11

The Minor Mode

DUALISM AND THE ROOT OF THE MINOR TRIAD

In this chapter the idea of *Harmonic Dualism*, an elegant but largely discredited tonal theory founded upon a conception of inverse partnership between the major and minor triads, chords, scales and keys, is re-examined and set in the wider context of a system of nested harmonic series. Though principally developed in Germany during the second half of the nineteenth century by Moritz Hauptmann, Arthur von Oettingen and Hugo Riemann, dualism has failed to find general long-term acceptance amongst music theorists, due in part to its central tenet appearing to be contradicted by natural science (i.e. the existence of *undertones* – fractional standing waves). However, below, one crucial aspect of the theory – the ‘root’ of a minor triad – is explored and developed within the context of mutable base numbers, leading to the conclusion that it might be possible to reconcile some elements of dualism with acoustics. This new approach to dualism is based on the mechanisms of the broader model of *Modulating Oscillatory Systems*.

In mathematics and the physical sciences I believe it is rare, perhaps even unknown, for a truly elegant idea to be wrong. Although this observation may not be quite so trustworthy for all disciplines, and leaving aside the difficult question of defining true elegance, sometimes a discipline may be in possession of what appears to be a beautiful idea, while only poorly and imprecisely understanding in what manner or context the concept might be rightly applied. I suspect the idea of *harmonic dualism* could fall into this category: an insight that flared brightly for a time before fading from view, its promise unfulfilled. The most fully developed and satisfactory account of dualism, a tonal theory that places the major and minor triads, scales and keys in an equal inverse relationship, was given by Arthur von Oettingen (1836–1920) in *Das duale Harmoniesystem* published in 1913. He worked on the theory of dualism throughout his academic life, publishing over the years a number of treatises and scientific papers on this and other topics. And as a physicist he no doubt appreciated the role that the principle of symmetry was assuming in many fields of endeavour, not least in mathematical group theory.

The origins of dualistic theory stretch back as far as Zarlino’s observation in *Le Istitutioni Harmoniche* (1558), that while the harmonic division of a string or monochord leads to the major triad, an arithmetic partition of a fundamental vibration yields the minor triad – illustrated in Figure 11.1. Since Gioseffo Zarlino’s initial foray, there has been sporadic interest in dualistic ideas over the centuries, with contributions from Rameau 1737, G. Tartini 1754, F. Vallotti 1778, M. Hauptmann 1853, A. v. Oettingen 1866/1913, H. Riemann 1905; and more recently V. d’Indy, P. Hindemith and S. Karg-Elert.[1&2] To be fair however, most contributors have approached the topic with different perspectives, yielding a variety of conclusions, with Hauptmann rather philosophic, Oettingen scientific and Riemann musicological, in focus.

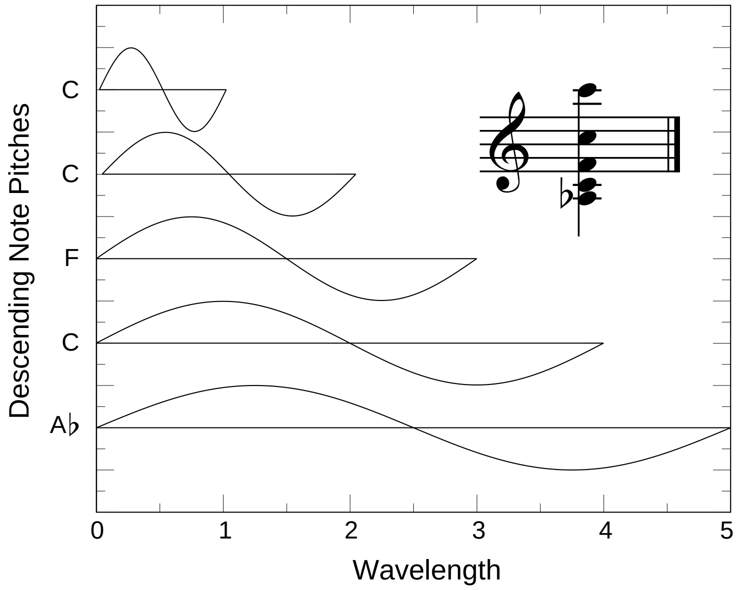


Figure 11.1 The arithmetic division of a string into five equal segments yields the notes of a minor triad, arrayed in intervals of an octave, fifth, fourth and major-third descending. The inverse of the familiar ascending harmonic series.

Rameau was the first to seriously develop Zarlino's initial observation by placing the generative procedure for the minor triad below the fundamental; and there the matter basically rested until Moritz Hauptmann's book on *The Nature of Harmony and Metre* emerged in 1853. Hauptmann accepted Rameau's top-down generation of the minor triad, but in place of Rameau's rather weak argument of the arithmetic principle being 'suggested' by the predominantly harmonic ordering of music, he substituted a highly abstract dialectic approach. Hauptmann identified the intervals of the octave, fifth and (major) third with the three fundamental categories of thesis, antithesis and synthesis, respectively, and used this association to fashion a wide-ranging theory of music, which worked outward from mirroring major and minor triads to mirrored scales and keys, and beyond, to encompass the rhythmic and metrical domain of music. Writing before Helmholtz's ground-breaking research into the acoustics of musical sound and physiology of the ear, the great virtue of this 'Hegelian (philosophical) triad' was that it provided a single unifying principle upon which he could erect a complete theory of music encompassing both the vertical and horizontal dimensions: pitch, harmony, rhythm and duration. However, by using only the (major) 'third principle' to reconcile the 'opposition' between the octave and fifth, he had to invert the relationships for the minor triad – and thus nineteenth century German dualism was born. The puzzling aspect of Hauptmann's treatise is that for a person so deeply knowledgeable and intuitively musical, he was prepared to allow philosophical sophistry to override musical experience. His musical instincts were reliable, as for example, in the derivation of the major scale and principal chords of the major key through the sharing of notes between subdominant, tonic and dominant chords (F – a – C – e – G – b – D), a 'bridging' principle similar to the role of 'conjunctions' in the MOS model. Hauptmann also identifies the minor triad with the ratios of the harmonic series – E-h10, G-h12 and B-h15 – and used them to produce a beautifully mirrored mathematical relationship:

Major C–e–G 4 : 5 : 6 = 4/1:5/1:6/1 = $(4:5:6)^{+1}$
Minor e–G–b 10:12:15 = 1/6:1/5:1/4 = $(6:5:4)^{-1}$

Perhaps it was this math, combined with a well developed sense of symmetry stemming from his early training in architecture, that drew him into a dualist interpretation, despite its weaknesses: 1) the theory's philosophical foundations made the concepts remote from musical practice and 2) the theory could not account for the root of the minor triad. Indeed, Hauptmann maintained, rather in the tradition of medieval philosophy, that music theory and musical practice were in some respects distinct and separate endeavors.

Moritz Hauptmann (1792–1868) was born in Dresden, the son of a provincial architect. His father, intending Moritz to follow in his footsteps, nurtured the talents of his gifted son with a thorough and wide-ranging education, which included mathematics, science, languages, philosophy and the fine arts. Notwithstanding his father's wishes, at the age of nineteen Moritz left Dresden to study music under Louis Spohr in Gotha. Very soon Hauptmann established himself as a professional musician, and some years later was playing the violin under Spohr's baton in the Vienna Theatre Orchestra and undertaking work back in Dresden. In 1815 Hauptmann was appointed music master to a noble Russian family and for the next five years travelled with them to St Petersburg and further afield in Russia. Returning to Dresden in 1820, Hauptmann rekindled his friendship with his old teacher and accepted a post in the Electoral Orchestra in Kassel conducted by Spohr. Hauptmann remained at Kassel for twenty years, playing the violin in the band, teaching and studying all aspects of music theory. Like most music theorists he dabbled in composition, producing an opera (Mathilde), two masses and sundry other choral and instrumental pieces. Though of a modest and unassuming character, his talent as a teacher and his insight as a theorist garnered a high reputation amongst many famous names: Carl Maria v. Weber, Meyerbeer and Mendelssohn. In 1842 Mendelssohn used his influence to help Hauptmann obtain the post of Cantor at the Thomasschule in Leipzig – occupied by J.S. Bach a century earlier – as well as an appointment at the newly founded Leipzig Conservatorium. It was no doubt this advancement which allowed Moritz, in the previous year, to marry Susette Hummel the cultured daughter of the Director of the Kassel Art Academy. It was to be a happy union blessed with three children. Also, the security and distinction of Hauptmann's new position allowed him to complete his long-considered treatise, *Die Natur der Harmonik und Metrik,* published in 1853. After twenty-five settled and productive years of work in Leipzig, Moritz Hauptmann died on the 3rd January, 1868.

The key observation of dualistic theory is that, by strict inversion, the major triad is transformed into a minor triad: thus major-third plus minor-third reckoned upward produces C-E-G, the major triad, but when applied downward (from the highest note) produces G-Eflat-C, the minor triad. By setting this inverse or mirrored relationship within the wider context of extended harmonic and arithmetic series, Oettingen proposed that the notes of the major triad have a focus in the fundamental tone, two octaves below their root (C), which he described as the *tonic* fundamental, while the tones of the minor triad find a mirroring focus in their first common overtone two octaves above their 'root' (the upper note of the C-minor triad, G) – the *phonic* 'fundamental'. While this beautiful symmetry is most appealing, there is a problem: we do not hear and recognise G as the *root* of a C-minor chord, but rather identify C as the root of both the C-major and C-minor chords.

This flaw at the heart of harmonic dualism is forcefully identified by the entry in the Harvard Dictionary of Music[1]: "The greatest shortcoming of the theory [dualism] lies in the fact that in a minor mode the triad is determined, not by its lowest, but its highest tone, ... This forced explanation is in contradiction to the most elementary facts of acoustics and of musical experience." Theory and observation appear to brutally collide. Although this didn't appear to worry Moritz Hauptmann, it has been a thorn in the flesh of most promoters of dualism. Arthur von Oettingen's approach to the problem was to suggest that our musical senses had been led astray by the forceful rootedness of the major triad and that musicians needed to be 're-educated' so as to understand, if not hear, the mirrored logic of the minor principle. Oettingen, a highly competent scientist, knew that he would need to look more to the fields of physiology and psychology for support, than to physics. Some other dualists, for example the great musicologist Hugo Riemann, were at times to go as far as to claim the existence of *undertones*,[2] fractional standing waves but such contrivances could not withstand scientific scrutiny. Whether dualists ignored the problem of the minor triad's root, tiptoed around it or rather rashly made unscientific assertions, the clash of theory and practical experience remained an imperturbable and unwelcome elephant ever at their elbow. Notwithstanding, dualism and dualists have been rather harshly treated by posterity and their elegant symmetrical approach to music much derided and shunned. Yet although the mathematical inversions and reflections of dualism failed to provide a complete and wholly satisfactory theory of tonal music – due to the asymmetry of human hearing which gathers and interprets ascending frequency relationships but not directly those of wavelength – their exploration of symmetry in tonal music remains most illuminating.

The Context of Nested Harmonic Series

Here I would like to suggest a context in which this clash might be resolved – a solution that attempts to incorporate v. Oettingen's *theoretical* arithmetic/minor principle within the harmonic/major system of *real experience*. The scheme does not challenge the monist concept of a fundamental bass or the view of the primacy of the major mode over the minor, but rather accommodates the minor within the major. However, at the outset, it must be admitted that the solution itself is rather theoretical in character, in the sense that although it embeds the minor mode within the reality of the ascending overtone series, whether this scheme bears any relation to how human beings actually hear and process musical sounds, is an open question. Primarily the focus here is on an abstract and mathematical approach to the problem, yet the question of musical cognition cannot wholly be ignored, as the organisational structure of tonal music ultimately depends on human choices, largely made in response to the perception of aural stimuli.

Essentially, *the context* is that of the 'natural' major and minor triads (h4, h5, h6 and h10, h12, h15 respectively) as they occur in the harmonic series. And though some small reference will be made to the broader setting of chord progressions in *Modulating Oscillatory Systems* (MOS), for the most part discussion will be limited to the relationships within one single chord/harmonic series.

To construct the desired context requires the expansion of the written chords found in scores outward into extended harmonic series, taking the process to even greater lengths than v. Oettingen originally envisaged. For a full dynamical MOS analysis, this expansion of the written chords into harmonic series is taken further than simply placing the individual tones in positions within a harmonic series where all the constituent notes in a chord can be expressed (allowing for a reasonable margin of error arising from scale temperaments) as harmonics of a single fundamental frequency. Indeed, to achieve a necessary flexibility, the expanded context of a single harmonic series has itself to be *nested* within another, broader harmonic series and this broader harmonic series *nested* within yet another, even more fundamental, harmonic series! To these three ascending levels of nesting the terms *fundamental/*

nesting series, *nested* series and *aggregated* series are applied. However, for the purposes of this discussion of natural major and minor triads, only the two upper levels are required – nested series and aggregated series. And although this may appear a great extension and extrapolation of what is only a clutch of written notes, in performance it must be born in mind that a wide spectra of vibration assaults the ear, and these myriad objective frequencies are then further enhanced and supplemented by subjective combination tones of the ear's own making. Indeed, it is probably a reasonable assumption that the processes of aural cognition operate on the whole frequency range delivered by the ear, even though the listener's conscious attention may be focused on the notes alone.

Also, like v. Oettingen, we consider the notes/tones of music, in principle, to be *just* or natural harmonics in intonation, with the ear's accepting tolerance being relied on to smooth over any irregularities. The convention of labelling the ratios of the harmonic series from the fundamental tone h1, h2, h3, etc., is adopted, with the addition of the upper case: H1, H2, H3, etc., to distinguish the more fundamental of two or three enmeshed harmonic series.

Nesting and the Harmonic Series

Before moving to the core material concerning the root of the minor triad, it might be helpful to recap a little on the nesting of harmonic series discussed in Chapter 9 and illustrated in Figure 9.7, where the idea of parent and child series was introduced. Figure 9.7 shows two nested series but just one level of nesting. However, if for example, the nested series built on G-384Hz were extended upward to incorporate six harmonics, the same pattern of nested child series could be repeated to provide a third level of *grandchild* series built on G-768Hz and D-1152Hz.

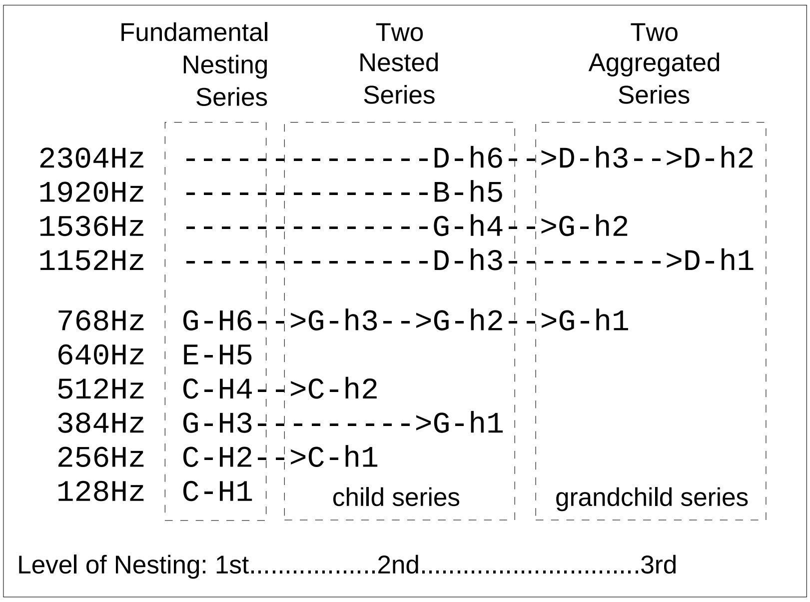
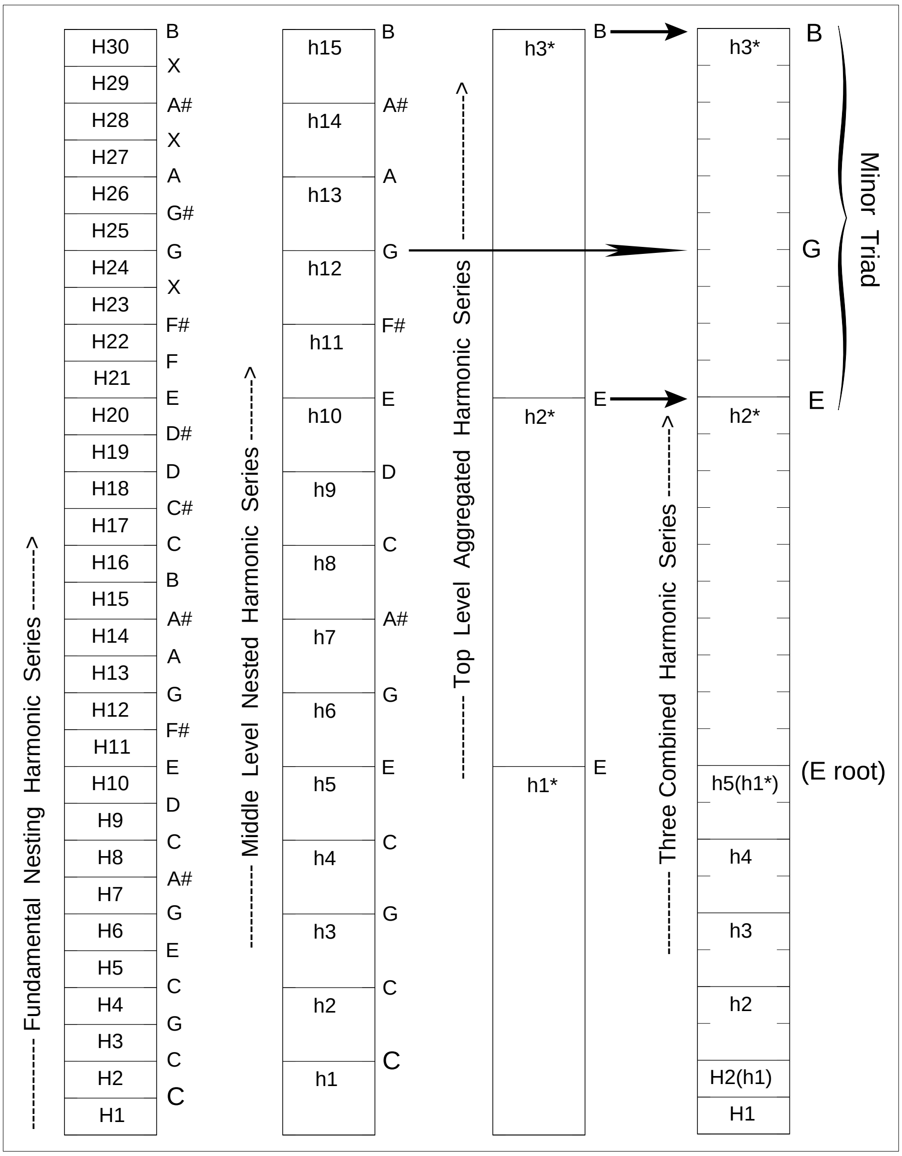
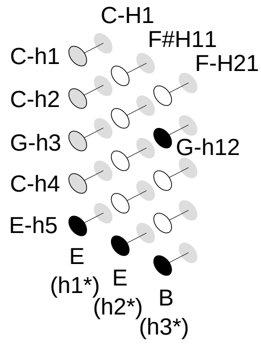


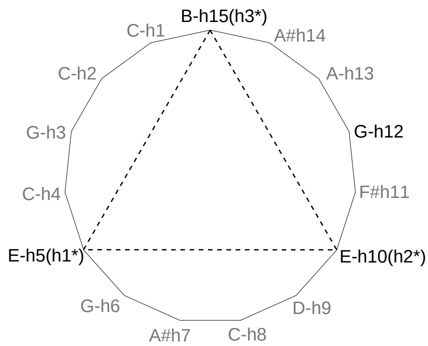
Figure 11.2 Three levels of nesting: Fundamental/Nesting, Nested and Aggregated series.

In Figure 11.2 the series built on G-384Hz (one of the two nested series illustrated in Figure 9.7) has been used as the basis for another level of nesting, but just as easily the nested series built on C-512Hz could have been used, or for that matter, any other higher partial H4, H5, H6, etc., of the fundamental nesting series. The third level series, the grandchildren of the fundamental, are termed *aggregated series*.





Above: The minor triad (E-G-B) as a three-dimensional number pattern. A fundamental series (all dots) C-h1 through B-H30, enfolding a nested series (black-ringed dots) C-H1 through B-h15, which in turn enfolds an aggregated series (black-filled dots) E-h1* through B-h3*.



Above: A geometric interpretation of the minor triad. A nested series of fifteen harmonics partitioned into three groups of five ratios each. The interval of a minor-third created by G-h12 remains outside the aggregations of five elements. (Fundamental series not shown.)

Figure 11.3 Three levels of nesting underlying the minor triad: fundamental/nesting, nested and aggregated series from left to right, with the combined system illustrated on the right of the box section (value thirty, MBN $3_5\,0_2\,0_1$).

The ‘ladders’ of Figure 11.3 illustrate the application of three levels of nesting to the hosting of the minor triad. However, this third level of nesting, the aggregated series as I have termed it, is to some extent a useful fiction in that the grandchild series could equally be viewed as nesting within the parent series and this would be true for any further levels one might care to construct. It is simply convenient to give two different series, both nested within a fundamental series, distinguishing names indicating a clearly defined relationship between them. Ultimately it is difficult to draw absolute distinctions beyond the two levels of *nesting* and *nested* series; indeed, the relationship between child and grandchild series could be redefined, where useful, to that of parent and child. Notwithstanding this caveat, the relative amplitudes of the notes/harmonics within a system could indicate that a particular structure of nested series was logical or natural. For example, in Figure 11.3 the ‘weight’ of the objective tones of the minor triad would tend to divide the fundamental into two nested layers with the majority of the energy (the fifth E-B) concentrated in the upper level aggregated series but with the minor-third, G, picking out the middle-level nested series, thus maintaining the perhaps shadowy existence of the extended, complex, nested series capable of accommodating all three notes (EGB).

It is principally within the bounds of these two outermost layers, the aggregated series and the next below, that the scheme incorporating the arithmetic/minor chords into the overtone series operates. The fundamental nesting series, which represents ‘the key’ within which the chord(s) are embedded, is not a crucial element here, as it is intended to concentrate solely on the core problem of the *root* of the minor chord in this chapter. Also, whereas the nested and aggregated series are closely linked together (i.e. they could be viewed as a nesting and nested pair in isolation) their relationship to the fundamental series is more loosely coupled, emerging over time as a sense of key develops out of the harmonic progression of the two upper layers – driven ultimately by the objective notes of the music itself.

‘Natural’ Major and Minor Triads

The natural major triad is found within the harmonic series at h4, h5 and h6, and it would not be difficult to find scores with written major chords that are in effect complete harmonic series, from fundamental to the upper note of the triad: h1 through h6. In contrast to this, the natural minor triad appears at a markedly higher position in the harmonic series: h10, h12 and h15. A chord of fifteen different tones of the harmonic series would be far rarer, though no doubt one or two might be found, and the complete harmonic series from h1 through h15 is a considerably more complex structure than h1 through h6.

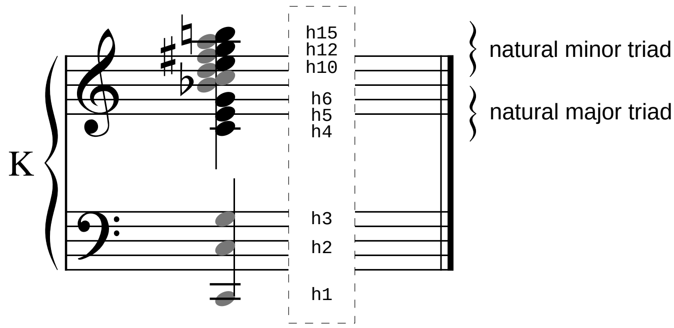


Figure 11.4 The harmonic series from fundamental (h1) up to the fourteenth overtone (h15), with the ‘natural’ triads illustrated in black notes (A#h14 omitted for lack of space).

However, despite their separation, the two natural triads share the feature of a lowest common multiple of *sixty* in their respective relative frequencies – harmonics 4, 5, 6, 10, 12 and 15 yield a LCM of 60. In other words both the major and minor triads have a Euler Metric of Sixty. In addition, h60 is v. Oettingen’s phonic ‘fundamental’ tone for this minor triad, with all the divisors of sixty forming a symmetrical pattern of intervals below the natural major triad and above the natural minor triad: octave, fifth, fourth, major-third and minor-third ascending from h1 and descending from h60. The descending pattern of frequencies from h60 might be viewed as an inverse harmonic series or arithmetic series of wavelength relationships. These relationships are illustrated in Figure 11.5.

The great stumbling block for dualism has been that whilst the major triad emerges naturally from the real *modes of vibration* of a physical object, the arithmetic/minor relationships have remained stubbornly theoretical. We don’t hear a phonic ‘root’, and nor do scientific experiments detect ‘undertones’. So it is to these real material phenomena, the ratios of the ascending harmonic series, that the elegant ideal of dualism must be adapted.

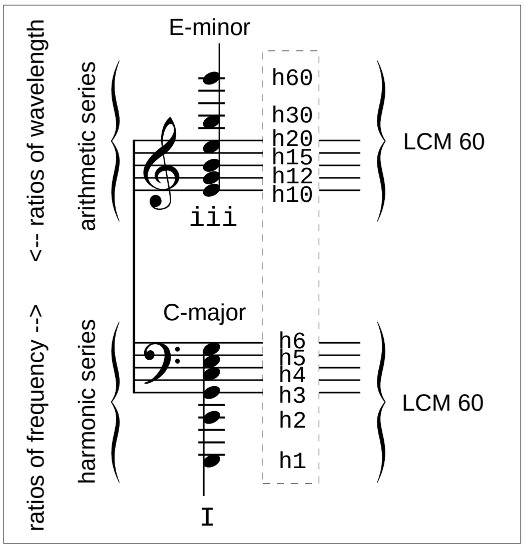


Figure 11.5 The ‘natural’ C-major and E-minor triads set within the wider context of an extended harmonic series.

THE PERCEPTION OF TONAL MUSIC

To digress for a while and venture upon the difficult terrain of the human perception of music – though being clear that the interpretation presented by the MOS model stands independently upon a mathematical foundation. Nevertheless, it might be useful to review a little of what is known about aural cognition and attempt some integration of the two strands. The following overview draws heavily upon James Beament’s most readable book *How We Hear Music*[3] but also differs from and extends his analysis in some regards.

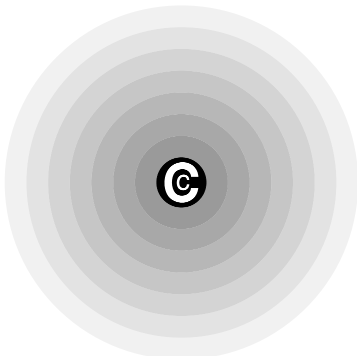


Figure 11.6 A schematic representation: a single complex tone consisting of fundamental and integral overtones creates the sensation of clear focused pitch (black center) with a ‘penumbra’ of tone quality attached (gray rings). The perception of pitch ‘C’ and rootedness ‘c’ coincide.

For a single note the ear easily identifies the repeating pattern of the period of the fundamental frequency (reinforced by the integer overtones) which is perceived as pitch. Also the configuration of overtones, their strengths and any significant frequency deviation from that of an ideal series is identified too, and conveyed as the tone color or timbre of the perceived pitch – Figure 11.6. (Beament gives timbre a particular meaning not adopted here.) The ear apparently resolves the lower overtones as more or less separate entities – about h2 through h10 – but as their envelopes of resonance overlap on the sound detecting membrane of the inner ear, they are signalled in combination, thus emerging as the generalised tone color component of consciously perceived musical sound. Although there is no clear dividing line, the higher harmonics beginning around h8, falling ever closer together, become increasingly difficult for the ear to detect, due to gross mutual interference. The random nerve impulses generated by this mutual interference appear to gradually overwhelm the ordered signalling of overtones. Such random patterns are the hallmark of noise. (This feature might also have a bearing on the apprehension of conjunctions between chords for the MOS model: generally, this mutual destructive interference between overtones in simple chords will still leave clear headroom in the 1000Hz to 4000Hz frequency band where the ear has greatest sensitivity – roughly the top two octaves of the piano keyboard – but in very complex chords it may prove somewhat more restrictive.)

But to what degree is it legitimate to separate pitch from tone at the level of perception? Overtones appear to contribute to the perception of pitch, as evidenced by the lower registers of many instruments, like the bassoon for example, where the fundamental harmonic is only weakly present and most of the energy in the objective sound is shared between h5, h6 and h7.[4] Nevertheless the listener perceives a strong and resonant fundamental! Equally, a poorly constructed instrument generating strong, ill-focused overtones does not produce a perception of clean clear pitched notes but rather a ‘pitch-band’, to use Beament’s term, as well as an infelicitous tone. The perception of a single fundamental pitch involves more than h1 alone – excepting tuning forks and scientifically generated sound. Though that said, the fundamental frequency of a single note does resonate a significant part of the ear’s detector membrane unhindered by overtones, thus signalling one pitch alone for that portion. Perhaps the perception of a single clear note, the perception of pitch with tone, should be thought of as the characteristic signature of one pure harmonic series. And one might ask, what does the ear identify in a chord? Do the notes of a triad act as three separate *pitches* or three powerful harmonics of *timbre*, or something of both?[5]

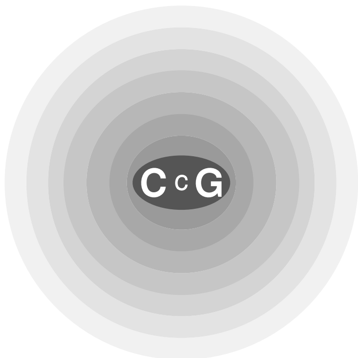


Figure 11.7 The interval of a fifth between two complex tones begins to ‘smear-out’ the strongly focused sensation of a single note into a perception that is somewhat more tone-like. This trend is especially marked if the component notes are commensurable in volume, tone and source (direction) so encouraging the ear to interpret them as a single entity. The sense of rootedness now falls below the interval (R-h1, C-h2, G-h3).

The ear–brain hearing system might usefully be divided into three stages, the ear mechanism, unconscious processing and conscious processing. Obviously, the system evolved long before the invention of tonal music, or speech for that matter, and its primary function was, and still is, to keep us safe, to warn of danger and give us an automatic omnidirectional awareness of changes in our surroundings – an acoustic early warning system. The ear harvests other information from the sounds that fall upon it beside pitch and tone color. It is very sensitive to the transient sound at the beginning of notes and to the volume of sounds. Plus, non-acoustic information from other senses, like sight, and expectations, assumptions, experience, etc., both conscious and unconscious, must all be taken into account. Most of the sound we hear and almost all of the sound that our hearing system was evolved to deal with consists of complex chaotically changing frequencies: transient frequencies, noises. With a few exceptions like some bird song, most natural sounds are perceived as containing little *fixed* pitch and tone information. A noise might be perceived as high-pitched, medium or low, or perhaps changing from one level to another, hollow, brittle, dull, etc., but change is the dominant feature. The ear did not evolve particularly to grasp settled pitch or tone because noises generally don't have simple settled frequency relationships of fundamentals or integral overtones – usually they are constantly varying. However there was, and is, an advantage in the ear and unconscious processors delivering a clear signal of possible danger at the conscious level of perception. That clear signal takes the form of a unified perception, a sound sensation with the attributes of direction, distance and character (volume, reverberation and patterns of frequency change). The ear and unconscious processors' particular strength is in matching up transients, from which they are able to distinguish between different simultaneous sources of sound and assign direction relative to the axis of the ears. Most natural sounds are composed almost entirely of transient frequencies and are recognised and remembered by their characteristic patterns of change. In contrast, musical sound generally consists of settled sustained frequencies recognised and distinguished by their unchanging frequency relationships. A fifth or triad is a categorical perception. Musical sound is an atypical subset of noise which contains a much reduced level of transients. Transients are generated at the start of a note's vibration, which is principally how the ear distinguishes between instruments and follows parts in contrapuntal music. Also, mini-transients are created by the random fractional instability in note frequency – a characteristic of sound generation in traditional instruments – this Beament terms 'timbre'. However, overall, musical sound is very different from natural noise, particularly in the integer relationship of its overtones, and our ears confirm that this is so. Regardless of this difference, the ear performs the same low level unconscious processing on musical sound as for noises – as far as the dearth of transients allows – and depending on how much information it has been able to glean, delivers one or more unified perceptions of a musical sound or sounds. Importantly, the sensation of commensurable musical sounds emanating from the same general direction will tend to be bundled together.

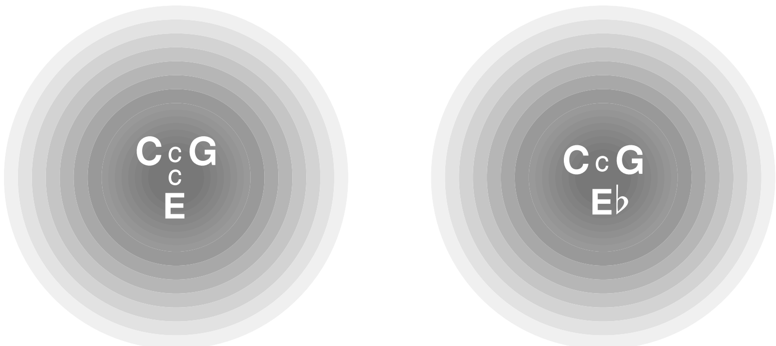


Figure 11.8 Three complex tones in the form of a triad further spread the perception of pitch, rendering the sensation predominantly tone-like in character. The sense of rootedness in the major triad is redoubled arising from both the fifth (R-h1, C-h2, G-h3) and the whole triad (R-h1, ... C-h4, E-h5, G-h6), while in the minor triad the fifth alone inculcates a sense of rootedness.

Depending on the amount of information available to the ear, the processes of aural cognition probably exhibit some degree of flexibility in the interpretation of aural stimuli. Three simultaneous notes of the triad with different transients, volumes and directions, like for example music played by a violin, clarinet and piano spread across a stage, might well be apprehended primarily as separate *packages of pitch*, three parts, each with an associated tone color. Though the musical listener will tend, unconsciously, to turn these separate sound sources into a perception of harmony – as well as enjoying the independent motion of the parts. The same three simultaneous notes played on the piano alone, thus produced with more or less similar transients, direction, volume and harmonic spectra, would, in contrast, tend to be perceived as one *package of tone color* (the sound quality of a triad) with the pitch element less clearly defined. Here the notes of the triad, which bear the frequency relationship of the harmonics h4, h5 and h6, in a major chord, are being combined, because like overtones of a single note, their resonances overlap on the ear's detector and thus are signalled as an intermingled group.[6] Somewhat akin to overtones; intervals, triads and chords have their individual pitches smeared out into something not unlike the perception of tone color – Figures 11.7/8. Again, equally, the musically experienced listener could unpick this package into parts, identifying the separate notes, though this does require an element of conscious effort. However, the overall impression created by a unified chordal sensation, is one of tone. The strident bare fifth chord C (h2) to G (h3) finds a match in the tone of the artificial 'harmonics' of an organ chorus with mixtures or the Quint registration: C-h1, h2, h4, G-h6 (eight, four, two, one-and-one-third foot), while the sweet major triad finds a parallel in the cornet registration: C-h1, h2, G-h3, h4, E-h5 (eight, four, two-and-two-thirds, two, one-and-three-fifths foot). For the tone quality of the minor triad, no matching flue or reed organ registration can be found, though interestingly, the harmonic spectrum of the bells in a zimbelstern stop, and that of the traditional European bell in general, do match the minor triad. Indeed, by nesting one harmonic series stepping in groups of five, inside a second fundamental series, the principal sequence of bell tones emerges: hum tone h5, fundamental h10, minor-third h12, fifth h15, nominal h20, major-third h25, upper fifth h30, minor-seventh h35 and octave h40. Pitch and tone are somewhat akin to the two faces of a coin: the harmonics of timbre bolster and color a single fundamental harmonic of pitch, while multiple simultaneous pitches in chords, resonating and overlapping each other on the ear's detector membrane, can take on something of the composite character of timbre. But what of the awareness of the whole chordal package, the identification of a *repeating pattern*, which, at the level of a single separate note is capable of generating a clear perception of pitch?

Given an E-minor triad (E-h10, G-h12, B-h15) just above middle C at 256Hz, the pattern of repetition, the interference pattern, would have a frequency of 32Hz, that is, the C-h1 fundamental of the triad's encapsulating series would repeat thirty-two times per second. However, when we hear a triad, minor or major, we do not apprehend a clear low pitch of 32Hz (or 64Hz for the major triad C-h4, E-h5, G-h6) analogous to the pitch of a single note – *but we do identify a root*. And the rootedness of chords, a real perception widely experienced, does possess a generalised pitch quality, here E for the minor triad and C for the major. (The nature and role of difference tones is discussed separately below.) What might be happening with the perception of chords could be somewhat analogous to our perception of noises, where there is only a rough sense of the pitch, yet a strong awareness of the sound's character and tone. However in contrast to the more normal fluctuating pitch of most noises in the subset of sounds we call chords the constituent frequencies are unnaturally stable and integral which produces a settled repeating pattern. These stable patterns are probably noticed at some level of aural cognition though not transmitted in the form of a pitch sensation. The overall repeating pattern, the fundamental, is there hovering like a shadow behind the scenes, and more explicitly, some of the harmonics of its series are present in the form of the objective notes/partials of the chord. However, this background, this broader awareness, assuming it does exist at some level of cognition, is only forceful enough, for whatever reason, to lend confirmation to our sense of the rootedness of dyads, triads and other more complex chords. This could operate rather as if a weak or meta-level reapplication of the extraction of pitch from broad harmonic spectra is occurring; and, as with the perception of a sound's direction, this recognition of the long-stop period in the sound is, likewise, silently appended to the explicit sensation of pitch/tone. This of course is guesswork and a guess that would seem to suggest that the root of the E-minor triad should be C and not E! But as shall be seen below, it is a step in the right direction, a step toward the root of the E-minor chord.

Attempting to produce a model of aural cognition at any level is necessarily a speculative exercise. To quote James Beament: "looking for a needle in a haystack is much easier than putting one in a cortex and finding useful evidence". Notwithstanding these words of caution, as the MOS model provides a blueprint for the processing of musical sound – in the form of mutable digit sequence exchanges – it is perhaps worthwhile attempting a brief sketch. Beament describes in some detail the nature of the coding of nerve pulses carrying information from the ear mechanism to the low-level unconscious processors. The salient point is, generally, that a range of individual nerves are involved in signalling a range of frequency information. Thus where the resonances of two or three or more frequencies overlap on the ear's detector membrane, nerves in that area will generate pulses, on a rather irregular basis, but linked to the various periods of resonance, which taken en masse equate to a metrical encoding of the sound's components. And though this metrical information is unlikely to be presented 'in phase', the repetition patterns of the integer components of harmonic series will be periodic.

This given, one might draw a rough analogy more than a model of the processing of musical sound, along the lines of an auditorium where each seat is provided with a small light connected to one of the nerves from the auditory pathway leading from the ear. Seating is arranged such that nerves that lie close together beside the detecting basilar membrane will be close together in the auditorium. The auditorium is full, and each individual is instructed to clap their hands once, each time their seat light flashes. The role of auditory perception is played by one individual standing on the stage listening to the aggregate of all clapping but also capable of focusing their attention on particular areas within the auditorium. I shall term this function, the sensor. The effect of this 'fuzzy focus' is to extract the lowest common multiple from areas of clapping.

As the signals arrive for a single note heard, particular areas in the auditorium will, taken overall, produce the clap rates of the individual harmonics detected, though no one contiguous group will be producing a single clap rate, save the portion of seats connected to the region of the detector vibrated by the fundamental alone. Thus the perception of tone color, the overtones, is melded together; however, in addition to the production of the tone color perception which could be thought of as the predominating subdivision(s) of the period (i.e. a meter) the overall repeating pattern of overtone claps will match with the clapping from the contiguous region signalling the fundamental only. This generates the perception of pitch. Even if the fundamental is weak or non-existent, sufficiently strong overtone claps may be able to produce the perception of a pitch in the sensor, by their combined period. For the perception of intervals, triads and chords, the same effects are at work but without any significant contiguous group of seats signalling one clap rate alone. Thus an overall perception of tone color also pervades the sensing of intervals and chords, with the exception of the octave interval which is more like that of a single pitch. Also there will be a period generated by intervals and chords (though often a rather long period) analogous to the pitch period, delineated by the repeating pattern of the sum of all claps, a period not directly perceived beyond some sense of 'rootedness' arising in the sensor – discussed further below.

For chord progressions, in this application of the MOS model, the crucial factor perceived between commensurable chord steps, like for example the dominant-seventh to tonic exchange, is that *in one area* *in the auditorium*, the seats receiving signals which overall delineate the clap rate of the conjunction frequency, *remain constant*. The clap rate does not change in this oasis of stability, being common to both harmonies (more strictly oases of stability, as conjunctions reappear at integer multiples within the range of hearing). Thus the chordal steps in tonal music are connected by a simple logic. The sensor has learned to scan all the seats searching for these continuities. By navigation via these stationary beacons, the ebb and flow of otherwise apparently unrelated aural perceptions can be tamed, classified and interpreted. Harmonic progression is rendered intelligible by the relationships of the mutable base number system.

The above overly simple sketch is but a brief and speculative excursion into largely unknown territory. There are two strands of thought being pursued in this chapter, the principal strand operating at the mathematical level of the interaction of waves in general, and a secondary strand working by implication from the first, concerning the much more intractable processes of the human ear and mind. The second strand is purely conjectural. And to conjecture a little further, perhaps one could argue for the site of the above described auditorium to be at the level of automatic unconscious processing. Such a location might then also account for the rather direct emotional response which music can undoubtedly illicit, acting along similar lines to the automatic responses to sounds indicating danger, surprise or contentment. Beyond the automatic processes in the auditory pathway the situation is even less certain. Neuroscience is in its infancy, with progress to date finding complex, distributed, adaptive processes, particularly at the higher levels of cognition. Hopefully, at some stage in the future a general understanding of the brain's processing of musical stimuli will become available. A current summary of the situation is: "Collectively, studies of patients with brain injuries and imaging of healthy individuals have unexpectedly uncovered no special brain 'center' for music. Rather music engages many areas distributed throughout the brain, including those that are normally involved in other kinds of cognition".[7]

AGGREGATION AND THE MINOR TRIAD

An interesting feature of any complete group of harmonic frequencies h1 to hn, of equal amplitude and uniform phase – in other words, a collection of waves of equal peak-to-trough displacement which all begin and end together – when brought together in an interference pattern, is that they describe exactly 'n' equal subdivisions of the fundamental period. For example, the rare chord of the first fifteen tones of the harmonic series would, under these conditions, subdivide its fundamental period into fifteen equal units – illustrated in Figure 11.9.

These fifteen subdivisions could be further grouped or aggregated into five groups of three oscillations (MBN $3_5\ 0_1$) or three groups of five oscillations (MBN $5_3\ 0_1$), the groupings corresponding to the harmonics E-h5 and G-h3, respectively. While this is a special case, it is useful because it highlights relationships. Also, as the processes of aural cognition do not register phase differences for the most part (Ohm's Acoustic Law[8]) – together with there being generally some level of equality in note intensities – any real situation, though less uniform, might well possess something of the regularity and proportionality exhibited in Figure 11.9 and the following graphs. As James Beament points out, irrespective of phase differences in the objective frequencies, the mechanisms of the ear–brain system identify underlying patterns; indeed his bar diagrams on pages 68 and 77 of *How We Hear Music*, chart relationships similar to the graphs presented below. (Figure 11.12 illustrates that even with phase shifts, underlying relationships still emerge.)

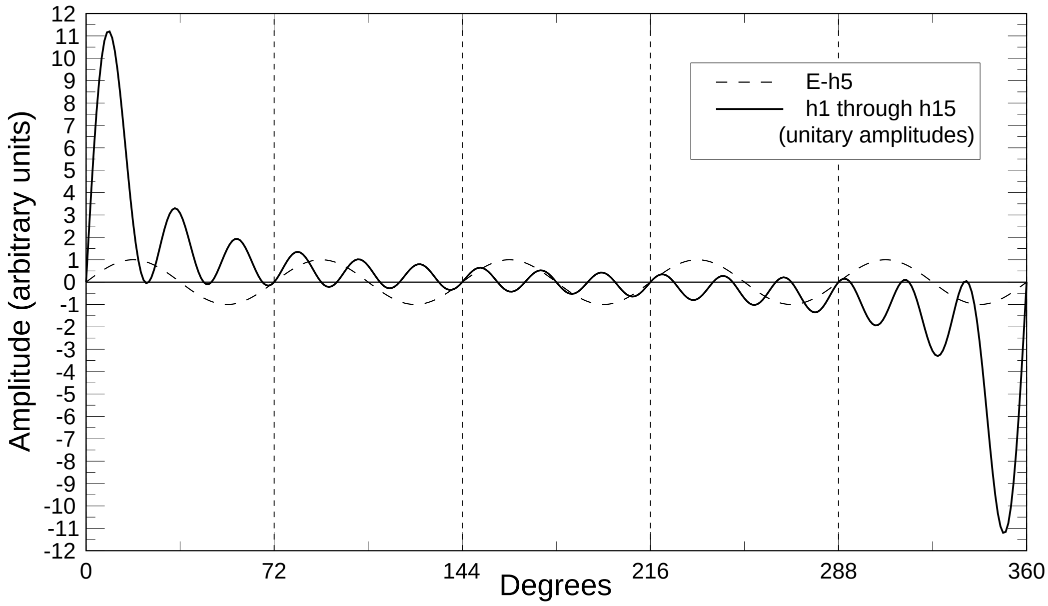


Figure 11.9 The interference pattern of all the harmonics from h1 through h15 (with equal amplitudes and uniform phase) – the rare and complex chord which encompasses the natural minor triad (Figure 11.4).

The Natural Minor Triad

Interestingly, if just the frequencies of the natural minor triad are considered in isolation (E-h10, G-h12 and B-h15 in this example) they likewise support subdivisions of five and three – within a given tolerance. However, this produces only twelve oscillations, aggregated as alternate groups of two and three cycles – three groups of two cycles plus two groups of three – in one full period for E-h5 (Figure 11.10), or a regular grouping of four cycles for G-h3 (Figure 11.11). A necessary feature of aggregated groupings, is that they are constituted of integral numbers of whole oscillations – complete cycles – so as to be supportive of resonances, though the aggregations themselves may contain uneven numbers of complete oscillations.

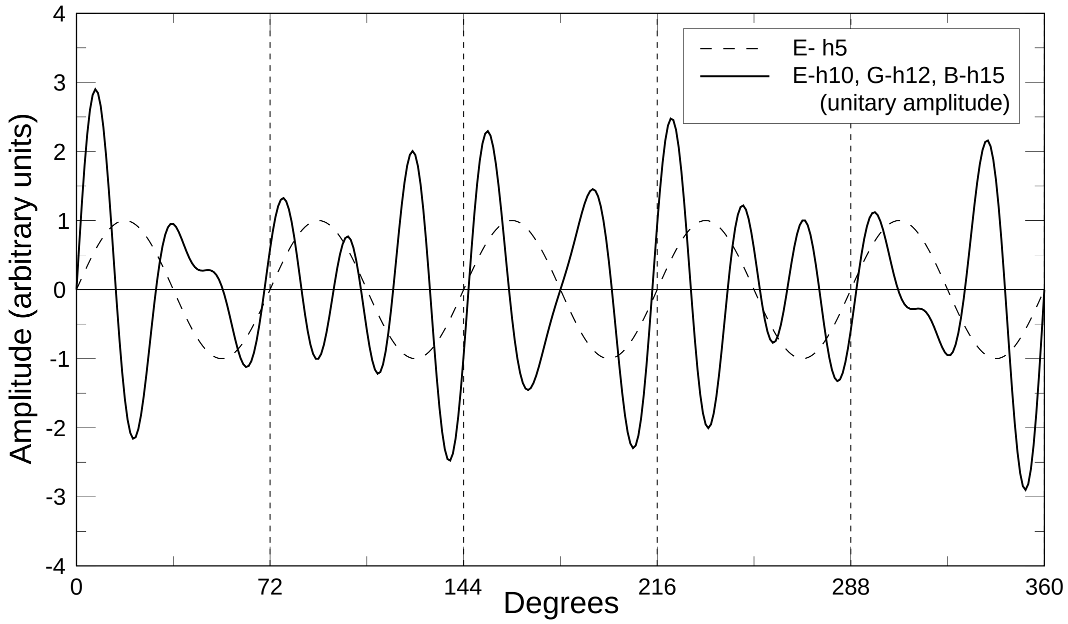


Figure 11.10 The interference pattern of an E-minor triad (notes E,G,B) supports internal subdivisions of five: E-h5.

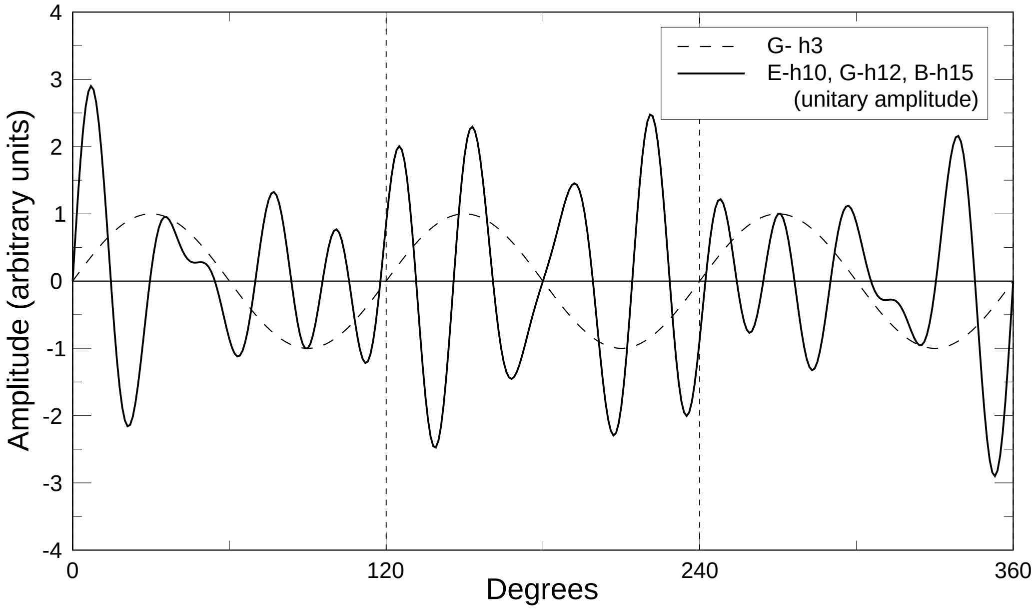


Figure 11.11 The interference pattern of an E-minor triad (E, G, B) supports internal subdivisions of three: G-h3.

Where more than one subdivision of the overall period is possible – as here with both subdivisions of five and three, plus two and a questionable four – the highest number of aggregations, E-h5 (i.e. highest frequency/energy), would probably emerge as the most prominent. That is because this arrangement, set at the bottom of a nested structure, would represent the lowest energy or ground state configuration for the system considered overall. However, the rather veiled tone of the minor triad perhaps stems from a shadowy perception of the period's subdivision into three aggregations of four complete cycles – G-h3. Also, the uneven numbers of oscillations within the period of E-h5 might, perhaps, contribute to its prominence, as well as 'locking' it into the period of C-h1, with the repeated sequence of 2-3-2-3-2 cycles.

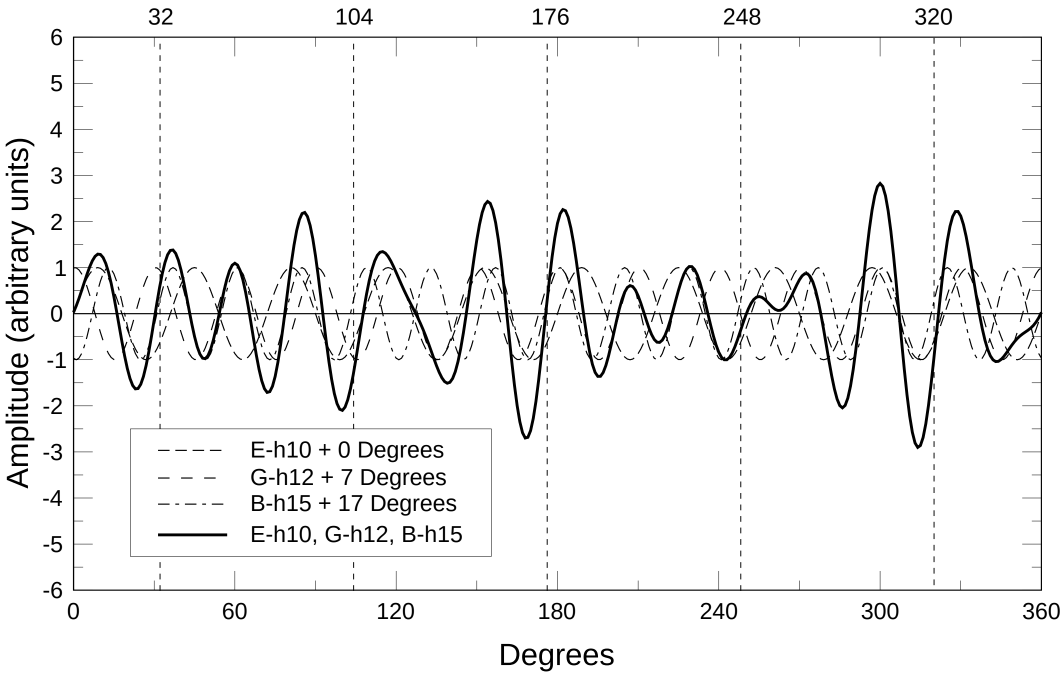


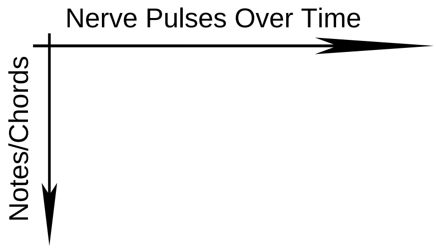
Figure 11.12 The interference pattern of the minor triad with randomly chosen phase shifts for frequencies E, G and B, maintains aggregations spaced at approximately 72 degrees, equivalent to the period of E-h5 (unit note amplitudes).

Nerve Pulse Streams

Now continuing the digression into the perception of tonal music. The nerve pulse relationships illustrated in Figure 11.13 are expressed in the form adopted by James Beament[3] (pages 68, 77 and 159) and are shown in uniform phase for clarity – though as made clear in Figure 11.12, phase differences between constituent frequencies does not alter the period of repetition. Also for clarity of sight in Figure 11.13 the horizontal time scale for the note G (nominally G above middle C) is kept constant for all chords, with the exception of the augmented triad Ex.8. To the left of each diagram the objective notes of the chord are written adjacent to their pulse streams and to the right of these schematic pulse streams their positions within one overriding harmonic series are shown. Tracing these harmonic series back to their fundamental frequencies yields the chord's period of repetition – in the form of nerve pulse structures. In MOS terms these all embracing harmonic series correspond to *middle-level nested series*, and there appears to be connections, or at least some parallels, between the MOS model of mutable numbers and the structure of nerve pulses in the auditory pathway. When discussing the structure of these pulse streams Beament states on page 159: *"the recognition of a major triad is the relationship of the chordal repetition rate to that of the components: ... Each kind of chordal sensation which can be named: major, minor, dominant seventh and so on, has a different repetition pattern related to the repetition rate of its components. ... The patterns are the only things which provide the similarities and the differences."*

The first example in Figure 11.13 is the perfect fifth dyad C-h2 to G-h3 which produces a repetition period equal to C-h1. The fifth interval – 2:3 ratio between nerve pulse trains – is the most basic and unequivocal signal that a harmonic series is present, and therefore also that a root may be divined in the form of the series' fundamental period. The more basic octave interval is somewhat ambiguous in that the octave's period of repetition is the same as the lower note's pitch period, and additionally, the upper note might be subsumed into the even numbered harmonics of the lower note. As discussed below, arguably, the interval of a fifth within a chord is the most significant element for the perception of a root to emerge. It is the first interval in the harmonic series that creates a differentiation between the pitch periods of the objective notes and the repetition period of those same notes in combination.

The second example contains the pulse streams of the major triad (CEG) and by the addition of a minor-seventh interval the dominant-seventh chord (CEGA#). Here the period of repetition for both chords is double that of the perfect fifth, however in both chords the embedded fifth interval C–G introduces the possibility of a nested repeating pattern at one-half of the full repetition rate – indicated by an asterisk in Figure 11.13. These two chords effectively have one root (C-h2) nested within another root (C-h1) and so evince a strong feeling of rootedness. Which is also to say that they have one harmonic series nested within another – for what is the rootedness of chords but an awareness of the relationships of the harmonic series within the structure of sound stimuli. The third example is of a diminished triad (EGA#), which is to some degree a form of abbreviated dominant-seventh, and it has the same repetition rate as the chords in Ex.2. Lacking a perfect fifth in its make up, the diminished triad poses the question of to what extent this full repetition period alone is able to support a sense of rootedness. Depending upon the surrounding context, the listener may or may not feel the chord to be an incomplete dominant-seventh, and so, may or may not thereby declare it to possess a root. Taken by itself, the equal symmetry of two minor-thirds (E–G, G–A#) allows for no precedence to emerge from the relationships between the pulse streams generated by the notes themselves and the diminished fifth (E–A#) finds no short harmonic series nested within the overall series to reinforce the rootedness of the fundamental C-h1 – as in the examples of the major triad and dominant-seventh. The rootedness of the diminished triad is marginal at best.




```
   |<---period--->| C-h1 Root
C  |      |       | C-h2
G  |    |    |    | G-h3
```
Ex.1 Perfect Fifth Interval

```
   |<----period of repetition--->| C-h1 Root
   |              *              | C-h2 Root
C  |      |       |       |      | C-h4
E  |    |     |     |     |      | E-h5
G  |   |    |    |    |    |     | G-h6
A# |   |   |   |   |   |   |     | A#h7
```
Ex.2 Major Triad and Dominant-Seventh Chord

```
   |<----period of repetition--->| C-h1 Root?
E  |    |     |     |     |      | E-h5
G  |   |    |    |    |    |     | G-h6
A# |   |   |   |   |   |   |     | A#h7
```
Ex.3 Diminished Triad

```
   |<------------------ period of repetition ----------------->| C-h1
   |           *           *           *           *           | E-h5 Root
E  |     |     |     |     |     |     |     |     |     |     | E-h10
G  |    |    |    |    |    |    |    |    |    |    |    |    | G-h12
B  |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   | B-h15
```
Ex.4 Minor Triad

```
   |<------------------ period of repetition ----------------->| C-h1 Rootless
E  |     |     |     |     |     |     |     |     |     |     | E-h10
G  |    |    |    |    |    |    |    |    |    |    |    |    | G-h12
A# |   |   |   |   |   |   |   |   |   |   |   |   |   |      | A#h14
C# |   |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |       | C#h17
```
Ex.5 Diminished-Seventh Chord

```
   |<-- period of repetition --->| C-h1
   |         *         *         | G-h3 Root
E  |     |     |     |     |     | E-h5
G  |    |    |    |    |    |    | G-h6
A# |   |   |   |   |   |   |     | A#h7
D  |  |  |  |  |  |  |  |  |     | D-h9
```
Ex.6 Half-Diminished Seventh / Minor Chord of the Added Sixth

```
   |<------------------ period of repetition ----------------->| C-h1
   |              *              *              *              | C-h4 Root
E  |           |           |           |           |           | E-h5
G  |    |    |    |    |    |    |    |    |    |    |    |    | G-h12
C  |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   | C-h16
```
Ex.7 First Inversion Common Major Chord

```
   |<--------------------  period of repetition ------------------>| C-h1 Rootless
C  |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   | C-h16
E  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |   | E-h20
G# | | | | | | | | | | | | | | | | | | | | | | | | |              | G#-h25
```
Ex.8 Augmented Fifth Triad

Figure 11.13 A diagrammatic representation of the repetition patterns of nerve pulse streams generated by the common chords of tonal music in the auditory pathway. It is generally presumed that the structural relationships between the pitch periods of the individual notes and the combined period of repetition underlies the characteristic sensations of different chord types.

The fourth example in Figure 11.13 delineates the pulse streams of the minor triad (EGB), which is of particular interest in this chapter, and here it can be seen that the period of repetition doubles yet again. Thus if the repetition rate of the diminished triad struggles to support a sense of rootedness, how much more must this be the case for the minor triad; empirically our ears confirm that this is so because the root of the E-minor triad is E and not the C-h1 repetition period of the minor chord's component notes. As with the major triad and the dominant-seventh chord, there is a harmonic series nesting within the all embracing period of repetition series but this time the 'short' series is not octave aligned with the absolute fundamental C-h1. The short 'abbreviated' series that emerges from the interrelationships of the pulse streams is built on the perfect fifth E-h10 to B-h15, and this produces the awareness of a root or 'nested fundamental' at E-h5 of the overall period of repetition. (In MOS terminology this 'abbreviated' series is a *top-level aggregated series*, held within the broader confines of the *middle-level nested series*, which in turn corresponds with the period of repetition series.) As can be seen by the spacing of asterisks, the root E-h5 is of an amenably short period and therefore more easily accessible to aural cognition than the distant overall repetition period. Also the objective note G, the minor third interval, is neutral in its effect upon the aggregated E-based series – in the sense that it is non-dissonant. The overall structure of nerve impulses generated by minor triad illustrates how the question of rootedness is intimately entwined with the core MOS concept of nesting harmonic series within each other.

Example five shows the nerve pulse structure of the rootless diminished-seventh chord (EGA#C#). Here again the period of repetition is extensive. However, though the three components of the diminished triad (EGA#) align themselves at the half period, what hints of rootedness that might arise from their incomplete dominant-seventh structure is counteracted by the semitonal dissonance of the last component C# against C-h1. In this regard the effect of the C# contrasts with the benign influence exercised by the minor third interval in the minor triad. No root is divined as aural cognition fails to find a viable short structure close to hand, while the complexity and length of the notes' combined pulse train leaves them too remote from the all embracing period of repetition (C-h1) for this to provide an effective root.

The sixth example is the half-diminished seventh chord (EGA#D) which may equally be re-spelt as a minor chord with added major sixth (GA#DE). Again as in the previous example the core of the half-diminished seventh chord is a diminished triad, but here, contrastingly, the fourth component note D produces a less disruptive pulse stream. Indeed, rather than adding to the ambiguity of the diminished triad, the pulse stream of note D turns the relationships around by forming a perfect fifth interval with the note G, so allowing aural cognition to construct an aggregated series built on G-h3 of the overall period of repetition. As with the minor triad, the period of this aggregated series (G-h3, G-h6, D-h9) is short and accessible; shorter than E-h5 of the minor triad, and though the all embracing period of repetition is half that of the minor triad, the root G-h3 is strong enough to override the weakly competing pull of C-h1. Upon the basis of this analysis of nerve pulse streams, it would appear that the more accurate of the two names attached to this chord is 'minor chord of the added sixth', or more simply 'minor sixth chord'. Finally, an additional point of interest is revealed if the pulse stream of note E is removed from the example altogether: the arrangement, sans E, forms an alternative minor triad structure of component pitch periods (GA#D) set against the background of the overall period of repetition. Effectively, within bounds of tolerance, there appear to be two pulse stream structures capable of producing the minor triad sensation – which for comparison can be multiplied out into: Ex.4 based on the 'period 5' :30:36:45 and Ex.6 based on the 'period 3' :30:35:45. It would be interesting to see if an experiment could be devised that might test this supposition.

Having opened up the topic of alternative pulse streams, example seven illustrates the effect a different arrangement of notes can have upon the relationship between the constituent pitch periods and the all embracing period of repetition. By rearranging the notes of a major triad into a first inversion configuration (EGC) the period of repetition is doubled in length (and if E were an octave lower it would quadruple the length). Nevertheless the hearer can still identify the chord as major, though also perceiving a ‘hollowness’ in the sound. This ‘false bass’ effect can perhaps be attributed to the bottom note (E) lying outside the aggregated series that can be created from the objective fourth interval (G–C). The root or fundamental frequency of this aggregated series (C-h4) is indicated by asterisks in the example. As clearly the low frequency of note E-h5’s pulse stream cannot find a place in this aggregated series, it is left hovering in the shadowy series based upon the period of repetition C-h1 – sounding half-right and half-wrong – and very willing to move on to a root position a soon as the harmony allows. Although the aggregated series probably helps to bolster the octave related repetition period series, as will the low E-h5, Ex.5 shows that long periods of repetition don't easily translate into a firm sense of rootedness. Indeed, in Ex.5 no meaningful root is sensed, however here in the case of a standard first inversion chord, the sense of rootedness is balanced between C-h4 and C-h1, while both are also somewhat obscured by E-h5 sounding in the bass.

The last example in Figure 11.13 illustrates the pulse streams of the rootless augmented triad (CEG#), which take the period of repetition out to new lengths, and although the major third (C–E) could presumably provide the seed around which an aggregated series could develop – in a similar fashion to the perfect fifth and fourth in previous examples. However, as with the diminished seventh chord, here again there is semitonal dissonance between the objective note G#h25 and the implied G-h12 of the putative aggregated series built on C-h4. In the absence of an aggregated series there is little prospect that the long period of repetition could effectively generate a perceptible root.

The mode of visual representation employed by James Beament to illustrate the pulse streams generated by intervals and chords in the auditory nerve appear to produce an analysis remarkably close to the structures developed in the modulating oscillatory model of mutable numbers. In particular the identification of the period of repetition of pitch simultaneities as a significant feature of chordal sensation runs parallel to the MOS conception and application of *nested harmonic series*. Essentially, these periods of repetition equate to the fundamental periods of *middle-level nested harmonic series* in MOS terms. By reproducing and extending Beament’s diagrams in Figure 11.13, hopefully, the closeness of these two approaches is made manifest. The extension of Beament’s nerve pulse diagrams employed in Figure 11.13 has involved the introduction of *top-level aggregated series*, based upon the presence of incomplete ‘configurations of partials’ amongst the pulse streams generated by the objective musical sound. For example, a prominent interval such as a perfect fifth, fourth or major-third might induce aural cognition to identify a shorter and more accessible period as the root of a chord in preference to the more distant and inaccessible full period of repetition – as in the minor triad, Figure 11.13 Ex.4. A general consequence of such a partitioning of the period of repetition into aggregated sub-periods would be the introduction of a metrical division in the overall period. While all dyads/intervals would have a unitary meter (Ex.1), major chords would tend to favor the power of two meters such as duple, quadruple, etc. (Ex.2 and Ex.7) and minor chords the odd-numbered meters such as triple, quintuple, etc. (Ex.6 and Ex.4). Rootless chords such as the diminished-seventh chord and the augmented triad (Ex.5 and Ex.8) would effectively be meterless – technically their long periods of repetition would form a unitary meter but as aural cognition delivers no sensation of rootness, presumably for this purpose, these periods lie beyond its reach.

Aggregations and Difference Tones

The combination tones and other similar physiological-cum-psychoacoustic phenomena are fraught with controversy. Some scholars doubt the existence of Helmholtz's summation tones, others classify them as distortion effects. However, out of all these phenomena, the difference tones, the easiest to hear, are generally accepted, though their precise cause may be disputed. For the minor triad, the lowest frequency difference combination tone formed from the objective notes E-h10 and G-h12, is C (h2), not E (h5), though if B-h15 is included, the additional difference tones G-h3 and E-h5 are also generated. These three difference tones potentially define three modes of aggregation for the overall period of repetition of the harmonic series carrying within its higher ratios the objective minor triad: a subdivision of two oscillations – C-h2, a subdivision of three oscillations – G-h3 and a subdivision of five oscillations – E-h5 (the latter two within a given tolerance). Also there is the full period or repetition pattern of the combined notes, of course, matching C-h1 an octave below the lowest difference tone – C-h2 (Figure 11.16). The subdivision of two oscillations (C-h2) is illustrated in Figure 11.15, and in Figure 11.11, subdivisions of three oscillations (G-h3) are graphed.

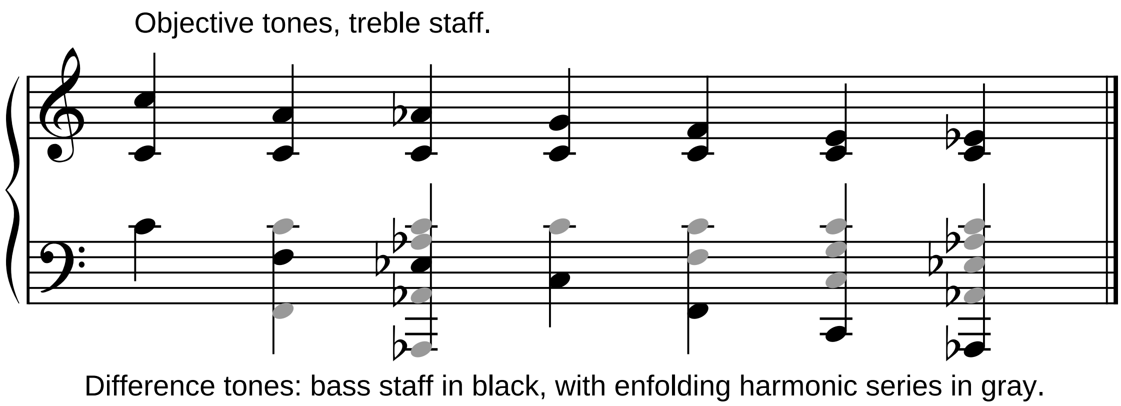


Figure 11.14 The most prominent difference tones set within the context their enfolding harmonic series. Third from the left is the 'counter example' of the minor-sixth, which doesn't define a root note. Also on the right, the crucial interval of a minor-third, C to Eflat, 'points to' the root of a series founded on Aflat, not its perceived root note C.

Difference tones often 'point to' the perceived roots of chords – any two adjacent overtones create a difference tone equal to the fundamental of the series. However, for reasons associated with the difference tone generated by the interval of a minor sixth (e.g. C–Aflat), difference tones are probably not the whole story concerning the creation of rootedness in chords. The interval of a minor sixth will generate a difference tone a major sixth below the lower of the two notes (i.e. interval C to Aflat generates Eflat below). The relationship between these notes is that of h3, h5 and h8 of the harmonic series, unlike the other prominent difference tones which reflect their fundamental tone. That is, apart from the minor sixth and minor third, the common difference tones do otherwise generate the fundamental note letters of their series and so would seem at least to run parallel to the rootedness-of-chords phenomena. For example, the interval of a major sixth will generate a difference tone a fifth below the lower of the two notes (i.e interval C to A generates F below in Figure 11.14) which yields the series F-h2, C-h3, A-h5. However, in the counter example of the minor sixth, the generation of the difference tone Eflat-h3 alongside the objective tones of C-h5 and Aflat-h8, produces the perception (at least for me) of an Aflat major chord with a root note of Aflat lying below the difference tone Eflat. Therefore, in this case the rootedness of the objective notes is not directly produced by the process that generates difference tones, wherever that process resides in the hearing system. Nevertheless they may be considered at least a contributing factor or perhaps a trigger, and, in the E-minor triad under discussion here, as difference tones are found to be pointing to three possible candidates – C, G and E – they clearly do not isolate a unique root. Yet still this is some advance on the situation, in that now as well as the period of repetition C-h1, E-h5 and G-h3 have also entered the field of 'rootedness'.

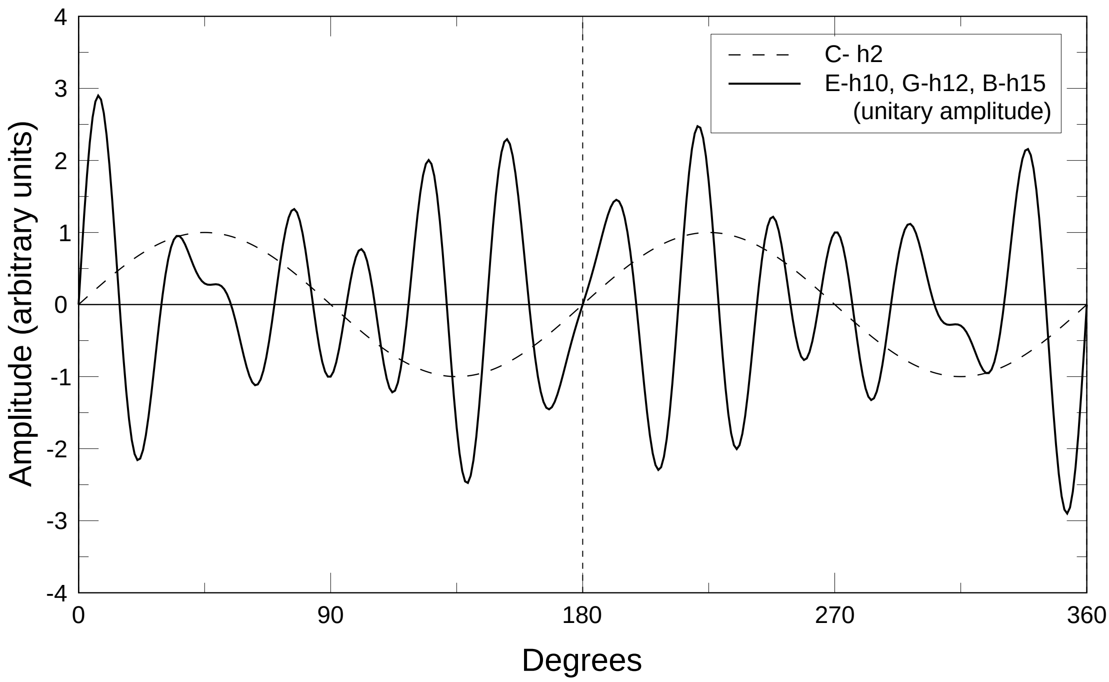


Figure 11.15 The interference pattern of an E-minor triad (notes E, G, B) supports internal subdivisions of two, C-h2 the period of the lowest difference combination tone.

To recapitulate a little upon the problem of the rootedness of chords, the crux of the matter is that our ears and mental processes, searching for intelligibility within the complex pattern of sound generated by the objective E-minor triad, with the particular aid of the period of repetition and the difference tones might seek out an accommodation between the complex extended series implied by the 10:12:15 frequency ratios found in the minor triad, and the very attractive low energy configuration of the ratio 2:3 (10:15) presented by the outer pair of notes – the fifth E to B. The accommodation consists of nesting a short and less complex low energy series based on E-h5 within the broader context of a harmonic series based on C-h1. (This argument would also apply to the alternative arrangement of nesting the relationships of the minor triad based on G-h3, illustrated in Figure 11.13, Ex.6.) Essentially this approach hints at a recursive application of the hearing system's procedure of sifting out relationships that fit into harmonic series and then combining them into a single perception. For clearly the tenacity with which aural cognition correctly categorises ill-tuned fifths, fourths and major-thirds indicates its strong inclination in this regard. The ear's tolerance, which stretches far beyond the small adjustments imposed by equal-temperament, is a testament to the importance these simple relationships play in cognition's ability to understand and interpret the flow aural stimuli. However, the more complex relationships of an extended interpretation of the minor triad's pitch relationships – E-h10, G-h12 and B-h15 – are perhaps rather too distant for the processes of aural cognition to readily grasp, especially when such a forceful alternative is to hand in the first three ratios of a series built on E-h5. The hearing system chooses the *simplest series*, based on a *short period* of repetition – the series founded on E-h5, *and from this choice we divine the root of the minor triad*. Nevertheless, the ambiguity inherent in the ratios does not go unnoticed; our perception of the minor triad is less stable and less satisfactory than the major triad, as demonstrated by the early adoption of the tierce de picardie cadential transformation,[9] early in the tonal era. This instability might perhaps be the hallmark of two series in contention. The extended context of a 'C-major' harmonic series could operate in a similar manner to our sense of key. For example, in the key of C major the dominant chord, G-major, for the most part is clearly heard as possessing the root note G, without this necessarily diminishing our sense of being in the key of C major. Equally, the broader context of an underlying harmonic series built on C-h1, could form a background against which the forceful objective E-minor triad's relationships are 'computed' – and if not within the mind itself, at least in terms of the abstract mathematical scheme of mutable base numbers. Indeed, though it is another strand of the story, in the MOS model, the movements of the underlying harmonic series which form this background – i.e. the motion of the middle-level nested harmonic series induced by a succession of harmonies – themselves define a sense of key, the *fundamental nesting series*.

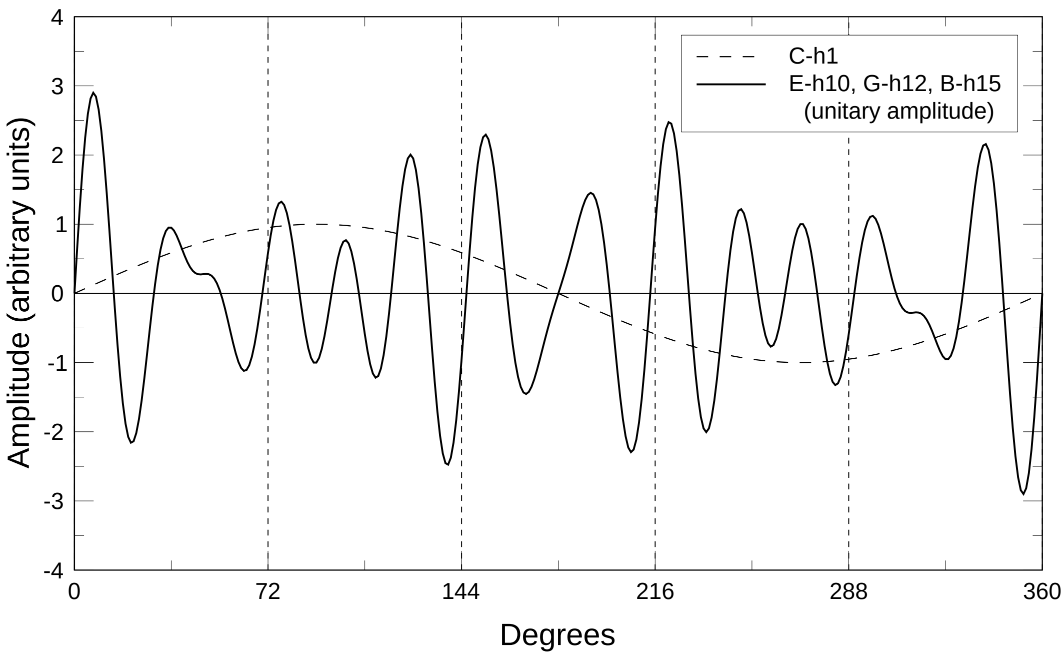


Figure 11.16 The overall period of the interference pattern of the natural minor triad is C-h1, the period of repetition.

But can this extended harmonic context truly exist in the mind, as a real part of the processes of aural cognition, given the strength of the perception of an E-rooted, and not a C-rooted triad? Perhaps our processing of the minor triad penetrates no further than the shortest period of repetition? A definitive answer to this question must probably await further developments in the field of neuroscience. Nevertheless, the simplicity and consistency of the MOS model and mutable base numbers may perhaps give some grounds to believe that an approach to the mysteries of aural cognition, along these or similar lines, might prove worthy of pursuit. Indeed, in James Beament's analysis of the mechanics of musical hearing discussed above, the period of repetition – which equates to the fundamental tone of a *middle-level nested harmonic series* in a modulating oscillatory system – is recognised as a real objective element in aural cognition.

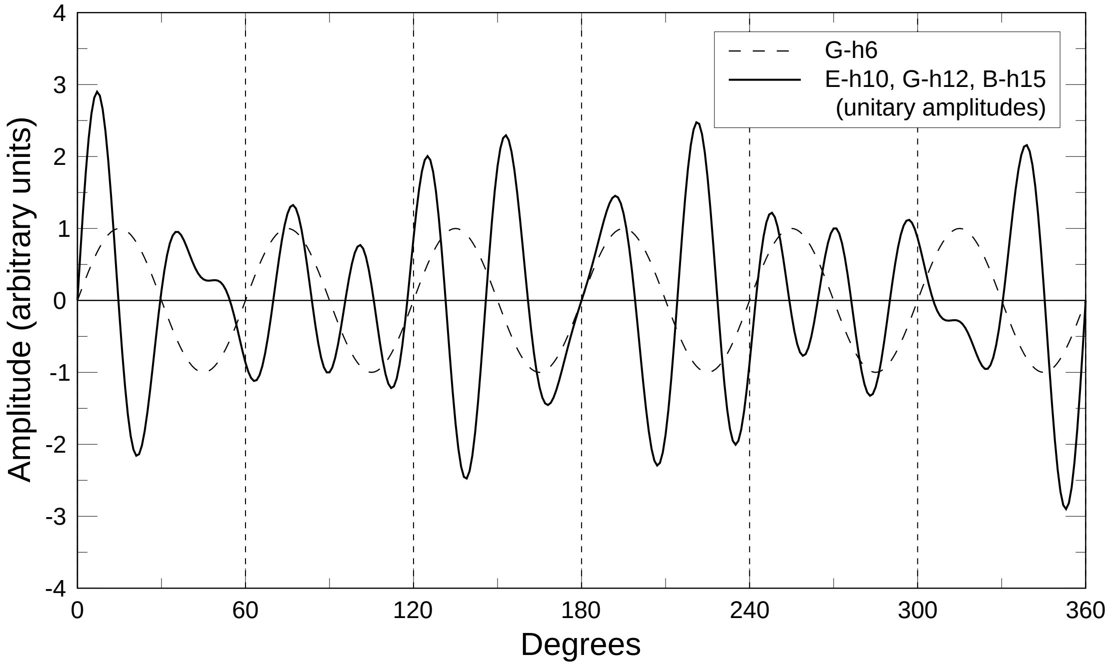


Figure 11.17 The interference pattern of a natural E-minor triad and G-h6 are in almost complete anti-phase at 60 and 300 degrees, destroying any possibility of matching complete oscillations to the pattern to G-h6.

It is interesting and perhaps significant that the run of consistent periods of aggregated oscillations within the E-minor triad's interference pattern stops at E-h5: G-h6, A#-h7, C-h8, etc. are not well supported. For example in Figure 11.17, G-h6 doesn't meld, in complete cycles, with the interference pattern around 60 and 300 degrees, though matching well on either side of 180 degrees.

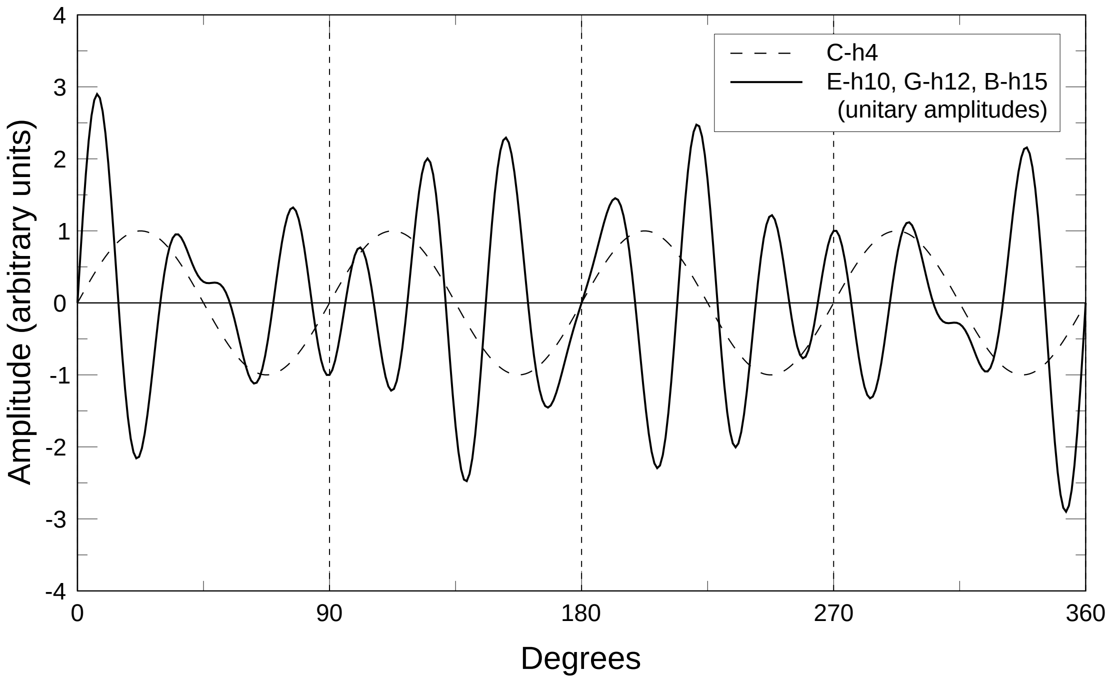


Figure 11.18 The questionable match of C-h4, with a discrepancy of five degrees at 90 and 270 degrees.

However, C-h4 with a five-degree discrepancy between the interference pattern at 90 and 270 degrees (Figure 11.18) is somewhat less wayward than the nine-degree discrepancy of G-h6. Notwithstanding the possibility of other scenarios, the main point is that the E-h5 resonance makes the uppermost clear match with the interference pattern generated by the objective tones of the E-minor triad. And though under normal conditions of transmission to the ear, the waves will form a complex pattern through the processes of reverberation, absorption, etc. (see Figure 11.12), the ear sifts and sorts this jumble, and in combination with the processes of aural cognition appears able to reconstitute something akin to the simple relationships illustrated in the graphs. Whatever the detailed mechanisms of musical cognition are eventually revealed to be, it is likely that they will involve processing acting over the whole frequency range accessible to the ear, a range spanning up to ten octaves, and, no doubt, will include both innate and learned elements. Current ideas and models are necessarily provisional.

The Minor-third

A most crucial note/interval in all of this is the minor-third, G-h12 in the E-minor chord, which, working in conjunction with the period of repetition C-h1, the lowest difference tone C-h2 and the twelve oscillations of the overall period, possibly encourages the ear and mind to entertain the existence of an extended ‘background’ harmonic series from C-h1 up to, and beyond, h12. However, the period of such an extended harmonic series is prone to break down into subdivisions or aggregations – as decreed by the law of entropy increase, the all-pervasive second law of thermodynamics. In the case of the minor triad, groups of five, the most energy efficient arrangement, become interwoven into the fabric of the underlying series built on C-h1 – Figure 11.19.

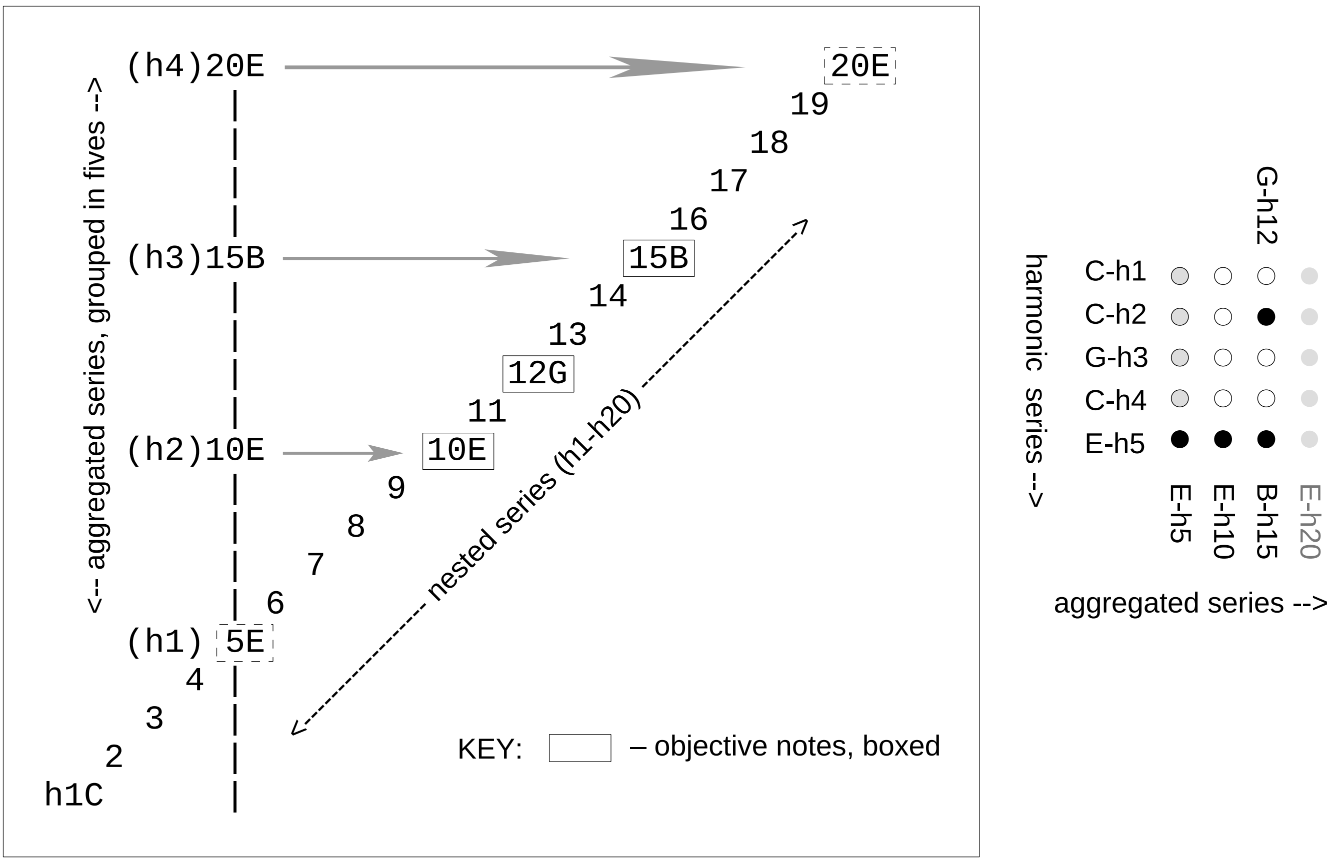


Figure 11.19 The E-minor triad/chord embedded within an extended harmonic series based on C-h1 and supported by an aggregated series grouped in fives.
Number pattern on right for value fifteen: MBN $3_5\, 0_1$ (and in gray MBN $4_5\, 0_1$).

A similar breakdown could also apply to a major triad too, where the configuration of objective notes sets the chord at a higher position within its series, but as the upper series would be based on C-h2, C-h4, etc., the perceived root would not be changed as is the case with the minor. (Though arguably the G-h3 based arrangement of the minor triad illustrated in Figure 11.13, Ex.6 is more economical, it is however a less accurate portrayal of the natural minor's internal relationships.)

Also, without this particle of 'grit', G-h12, lodged in the background nested series, the aggregated series could go on itself to form an 'E-major' series – E-h5, E-h10, B-h15, E-h20, G#-h25 and so on – releasing energy/complexity, by destroying the nested series, so as to become itself effectively a nested 'major' series itself, built on 'H5n' of the fundamental nesting series. It is the frequency G-h12 in the pattern of the E-minor chord's notes that holds the middle-level nested and top-level aggregated series apart, maintaining the separation that allows this dual processing to occur. The note G-h12 cannot be accommodated within the aggregated series where the other objective tones reside, but forces another 'compatible' series into focus – the series based on the overall period of repetition. Thus the aggregations or groups of five harmonics (carved out of the nested series), form a skeleton E-based aggregated series – E-h5, E-h10, B-h15 and E-h20 – within which G-h12 from the underlying nested series perches (displacing the aggregated series' rightful progeny G#-h25), like a *cuckoo* in a sparrow's nest.

Conclusion and Example Chord Progression

Whether considered purely as an abstract mathematical model, or more tentatively, as a possible template for the aural processing and cognition of musical stimuli, the overall effect of the scheme of aggregation within nested series described above leads to a superimposing of the minor triad (and key) upon the relationships of an underlying natural 'major' harmonic series. Through this adaptation, the theoretical minor/arithmetic principle of traditional dualistic theory, could possibly be embedded within the real relationships of an ascending 'major' overtone series, and thus perhaps, a beautiful though now neglected idea might draw new breath in *the context of dual nested harmonic series.*

A hint or clue to this duality of nested relationships lies is the instability of the dominant chord in the minor key: the chord of E in the key of A-minor. The magical effect of the E-major chord replacing the normal E-minor harmony, that we feel and respond to, could be explained by the processing of minor chords within surrogate 'major' series. The delight we take in the sweet transformation to a major third above the root – which represents a reduction in stress and complexity further enhanced by the underlying (nested) 'major' series shifting its ground from H4 to H5 of the fundamental series – is a real clue to the dynamics of the minor triad. As an example of this relational dividend, the replacement of the 'meta-stable' E-minor chord (in the key of A-minor) with the more stable and less complex configuration of an E-major chord is illustrated in Figure 11.20, where the 4:3 (2:3) relationship of the objective tonic-dominant chord progression of A-minor to E-major, at the level of aggregated series, provokes the underlying nested harmonic series to make a 4:5 (major-third) adjustment to accommodate the objective note E-160.0Hz, which the series based on F-21.33...Hz does not possess. This example is taking us into the arena of dynamic *Modulating Oscillatory Systems.* A fuller description of the terms and concepts of the MOS model will have been encountered in Chapter 9, and in Chapter 12/Example S these procedures are applied to a complete example composition: the first Prelude from the Well-tempered Clavier, written by J.S. Bach.

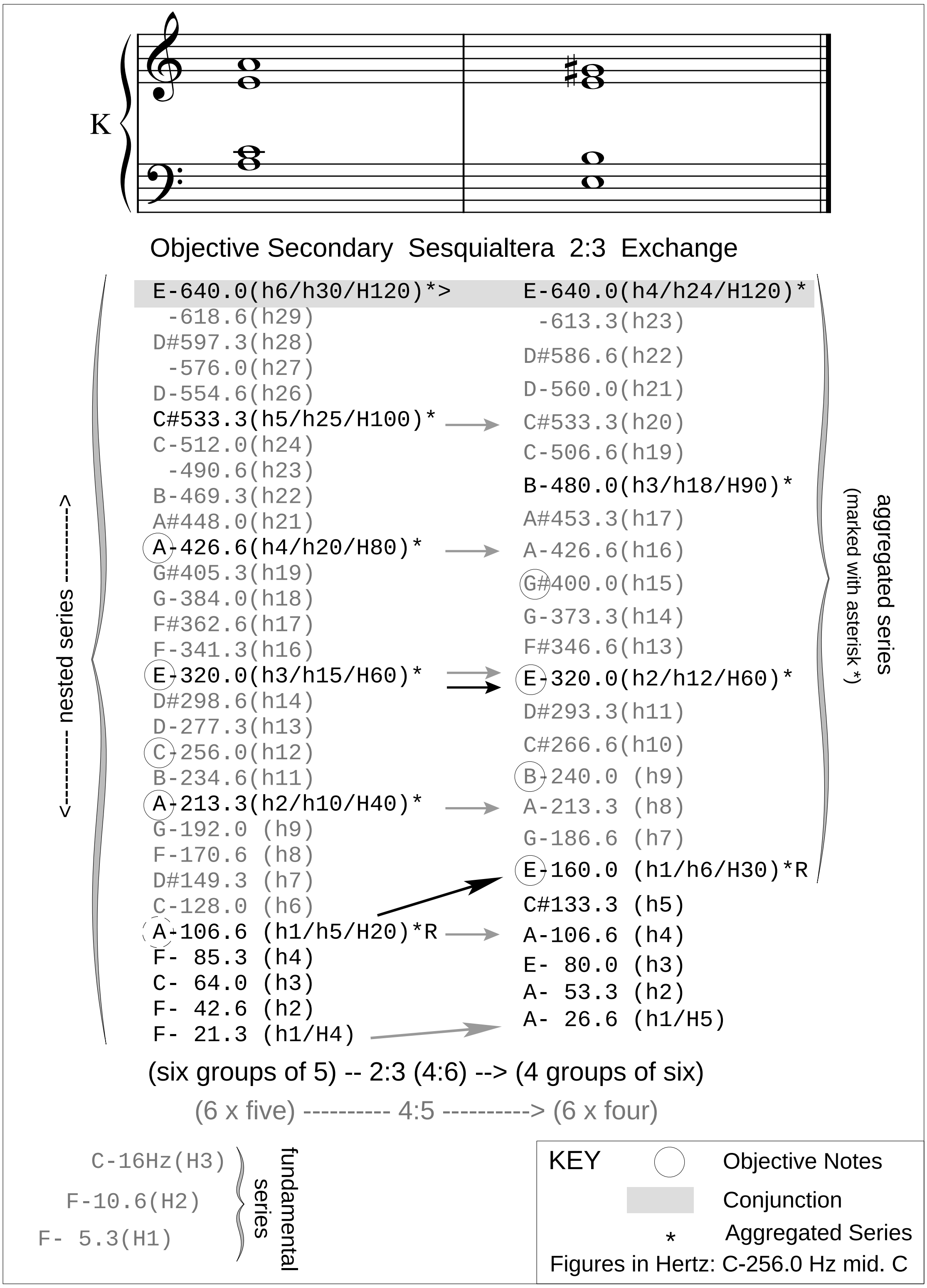


Figure 11.20 On the surface level of objective tones, a 4:3 (2:3) tonic-dominant progression with a fourth/fifth root movement, sets in motion an underlying 4:5 step of a major-third at the lower level of the nested harmonic series – from F-22.3(h1/H4) to A-26.6(h1/H5). In mutable numbers this chord progression of A-minor to E-major would be represented by the two digit sequences: MBN $6_5\,0_4\,0_1$ --> $4_6\,0_5\,0_1$ both of which yield the value one hundred and twenty – Decimal $1_{10}\,2_{10}\,0_1$ in the generalised format.

Notes

1. Apel, W., *Harvard Dictionary of Music* (Harvard University Press, Cambridge, Mass, 1966)

2. Jorgenson, D.A., *Résumé of Harmonic Dualism*, (Music and Letters, XLIV, 1963) p31.

3. Beament, J., *How We Hear Music*, (The Boydell Press, Woodbridge, Suffolk, UK, 2005).

4. Taylor, C., *The Science of Musical Sound*, in *Music and Mathematics*, Eds. J. Fauvel, R. Flood, & R. Wilson, (Oxford University Press, 2003) p59.

5. Beament (2005, ibid.) gives an interesting account of notes being apprehended as either pitch or tone in what he calls the 'Three-tone Paradox', section 7.6, p83.

6. This raises the question of what happens when the notes in a chord are sufficiently widely spaced so that their fundamental resonances do not overlap: do we hear a chord or three separate pitches? Given the flexibility of aural cognition, we generally hear what we expect to hear – usually perceiving the intermixed tone of the chord – especially as the overtones of the lower notes will overlap the upper notes, in a similar way to the fundamentals of a triad. Learning and experience, to a great extent, influence perception, even to the point of sweeping up melodic elements into chords, as for example in the unaccompanied suites by J.S. Bach.

7. Weinberger, N.M., *Music and the Brain*, (Scientific American, Nov. 2004) p68.

8. Jeans, J., *Science and Music,* p86 (Cambridge University Press, 1937; Dover, New York, 1968)

9. The 'Picardy-third' replacement of the expected tonic minor chord, at the final cadence of a composition in a minor key, by the tonic major chord. The practice has existed since 1500, and the name suggests, perhaps, an association with the Franco-Flemish polyphonic school.

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