with a complex but generally recognisable signature. The somewhat subtle difference between the ‘tone’ of a tiger and a stag, within the broad gamut of natural sound, could be distinguished by reading the pattern, and pattern of change, of a band of frequencies. Over time the brain builds up a library of sound spectra or signatures which may be compared with the live ‘sound feed’ from our ears and so sounds are usually instantly identifiable. Perhaps because it was also imperative that simultaneous sources of noise could be separately distinguished with regard to direction, distance and character the mechanism of aural processing developed the ability to extract and package individual sources of sound into single sense impressions. These basic characteristics evolved long before human language or music, setting the parameters within which these later cultural developments could take place. Both language and tonal music have developed in a way which relies on the (already present) perception of the higher partials of sounds, as well as the fundamental frequencies. For example, a ‘tinny’ mobile phone connection, with a narrow bandwidth, makes it difficult to recognise the voice of the caller, while over a modern landline with a four thousand hertz bandwidth we can recognise the speaker’s voice within an instant, due to the extra information made available to the ear from the spectrum of higher frequencies. When music arrived, and in particular when musical sound with approximately ideally placed harmonics arrived (e.g. h1, h2, h3, etc.), the ear processed the sound as it would any other sound, but because of the ideal and fixed harmonic structure, the amalgam of frequencies packaged together to create the characteristic signature or ‘tone’ of the sound source had a peculiar attribute – a clear sense of *pitch*. Compared to most noises, a musical sound, a note, has a very simple and constant structure. All the higher partials align with and thus bolster the perception of the fundamental frequency. (The ear hardly notices phase differences.) Often, but not always, the fundamental will be the strongest frequency, and on the ear’s detector membrane the fundamental frequency will be less crowded than the higher harmonics, which fall ever closer together. So the fundamental frequency of the spectrum of frequencies found within a note is normally favored in terms of perceptual focus as it unites the higher partials within itself while allowing their relative strengths to color its perception. Thus pitch and tone are entwined. All sounds produce a signature, a signature created by amalgamating all the frequencies from a single source into a characterful sensation. When the amalgamated frequencies happen to align in whole number multiples then two clear sense impressions can emerge out of the sound signature, the sensations of clearly defined pitch and tone, a musical note. Indeed, in laboratory tests, this sense of pitch can be generated entirely from harmonics of timbre (i.e. the harmonics from h2 upward) without the presence of a fundamental pitch at all. And the perceived strength of the lower register notes of many instruments is crucially dependent upon this intertwining of fundamental pitch with the spectrum of higher integral harmonics. In contrast, notes played on poorly made instruments producing non-ideal harmonic spectra (i.e. h1, h2.1, h3.07, h3.9, etc.) are difficult for the mechanisms of hearing to resolve into a clean pitch and clear tone impression. Noises generally diverge even farther from any perception of clearly defined pitch and tone: they may vaguely present some sense of pitch and tone, ranging for example, from the low rumble of thunder, through the random ‘white’ noise of tyres on a wet road, to the high jangle of cutlery.

So now it becomes possible to understand the justification for extending upward the aggregated series beyond the objective notes into the range of the harmonics of timbre: *the notes themselves are not single fundamental frequencies but extended harmonic series in their own right*, harmonic series with which the ear is vitally concerned. For it is from this filigree net of partials that much information, the subtle nuances and fine detail, will be extracted by the hearing mechanism. And though only the lowest conjunction between adjacent aggregated series is generally represented in the analysis below (gray band), the ear and processes of aural cognition would cast their net far more broadly, capturing multiples of these lowest conjunctions, occurring at higher levels within the range of hearing (illustrated on pages 12–13 and Example S, pages S.11–12). However there is also the limiting factor of crowding on the ear’s detector to be taken into consideration when applying mutable numbers to the processes of aural cognition. More simultaneous notes means more harmonic resonances jostling together on the ear’s basilar membrane: gradually the interference between ever more tightly packed harmonics inevitably introduces a destructive irregularity in the nerve impulses generated. Though balancing this loss of information from the top end, more extensive chords will correspondingly manifest a larger degree of information in their note (i.e. written) pitches.

The human ear reaches its most sensitive and discriminating level in the frequency range around the top octave of the piano, approximately 2000 to 4000 hertz. Interestingly, this is precisely the hunting ground of the higher resolvable partials (around h4 to h8) generated by notes in the octave above middle C: the cockpit of harmony. Scores are misleading in this regard as they present on the staff only the ‘h1’ fundamental of each note, when in performance the ear is actually deluged by a swirling sea of un-notated overtones, each note its own harmonic series. Though unseen by the eye reading the score, the ear unobtrusively sweeps up and sifts all the components of musical sound within its frequency range, and in so doing, finds the material out of which mutable numbers are made, and so also the means by which mutable number digit sequences (chords) may be logically and satisfyingly conjoined into commensurable harmonic progressions.

To digress a little, one question raised by Beament is that of the relatively low frequency base of traditional musical instruments. Why, if the ear’s most sensitive range lies between two and four thousand hertz, do we make music with instruments founded on the frequency range around middle C (256 hertz) and lower? In addition, Beament points out that almost all instrumental development has focused on lowering the range of instruments and not raising them! His conclusion is that this is down to physics: historically our technology has not favored the manufacture of very small instruments. While this is no doubt true, there is more to this question, which perhaps finds an answer in the above discussion of the broad band of frequencies represented by a single written note. The successful conduct of tonal music involves more than just the written notes: it requires the headroom of the ear’s core range of hearing up to 8 kilohertz (8000Hz), so as to accommodate the higher partials as well as note-fundamentals. (In addition to which instruments and the voice also produce a rather more static set of accompanying resonances over a wide frequency band described as the *formant*.) Musical instruments and voices at the dawn of the tonal era were mostly of a rather high register; however, as the system of tonal computation gradually developed, the existing instrumental and (to lesser degree) vocal ranges, which already populated the most sensitive 2–4kHz range and above with partials, were augmented by lower register equivalents. Any upward movement in the pitch base of instruments would move the generally stronger lower partials into the ear's most sensitive range, which would produce a shriller effect as well as moving the higher resolvable partials into the less acutely sensed region beyond 4kHz. In contrast, a downward movement in the pitch of instruments, within reasonable limits, does not suffer in the same way as regards shrillness, and the squeezed scaling of larger instruments (longer but relatively narrower proportions) tend to produce stronger harmonics an octave higher than their more diminutive cousins. There are, of course, physical limits to this downward shift, and restrictions emerge upon the placing of closely spaced intervals between bass parts, due to the increased overlapping of resonances on the ear's detector (basilar) membrane. This bunching-up occurs because of the non-linear mapping of resonances upon the membrane at lower frequencies.

C*ombination tones* also possibly have a role to play in the deciphering of harmonic progressions and the processing of aural information in general. These aural sensations or 'ghost' notes generated within the mechanisms of the ear and auditory pathway, bear frequency relationships of *subtraction* or *addition* to the actual objective notes of the music. Combination tones do not exist in the objective sound but are 'fictitious' creations of the hearing mechanism. Interestingly, James Jeans[3] points out that a major triad – for example, CEG – will generate *difference and summation tones* which cover the range h1, h2, h3, (C)h4, (E)h5, (G)h6, h7, h8, h9, h10, h11, h12, h13, h14, h15, h16, h17 and h18.

The lower frequency and more prominent difference tones, first understood by Giuseppe Tartini and described in *Trattato dei principii dell' armonia musicale,* 1754, may provide a link from the objective notes down to a more fundamental frequency, that is, a link between the foreground aggregated series and the background nested series. While a major harmony will have a lowest difference tone which coincides with its root (e.g. C-major chord: E-h5 minus C-h4 yields C-h1), in the case of minor chords this lowest difference tone is notionally a major-third below the root (e.g. D-minor chord: F-h12 minus D-h10 yields A#h2). Figure 12.1 below illustrates the superposition of the D-minor harmony – the aggregated series – upon the underlying A#(major) series. In the analysis of the Prelude, difference tones are indicated with diamond symbols. Tartini used difference tones for many years as a method of fine tuning double-stopped notes on the violin but was beaten by others into print, though correctly, history honours these tones with his name.

Remaining at the pitch level of difference tones for a moment, there is another possible source of low frequency periods in the hearing mechanism, and though it is a highly speculative proposal, it is worth remarking on. This source of low frequency periods is the method of signalling employed by individual nerves in the auditory pathway between the ear and the conscious mind. Though en masse an adjacent group of nerves attached to the ear's detector membrane might in combination signal every cycle of a particular frequency, under normal circumstances any single nerve will signal only occasionally and at rather random intervals of whole numbers of cycles. A single nerve's signalling will keep in step with the particular frequency but will rarely signal every successive cycle. Thereby if the nerve pulses are assessed individually rather than all together, the result will be an array of periods *f, f/2, f/3, f/4, f/5 ...*; for any given frequency *f*. In other words an integral wavelength series of 'undertones', or 'arithmonics' as I have termed them. The situation is somewhat complicated by the fact that the higher the single stimulus frequency the greater the array of periods individually signalled is likely to be. For example, the array of individual pulses produced by the single frequency of top C (1048 hertz) is unlikely to contain any signals equal to *f*, and is more likely to resemble f*/2, f/3, f/4, f/5, f/6, ...* . However, as we are concerned with periods generated in individual nerves by much lower objective frequencies, that is, the bottom notes in chords, this feature does not significantly come into play. Indeed, for these lower objective frequencies the nerve pulses will tend somewhat to bunch up at the bottom end of the array, i.e. *f, f/2, f/3*. Notwithstanding this bunching, the outcome remains that when assessed individually, the nerve impulses generated by the lowest notes in chords may give rise to periods an octave, a twelfth, two octaves, and more below the sounding note.

Briefly moving beyond the hearing mechanism, it is interesting to note that there are to be found in the brain a number of wide-scale neural oscillations operating over the frequency range from 1Hz to 70Hz – which are believed to arise from the synchronization of broadly spread groups of neurons. In addition to these relatively prominent electrical signals, smaller ensembles of neurons are known to oscillate at frequencies ranging up to approximately 500Hz, and that these faster local oscillations may be entrained and synchronised by the more widely ranging lower frequency pulses.

Whether this possible source of ‘undertone’ periods plays any role in aural processing I do not know. And more broadly, the applicability of the lower parts of a MOS analysis to the processes of aural cognition is little more than the speculative projection of a formal mathematical structure onto what are deep and mostly uncharted physiological, neurological and psychological waters. The MOS model is ultimately mathematics, mathematics which might or might not find some reflection in mechanisms and processes of the human ear and mind.

Mirroring the low frequency difference tones are the fainter summation tones, which are produced in great abundance by most chords and hover like ethereal clouds above the objective notes. The summation tones were discovered by Helmholtz in the nineteenth century, though to this day they remain controversial and are not accepted by all authorities in the field of acoustics, the core difficulty being that the subjective nature of combination tones means that they are not amenable to objective observation and verification. However, acting upon the assumption that they could be produced by the mechanism of hearing, out of these clusters of tones there will usually be one, two or a few which coincide with the *conjunctions* between adjacent chords/series, and as often as not, parts of, and whole, harmonic phrases. The effect of the summation tones and the much more generally acceptable ‘harmonics of timbre’ is to provide, and/or highlight, linkages between chord progressions – threads of continuity for the ear to grasp, as the harmony steps through sequences of commensurable chords (i.e. computations governed by the modulation algorithm). Summation tones and/or harmonic partials, which often coincide with conjunctions between adjacent chords, are marked by squares in the pages below.

As noted in Chapter 9, aggregated series are simply another layer of nested series – it is just convenient to give them a separate name to avoid confusion. In Figure 12.1 below, the music – the objective notes, their harmonic partials and combination tones – are contained within the two upper nested layers, while the *sense of key* generated by hearing (aurally processing) the music, is represented by the fundamental nesting series.

In principle, all pitches in a composition are ultimately harmonics of this one underlying series, the lowest level fundamental series – H1 to Hn – although the whole system is ultimately portrayed as one gigantic series in the MOS model, with for example the first measure fundamental series in the analysis below stretching up to an astronomic partial (H1280). From the ear’s point of view, this conjunction (H1280) is no more than the objective second harmonic (h2) of the top melody note, a frequency easily within reach of the ear and one which quite possibly has more energy than the written note itself. However, the precise limits of aural processing are unknown, though they cannot be unlimited; at some point the mechanisms of the MOS model almost certainly must step across a border between a palpable description of music cognition into the realm of mathematical modeling. Exactly where this border lies, both to the 'north' and 'south' of the written score is debatable. To use a computational analogy, the fundamental series could be likened to the hardware, upon which the music – the software layers of the nested and aggregated series – runs.

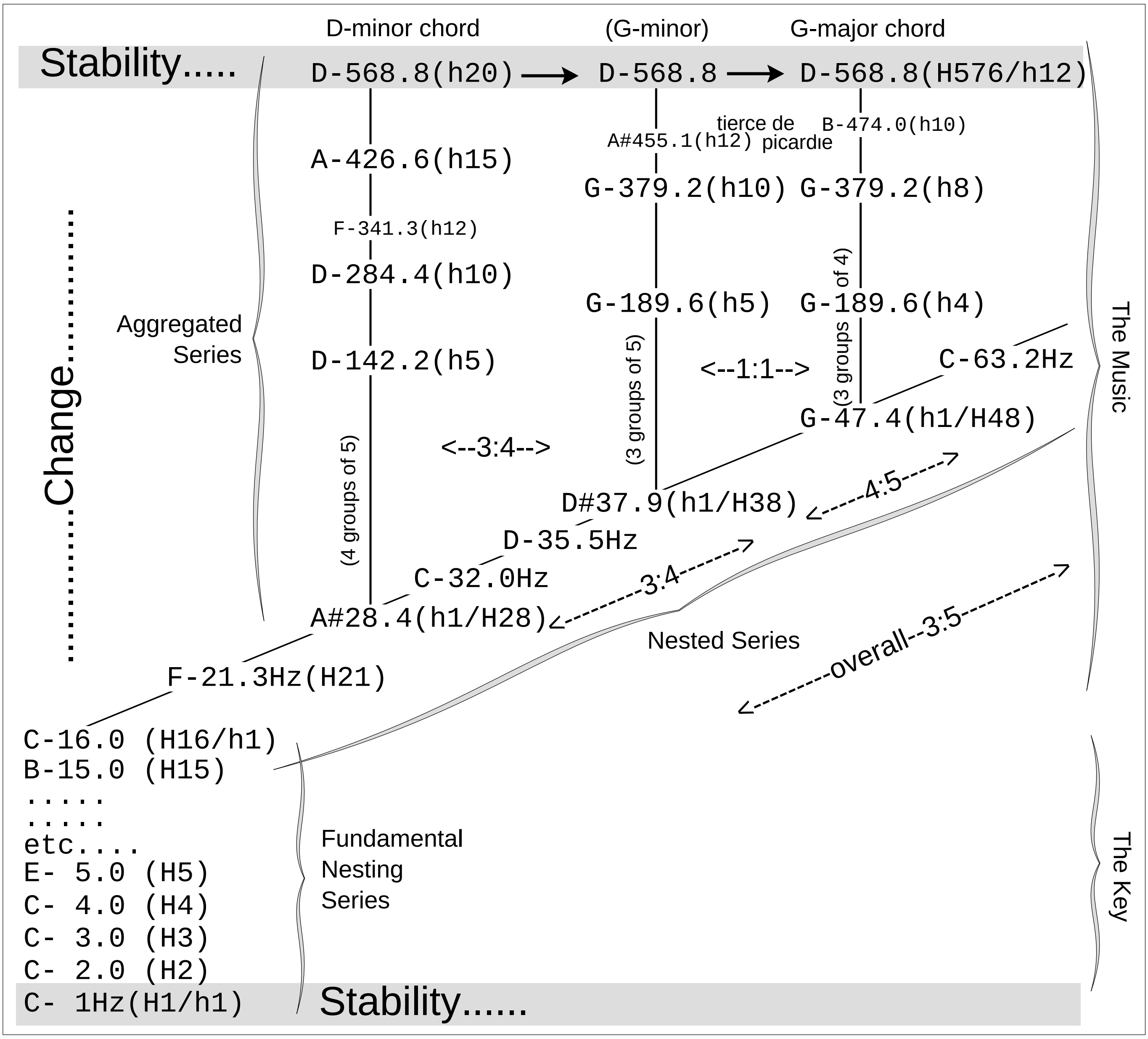


Figure 12.1 Diagram of three layers of nesting: fundamental, nested and aggregated series. In the following analysis, as the fundamental series remains constant throughout, i.e. key of C-major, it has been omitted. The position of the minor-third outside the ratios of the host 'major' series is crucial in maintaining the superposition of the minor chord within the underlying series, in that the relatively complex and extended middle level nested series, up to and beyond h12, cannot be rationalised to simpler multiples.

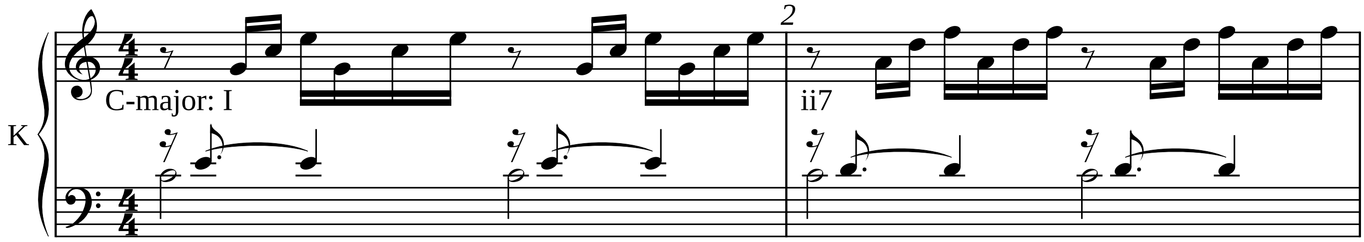
Between the relative stability of the upper and lower levels of the matrix of nested harmonic series, there lies a range of possible connections – the set of mutable base digit sequences appropriate to the value being represented by the harmony at that point. By means of moving from one intermediate configuration to another, chord progressions can be 'rationalised' into sequences of *coherent* steps providing an enchanting variety of aural transformations held within the framework of a broader, stable unity.

PRELUDE No.1 IN C – J.S. BACH

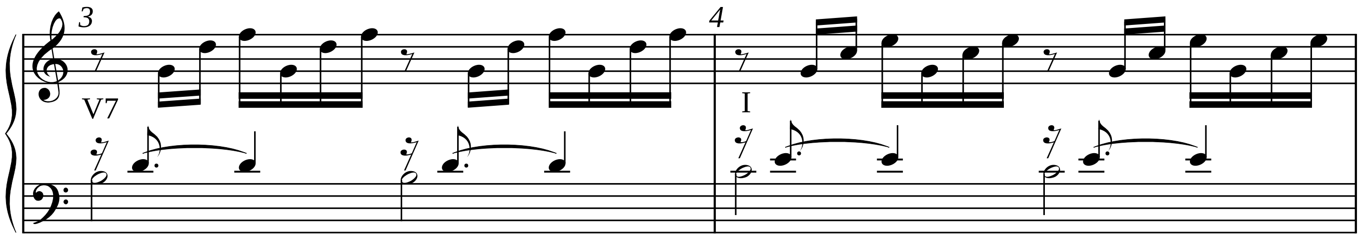
Here are presented the opening four measures of a MOS/MBN analysis of the first prelude from the Well-tempered Clavier, Book 1; given both in terms of a dynamical modulating oscillatory system (pages 10 and 11) and, immediately below, in the equivalent formalism of mutable-base numbers (in ascending factor format). The complete MOS/MBN analysis is presented in Example S.

from the Well-tempered Clavier (Forty-eight Preludes and Fugues – 1722)
with Mutable Base Digit Sequences and their values in decimal appended.

J.S.Bach (1685–1750)



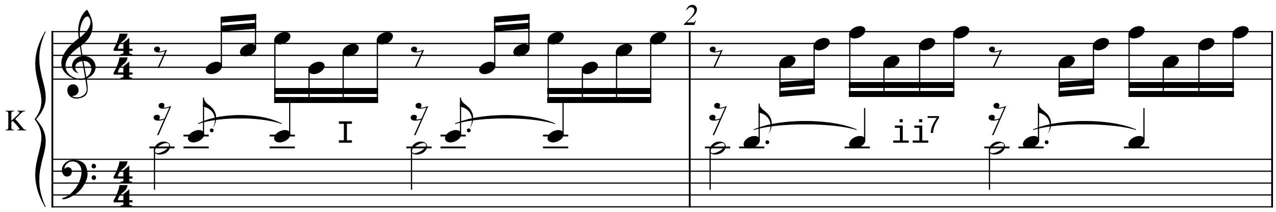
		1	2	3
	Value (decimal):	1280-E	1280-E	1152-D
(Pitch: middle C = 256Hz)	Mutable Digit Sequences:	10 4 32 1 (C-h1 = 1Hz)	9 5 28 1.016	8 5 28 1.029



1152-D	768-G	768-G	768-G	1280-E
6 4 48 1	4 4 48 1	3 4 64 1	6 4 32 1	10 4 32 1

Piano Sonata K545 (measures 63–66) – W.A. Mozart

Pages 12–13, overleaf, present an analysis of a short section from a Mozart piano sonata in the terms of a dynamical modulating oscillatory system. In this example, many extra conjoined frequencies are outlined in broken line bands. Some issues regarding the diminished triad and the dominant-seventh chord, arising from the analysis, are discussed in the remaining pages.



Conjunctions

Objective Notes

Objective Harmonics and/or Subjective Summation Tones (not all shown)

Difference Tones (not all shown)

E-1280(h40)*>	E-1280.0(h45)
-1248(h39)	-1251.5(h44)
D#1216(h38)	D#1223.1(h43)
-1184(h37)	-1194.6(h42)
D-1152(h36)*	-1166.2(h41)
-1120(h35)	D-1137.7(h40)*>
C#1088(h34)	-1109.3(h39)
-1056(h33)	C#1080.8(h38)
C-1024(h32)*	-1052.4(h37)
- 992(h31)	C-1024.0(h36)
B- 960(h30)	- 995.5(h35)*
- 928(h29)	B- 967.1(h34)
A# 896(h28)*	- 938.6(h33)
- 864(h27)	A# 910.2(h32)
A- 832(h26)	- 881.7(h31)
G# 800(h25)	A- 853.3(h30)*
G- 768(h24)*	- 824.8(h29)
- 736(h23)	G# 796.4(h28)
F# 704(h22)	G- 768.0(h27)
F- 672(h21)	- 739.5(h26)
E- 640(h20)*	F# 711.1(h25)*
D# 608(h19)	F- 682.6(h24)
D- 576(h18)	- 654.2(h23)
C# 544(h17)	E- 625.7(h22)
C- 512(h16)*	D# 597.3(h21)
B- 480(h15)	D- 568.8(h20)*
A# 448(h14)	C# 540.4(h19)
A- 416(h13)	C- 512.0(h18)
G- 384(h12)*	B- 483.5(h17)
F# 352(h11)	A# 455.1(h16)
E- 320(h10)	A- 426.6(h15)*
D- 288 (h9)	G# 398.2(h14)
C- 256 (h8)*	G- 369.7(h13)
A# 224 (h7)	F- 341.3(h12)
G- 192 (h6)	E- 312.8(h11)
E- 160 (h5)	D- 284.4(h10)*
C- 128 (h4)*R	C- 256.0 (h9)
G- 96 (h3)	A# 227.5 (h8)
C- 64 (h2)	G# 199.1 (h7)
C- 32 (h1/H32)	F- 170.6 (h6)
	D- 142.2 (h5)*R
	A# 113.7 (h4)
	F- 85.3 (h3)
	A# 56.8 (h2)
	A# 28.4 (h1/H28)

top level Aggregated Series in black

(10 groups of 4)

--- 9:10 -->

(9 groups of 5)

(sesquinona modulation)

(-1 group of 5)

(8 groups of 5)

----3:4------->

(sesquitertia modulation)

middle level Nested Series in gray

(5 x eight) -----------------9:8---------> (5 x nine)->(-5 x one)

(sesquioctava 9:8 modulation) -->(8 x five) -----3:5------>

NOTES

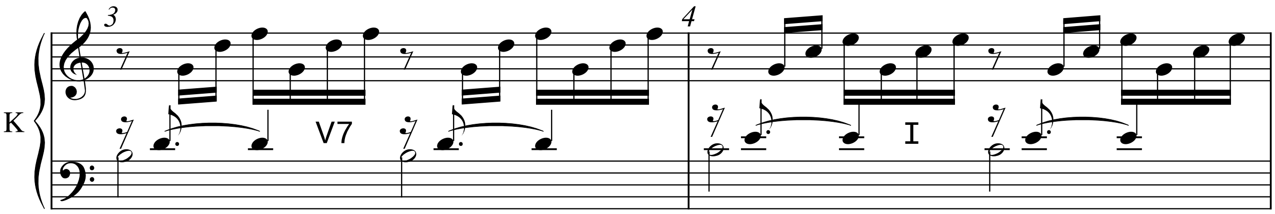
1) The objective notes of the prelude are circled, eg. C

2) The units of absolute frequency are in hertz, with mid C=256Hz. Logical frequency is given as (hn).

3) Foreground/aggregated series are shown & described in black. Background/nested series are gray. The fundamental/nesting series built on C-1Hz (H1) is omitted. It is the upper level aggregated series (in black type) which identifies the roots of chords (*R) through their fundamental tones.

Mutable Base Number: $10_4 0_{32} 0_1$ -------------------> $9_5 0_{28} 0_{1.0158...}$

MBN: $8_5 0_{28} 0_{1.0158...}$------>




```
G-1517.0(h32)*
 -1469.6(h31)
F#1422.2(h30)
 -1374.8(h29)
F-1327.4(h28)*
E-1280.0(h27)
 -1232.5(h26)
D#1185.1(h25)
D-1137.7(h24)*
 -1090.3(h23)
C#1042.9(h22)
C- 995.5(h21)
B- 948.1(h20)*
A# 900.7(h19)
A- 853.3(h18)
G# 805.9(h17)
G- 758.5(h16)*>
F# 711.1(h15)
F- 663.7(h14)
E- 616.2(h13)
D- 568.8(h12)*
C# 521.4(h11)
B- 474.0(h10)
A- 426.6 (h9)
G- 379.2 (h8)*
F- 331.8 (h7)
D- 284.4 (h6)
B- 237.0 (h5)
G- 189.6 (h4)*R
D- 142.2 (h3)
G-  94.8 (h2)
G-  47.4 (h1/H48)
```

```
G-1517.0(h24)*
 -1453.8(h23)
F#1390.6(h22)
F-1327.4(h21)
E-1264.1(h20)*
D#1200.9(h19)
D-1137.7(h18)
C#1074.5(h17)
C-1011.3(h16)*
B- 948.1(h15)
A# 884.9(h14)
A- 821.7(h13)
G- 758.5(h12)*>
F# 695.3(h11)
E- 632.0(h10)
D- 568.8 (h9)
C- 505.6 (h8)*
A# 442.4 (h7)
G- 379.2 (h6)
E- 316.0 (h5)
C- 252.8 (h4)*R
G- 189.6 (h3)
C- 126.4 (h2)
C-  63.2 (h1/H64)
```

```
G-1536.0(h48)**
- 1472.0(h46)
F#1408.0(h44)*
F-1344.0(h42)
E-1280.0(h40)**>
 -1248.0(h39)
D#1216.0(h38)
 -1184.0(h37)
D-1152.0(h36)*
 -1120.0(h35)
C#1088.0(h34)
 -1056.0(h33)
C-1024.0(h32)**
 - 992.0(h31)
B- 960.0(h30)
 - 928.0(h29)
A# 896.0(h28)*
 - 864.0(h27)
A- 832.0(h26)
 - 800.0(h25)
G- 768.0(h24)**
 - 736.0(h23)
F# 704.0(h22)
F- 672.0(h21)
E- 640.0(h20)*
D# 608.0(h19)
D- 576.0(h18)
C# 544.0(h17)
C- 512.0(h16)**
B- 480.0(h15)
A# 448.0(h14)
A- 416.0(h13)
G- 384.0(h12)*
F# 352.0(h11)
E- 320.0(h10)
D- 288.0 (h9)
C- 256.0 (h8)**
A# 224.0 (h7)
G- 192.0 (h6)
E- 160.0 (h5)
C- 128.0 (h4)*R
G-  96.0 (h3)
C-  64.0 (h2)
C-  32.0 (h1/H32)
```

-------------3:4-----> (6 groups of 4)
(-2 groups of 4)
(4 groups of 4) ------------------3:4---------------> (3 groups of 4)
(primary sesquitertia exchange)

(10 groups of 4) -6:5->
(+4 groups of 4)
---> (6 groups of 4)

and recalibrate
---2:1--->
to whole numbers

-----3:5------> (8 x three)
(-8 x one)
(4 x four) -------------------3:4--------------------> (4 x three)
(quaternary sesquitertia exchange)

(20 x two) --3:2-->
(+16 x one)
-------> (8 x three)

MBN: $4_4 0_{48} 0_{0.9876...}$ ------------------> $3_4 0_{64} 0_{0.9876...}$ MBN: $10_4 0_{32} 0_1$ -->

---------> $6_4 0_{48} 0_{0.9876...}$

Cmaj: Ib (3:4) IV (11:8) viib (3:4) iii (3:4)

```
E-1280.0(h80)*
 -1264.0(h79)
 -1248.0(h78)
 -1232.0(h77)
D#1216.0(h76)*
 -1200.0(h75)
 -1184.0(h74)
 -1168.0(h73)
D-1152.0(h72)*
 -1136.0(h71)
 -1120.0(h70)
 -1104.0(h69)
C#1088.0(h68)*
 -1072.0(h67)
 -1056.0(h66)
 -1040.0(h65)
C-1024.0(h64)*
 -1008.0(h63)
 - 992.0(h62)
 - 976.0(h61)
B- 960.0(h60)*
 - 944.0(h59)
 - 928.0(h58)
 - 912.0(h57)
A# 896.0(h56)*
 - 880.0(h55)
 - 864.0(h54)
 - 848.0(h53)
A- 832.0(h52)*
 - 816.0(h51)
G# 800.0(h50)
 - 784.0(h40)
G- 768.0(h48)*>
 - 752.0(h47)
 - 736.0(h46)
F# 720.0(h45)
F# 704.0(h44)*
 - 688.0(h43)
F- 672.0(h42)
 - 656.0(h41)
E- 640.0(h40)*
 - 624.0(h39)
D# 608.0(h38)
 - 592.0(h37)
D- 576.0(h36)*
 - 560.0(h35)
C# 544.0(h34)
 - 528.0(h33)
C- 512.0(h32)*
 - 496.0(h31)
B- 480.0(h30)
 - 464.0(h29)
A# 448.0(h28)*
 - 432.0(h27)
A- 416.0(h26)
 - 400.0(h25)
G- 384.0(h24)*
 - 368.0(h23)
F# 352.0(h22)
F- 336.0(h21)
E- 320.0(h20)*
D# 304.0(h19)
D- 288.0(h18)
C# 272.0(h17)
C- 256.0(h16)*
B- 240.0(h15)
A# 224.0(h14)
A- 208.0(h13)
G- 192.0(h12)*
F# 176.0(h11)
E- 160.0(h10)
D- 144.0 (h9)
C- 128.0 (h8)*
A# 112.0 (h7)
G-  96.0 (h6)
E-  80.0 (h5)
C-  64.0 (h4)*Root
G-  48.0 (h3)
C-  32.0 (h2)
C-  16.0 (h1/H16)
```

```
E-1280.0(h60)*
 -1258.6(h59)
 -1237.3(h58)
D#1216.0(h57)
 -1194.6(h56)*
 -1173.3(h55)
D-1152.0(h54)
 -1130.6(h53)
 -1109.3(h52)*
C#1088.0(h51)
 -1066.6(h50)
 -1045.3(h49)
C-1024.0(h48)*
 -1002.6(h47)
 - 981.3(h46)
B- 960.0(h45)
 - 938.6(h44)*
 - 917.3(h43)
A# 896.0(h42)
 - 874.6(h41)
A- 853.3(h40)*
 - 832.0(h39)
G# 810.6(h38)
 - 789.3(h37)
G- 768.0(h36)*>
 - 746.6(h35)
F# 725.3(h34)
 - 704.0(h33)
F- 682.6(h32)*
 - 661.3(h31)
E- 640.00(h30)
 - 618.6(h29)
D# 597.3(h28)*
D- 576.0(h27)
C# 554.6(h26)
 - 533.3(h25)
C- 512.0(h24)*
 - 490.6(h23)
B- 469.3(h22)
A# 448.0(h21)
A- 426.6(h20)*
G# 405.3(h19)
G- 384.0(h18)
F# 362.6(h17)
F- 341.3(h16)*
E- 320.0(h15)
D# 298.6(h14)
D- 277.3(h13)
C- 256.0(h12)*
B- 234.6(h11)
A- 213.3(h10)
G- 192.0 (h9)
F- 170.6 (h8)*
D# 149.3 (h7)
C- 128.0 (h6)
A- 106.6 (h5)
F-  85.3 (h4)*R
C-  64.0 (h3)
F-  42.6 (h2)
F-  21.3 (h1/H21)
```

```
E-1280.0(h90)
 -1265.7(h89)
 -1251.5(h88)
 -1237.3(h87)
 -1223.1(h86)
 -1208.8(h85)
D#1194.6(h84)
 -1180.4(h83)
 -1166.2(h82)
 -1152.0(h81)
D-1137.7(h80)
 -1123.5(h79)
 -1109.3(h78)
 -1095.1(h77)
C#1080.8(h76)
 -1066.6(h75)
 -1052.4(h74)
 -1038.2(h73)
C-1024.0(h72)>
 -1009.7(h71)
 - 995.5(h70)
 - 981.3(h69)
B- 967.1(h68)
 - 952.8(h67)
 - 938.6(h66)
 - 924.4(h65)
A# 910.2(h64)
 - 896.0(h63)
 - 881.7(h62)
 - 867.5(h61)
A- 853.3(h60)
 - 839.1(h59)
 - 824.8(h58)
 - 810.6(h57)
G# 796.4(h56)
 - 782.2(h55)
G- 768.0(h54)
 - 753.7(h53)
 - 739.5(h52)
 - 725.3(h51)
F# 711.1(h50)
 - 696.8(h49)
F- 682.6(h48)
 - 668.4(h47)
 - 654.2(h46)
E- 640.0(h45)
 - 625.7(h44)
 - 611.5(h43)
D# 597.3(h42)
 - 583.1(h41)
D- 568.8(h40)
 - 554.6(h39)
C# 540.4(h38)
 - 526.2(h37)
C- 512.0(h36)
 - 497.7(h35)
B- 483.5(h34)
 - 469.3(h33)
A# 455.1(h32)
 - 440.8(h31)
A- 426.6(h30)
 - 412.4(h29)
G# 398.2(h28)
G- 384.0(h27)
 - 369.7(h26)
F# 355.5(h25)
F- 341.3(h24)
 - 327.1(h23)
E- 312.8(h22)
D# 298.6(h21)
D- 284.4(h20)
C# 270.2(h19)
C- 256.0(h18)
B- 241.7(h17)
A# 227.5(h16)
A- 213.3(h15)
G# 199.1(h14)
G- 184.8(h13)
F- 170.6(h12)
E- 156.4(h11)
D- 142.2(h10)
C- 128.0 (h9)
A# 113.7 (h8)
G#  99.5 (h7)
F-  85.3 (h6)
D-  71.1 (h5)
A#  56.8 (h4)
F-  42.6 (h3)
A#  28.4 (h2)
A#  14.2 (h1/H14)
```

<---------- no internal nesting ---------->

dominant7th without root in groups of three

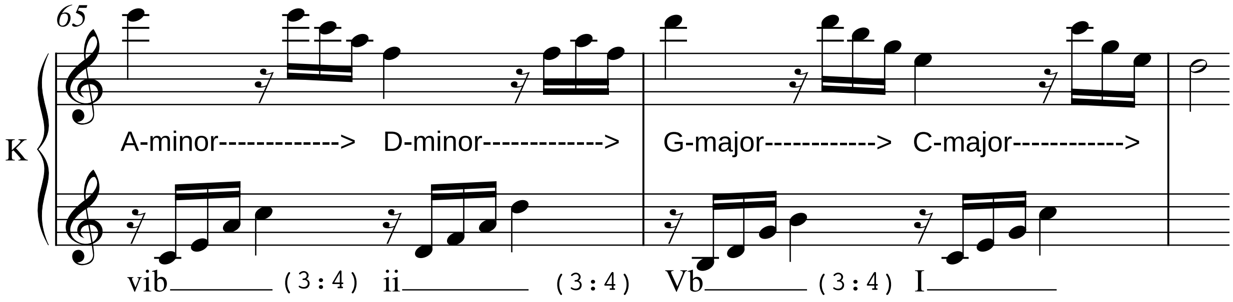
```
  E-1280.0(h80)*>
   -1264.0(h79)
   -1248.0(h78)
   -1232.0(h77)
  D#1216.0(h76)
   -1200.0(h75)*
   -1184.0(h74)
   -1168.0(h73)
  D-1152.0(h72)
   -1136.0(h71)
   -1120.0(h70)*
   -1104.0(h69)
  C#1088.0(h68)
   -1072.0(h67)
   -1056.0(h66)
   -1040.0(h65)*
  C-1024.0(h64)
   -1008.0(h63)
   - 992.0(h62)
   - 976.0(h61)
  B- 960.0(h60)*
   - 944.0(h59)
   - 928.0(h58)
   - 912.0(h57)
  A# 896.0(h56)
   - 880.0(h55)*
   - 864.0(h54)
  A- 848.0(h53)
  A- 832.0(h52)
   - 816.0(h51)
  G# 800.0(h50)*
   - 784.0(h49)
  G- 768.0(h48)
   - 752.0(h47)
   - 736.0(h46)
  F# 720.0(h45)*
  F# 704.0(h44)
   - 688.0(h43)
  F- 672.0(h42)
   - 656.0(h41)
  E- 640.0(h40)*
   - 624.0(h39)
  D# 608.0(h38)
   - 592.0(h37)
* D- 576.0(h36)
   - 560.0(h35)*
  C# 544.0(h34)
*  - 528.0(h33)
  C- 512.0(h32)
   - 496.0(h31)
* B- 480.0(h30)*
   - 464.0(h29)
  A# 448.0(h28)
*  - 432.0(h27)
  A- 416.0(h26)
   - 400.0(h25)*
* G- 384.0(h24)
   - 368.0(h23)
  F# 352.0(h22)
* F- 336.0(h21)
  E- 320.0(h20)*
  D# 304.0(h19)
* D- 288.0(h18)
  C# 272.0(h17)
  C- 256.0(h16)
* B- 240.0(h15)*
  A# 224.0(h14)
  A- 208.0(h13)
* G- 192.0(h12)
  F# 176.0(h11)
  E- 160.0(h10)*
* D- 144.0 (h9)
  C- 128.0 (h8)
  A# 112.0 (h7)
* G-  96.0 (h6)
  E-  80.0 (h5)*R
  C-  64.0 (h4)
*R G-  48.0 (h3)
  C-  32.0 (h2)
  C-  16.0 (h1/H16)
```

(15 groups of 4) ----->

(12 groups of 4) --------3:4--------> (9 groups of 4)

(16 groups of 5) --3:4--------------------------->

(12 x four) -------3:4-------> (12 x three)(18 x two) ---3:2---> (18 x three)(8 x nine) -----8:9-----> (8 x eight)(20 x four) --3:4--------------------------->




```
E-1280.0(h60)*>
-1258.6(h59)
-1237.3(h58)
D#1216.0(h57)
-1194.6(h56)
-1173.3(h55)*
D-1152.0(h54)
-1130.6(h53)
-1109.3(h52)
C#1088.0(h51)
-1066.6(h50)*
-1045.3(h49)
C-1024.00(h48)
-1002.6(h47)
- 981.3(h46)
B- 960.0(h45)*
- 938.6(h44)
- 917.3(h43)
A# 896.0(h42)
- 874.6(h41)
A- 853.3(h40)*
- 832.0(h39)
G# 810.6(h38)
- 789.3(h37)
G- 768.0(h36)
- 746.6(h35)*
- 725.3(h34)
- 704.0(h33)
F- 682.6(h32)
- 661.3(h31)
E- 640.0(h30)*
- 618.6(h29)
D# 597.3(h28)
D- 576.0(h27)
C# 554.6(h26)
- 533.3(h25)*
C- 512.0(h24)
- 490.6(h23)
B- 469.3(h22)
A# 448.0(h21)
A- 426.6(h20)*
G# 405.3(h19)
G- 384.0(h18)
F# 362.6(h17)
F- 341.3(h16)
E- 320.0(h15)*
D# 298.6(h14)
D- 277.3(h13)
C- 256.0(h12)
B- 234.6(h11)
A- 213.3(h10)*
G- 192.0 (h9)
F- 170.6 (h8)
D# 149.3 (h7)
C- 128.0 (h6)
A- 106.6 (h5)*R
F-  85.3 (h4)
C-  64.0 (h3)
F-  42.6 (h2)
F-  21.3(h1/H21)
```

```
E-1280.0(h45)*
-1251.5(h44)
-1223.1(h43)
-1194.6(h42)
-1166.2(h41)
D-1137.7(h40)*>
-1109.3(h39)
C#1080.8(h38)
-1052.4(h37)
C-1024.0(h36)
- 995.5(h35)*
B- 967.1(h34)
- 938.6(h33)
A# 910.2(h32)
- 881.7(h31)
A- 853.3(h30)*
- 824.8(h29)
G# 796.4(h28)
G- 768.0(h27)
- 739.5(h26)
F# 711.1(h25)*
F- 682.6(h24)
- 654.2(h23)
E- 625.7(h22)
D# 597.3(h21)
D- 568.8(h20)*
C# 540.4(h19)
C- 512.0(h18)
B- 483.5(h17)
A# 455.1(h16)
A- 426.6(h15)*
G# 398.2(h14)
G- 369.7(h13)
F- 341.3(h12)
E- 312.8(h11)
D- 284.4(h10)*
C- 256.0 (h9)
A# 227.5 (h8)
G# 199.1 (h7)
F- 170.6 (h6)
D- 142.2 (h5)*R
A# 113.7 (h4)
F-  85.3 (h3)
A#  56.8 (h2)
A#  28.4(h1/H28)
```

```
G-1517.0(h32)*>
-1469.6(h31)
F#1422.2(h30)
-1374.8(h29)
F-1327.4(h28)*
E-1280.0(h27)
-1232.5(h26)
D#1185.1(h25)
D-1137.7(h24)*
-1090.3(h23)
C#1042.9(h22)
C- 995.5(h21)
B- 948.1(h20)*
A# 900.7(h19)
A- 853.3(h18)
G# 805.9(h17)
G- 758.5(h16)*
F# 711.1(h15)
F- 663.7(h14)
E- 616.2(h13)
D- 568.8(h12)*
C# 521.4(h11)
B- 474.0(h10)
A- 426.6 (h9)
G- 379.2 (h8)*
F- 331.8 (h7)
D- 284.4 (h6)
B- 237.0 (h5)
G- 189.6 (h4)*R
D- 142.2 (h3)
G-  94.8 (h2)
G-  47.4 (h1/H48)
```

```
G-1517.0(h24)*
-1453.8(h23)
F#1390.6(h22)
F-1327.4(h21)
E-1264.1(h20)*
D#1200.9(h19)
D-1137.7(h18)
C#1074.5(h17)
C-1011.3(h16)*
B- 948.1(h15)
A# 884.9(h14)
A- 821.7(h13)
G- 758.5(h12)*
F# 695.3(h11)
E- 632.0(h10)
D- 568.8 (h9)
C- 505.6 (h8)*
A# 442.4 (h7)
G- 379.2 (h6)
E- 316.0 (h5)
C- 252.8 (h4)*R
G- 189.6 (h3)
C- 126.4 (h2)
C-  63.2 (h1/H64)
```

(8 groups of 4) ---------3:4--------> (6 groups of 4)

(8 groups of 5) ---------3:4-------> (6 groups of 4)

----------------3:4--> (12 groups of 5) ---------3:4--------> (9 groups of 5)

(8 x four) --------------3:4-----------> (8 x three)

----------------3:4---> (20 x three) (8 x five) -------------3:5------------> (8 x three)

(15 x four) -------------3:4----------> (15 x three)

EXTRACT: MOZART PIANO SONATA K545

On pages 12–13 is an example chord progression from Mozart's Piano Sonata in C major, K545 measures 63–66. This short section consists of a succession of chords descending the spiral/cycle fifths, via a sequence of sesquitertia 3:4 exchanges. That is, a chain of subdominant relationships. There are two exceptions: in columns two/three the system steps from major to minor chords via a diminished chord, and the adjustment back to major chords in the columns seven/eight via a sesquiquarta 4:5 exchange.

Conjunctions

Highlighted in broken line bands are the principal conjunctions of the whole phrase – threads of continuity binding the sequence of chords together. The higher one looks, the more encompassing the connections become; however, there is also a reasonably clear discontinuity between the conjunctions of the chords at the beginning and end of the phrase. Of the conjunctions shown, C-1024.0 Hz extends furthest, but overall there is a shift between the conjunctions up to column five/six (A-minor/F-series) and those from column seven onward (D-minor/A#series). These two 'sides' to the conjunctions (i.e. subdominant <- tonic -> dominant) echo the division between the static nature of chords I/IV in contrast to the dynamism of chords ii/V (Goldman, 1968: 68–71), though overall, the overlapping of conjunctions at the divide allows a predominant sense of continuity to emerge out of this dichotomy. The conjunctions around the subdominant bind the 'musical action' to the tonic key. However, once the harmony steps across the rubicon to chord ii/II, the new alignment of conjunctions operates to draw the music toward the dominant chord (V), which in turn will tend to redefine the tonic as chord IV in the key of chord V – the subdominant relationship to chord V, built on frequencies somewhat lower than the original tonic. For example, in measure 4 of the Prelude and measure 66 below, the 'flexing' brought about through the computations have left the chord of C-major standing on a fundamental tone of 63.2Hz, whereas it began the phrase at 64Hz.

Diminished Triad/Dominant-seventh Chord

Columns 2, 3 and 4 are particularly noteworthy, in that the diminished chord configuration of D-h10, F-h12 and B-h17 offers no prospect of internal nesting (above groups of two) as all notes are formed from different prime number vibrational patterns, producing a restless chord without any perceivable root. That is, D-h2×$\underline{5}$, F-h2×2×$\underline{3}$ and B-h$\underline{17}$; and if G#h14 is added, producing a full diminished-seventh, G#-h2×$\underline{7}$. Then, apart from the factor two which doesn't change the outcome, the numbers three, five, (seven) and seventeen have no common ground upon which to construct an aggregated series, and this disinclination toward nesting remains true whatever background series is chosen.

Immediately below are graphs of some of the combined patterns which the notes produce. Each note is treated as a single frequency oscillator of equal unitary amplitude.

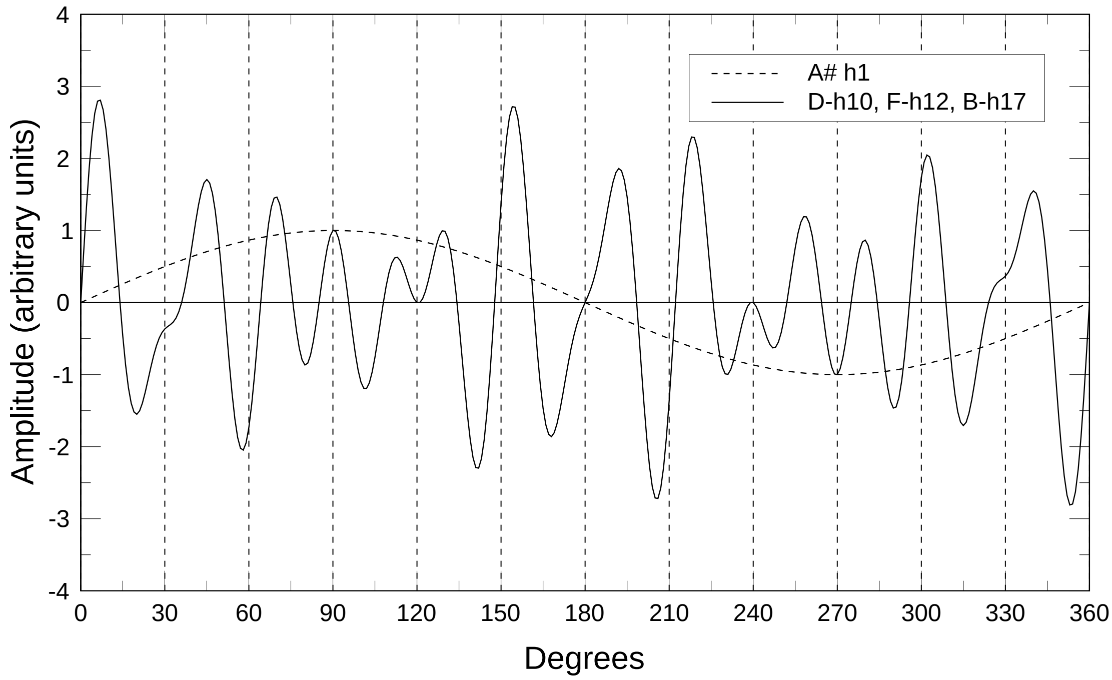


Figure 12.2 The B-diminished triad in column three (notes D, F, B) supports no internal pattern of subdivision.

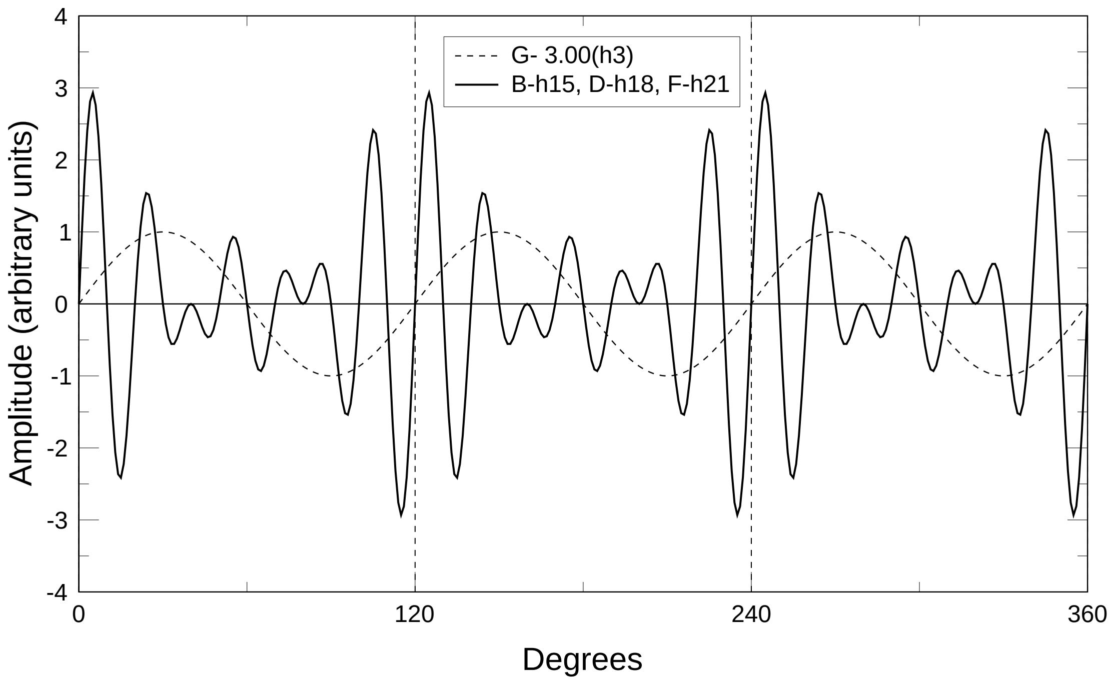


Figure 12.3 Rootless Dominant7th chord in column four (notes B, D, F) supports internal subdivisions of three: G-h3.

The restless configuration of chord vii can find sanctuary in the guise of a rootless dominant-seventh chord, in the form of an aggregated series in groups of three, nested within a background series built on the tonic frequency. This is illustrated with arrows superimposed to the left of column four in the example below (measure 64) and in the graph Figure 12.3.

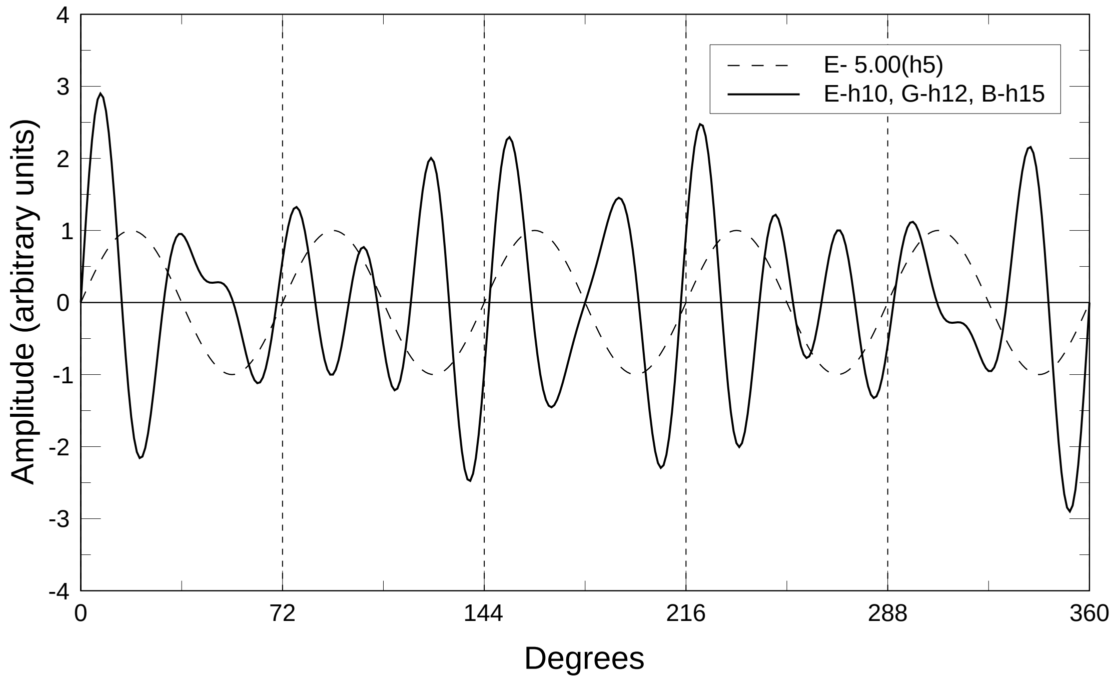


Figure 12.4 E-minor chord in column four (notes E, G, B) supports internal subdivisions of five: E-h5. This pattern also supports internal groups of three as well – shown in Figure 12.5 below.

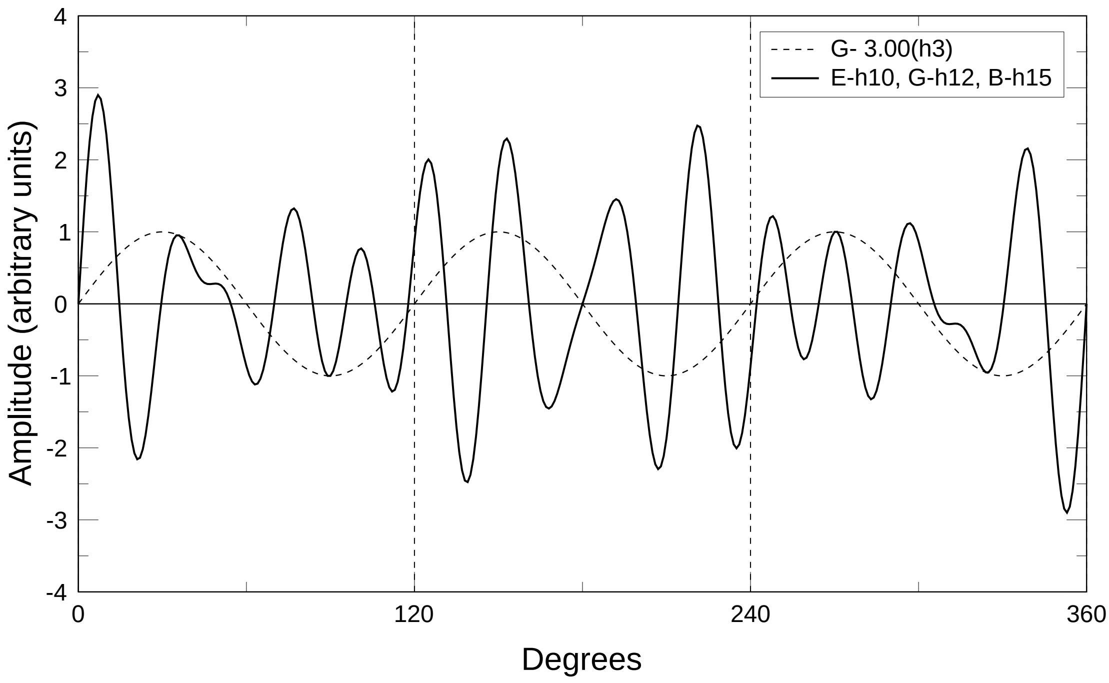


Figure 12.5 E-minor chord in column four (notes E, G, B) also supports internal subdivisions of three: G-h3.

Interestingly, when one series is nested in groups of three within another series, h21 of the nested series equates to h63 of the nesting series, while the 'correct' subdominant scale degree emerges from the nesting series view of that note as h64, that is h64/3 = h21.333.... For example, if a C-based series aggregated in groups of three is nested within an F-based series, then F-h21 in the C-based series occurs at the unmarked location of ratio 'h63' in the nesting F-based series, while one ratio above lies F-h64: *via the MOS model's stratagem of nesting, it is possible to generate the subdominant scale degree from the harmonic series.*

Notes

1. Goldman, R.F., *Harmony in Western Music*, (Barrie & Rockliff, London, 1965)

2. Beament, J., *How We Hear Music*, (Boydell Press, Woodbridge, UK, 2005)

3. Jeans, J., *Science and Music*, (Dover, New York, 1937/1968) page 237.