10

Chords and Meters

NUMBERS IN MUSIC

The analysis of chord structure and metrical relationships presented in this chapter is founded on twin propositions. Firstly, that the structure of tonally organised musical sound can best be understood as existing within the context of extended harmonic series, which encompass the notes/chords and rhythms/ meters of a composition in whole number frequency relationships; that is, extended harmonic series from fundamental, up to and beyond, the highest sounding note or shortest durational value, such that each note in a composition corresponds with a particular partial within an underlying series. (The tolerance of the ear is relied upon to smooth over the slight irregularities introduced through scale temperament.) In this way, any chordal arrangement of notes or rhythmic figure might be viewed as a *configuration of partials*: sequences of integer frequency relationships which fit particular positions within the host harmonic series.

The second, and more speculative proposition, is that in the domain of pitch/timbre the 'ear' makes, where possible, a 'shorthand' version of these often extended series, the structure of which is strongly influenced by the pattern of objective tones within the chord. In the context of the basic MOS pair of nested/nesting series, it is these abbreviated *foreground* structures that are recognised as encapsulating the *perceived harmonic gist* of the objective notes, while the complete extended series of integer frequency relationships required to encompass all objective tones might be described as the *background* – the stage scenery – against which the more potent and dynamic foreground structures move. Alternatively, and more mundanely, it is perfectly possible to interpret the objective musical sound – the notes and timbre of tonal compositions – as the *foreground* and relegate the whole number relationships of the underlying *background* (implied by the structure of the musical sound) to some mathematical/theoretical domain.

Cataloged below, are the basic configurations of the principal chord and meter types, given in graph format with accompanying text. The background harmonic series are illustrated in 'white' dots (i.e. 'open' points) and the foreground series in black. In these graphs, the vertical y-axis counts off the harmonics, from h1 fundamental at the bottom, in ascending order and the horizontal x-axis indicates the increasing energy and complexity of the system from left to right, in arbitrary units. For chords the key/tonal center is C major throughout, with each chord type rendered at the lowest position within the harmonic series at which its particular *configuration of partials* matches the underlying background series (which is always a harmonic series founded on C-h1). These *base* positions are by no means the only possible matches. Apart from upward octave transposition, given the tolerance of the ear, there will probably be a range of other matches between configurations of partials (chords) and the background harmonic series acceptable to the ear – given the context of the surrounding chord progressions. Meters are cataloged separately following on from the section dealing with chord types.

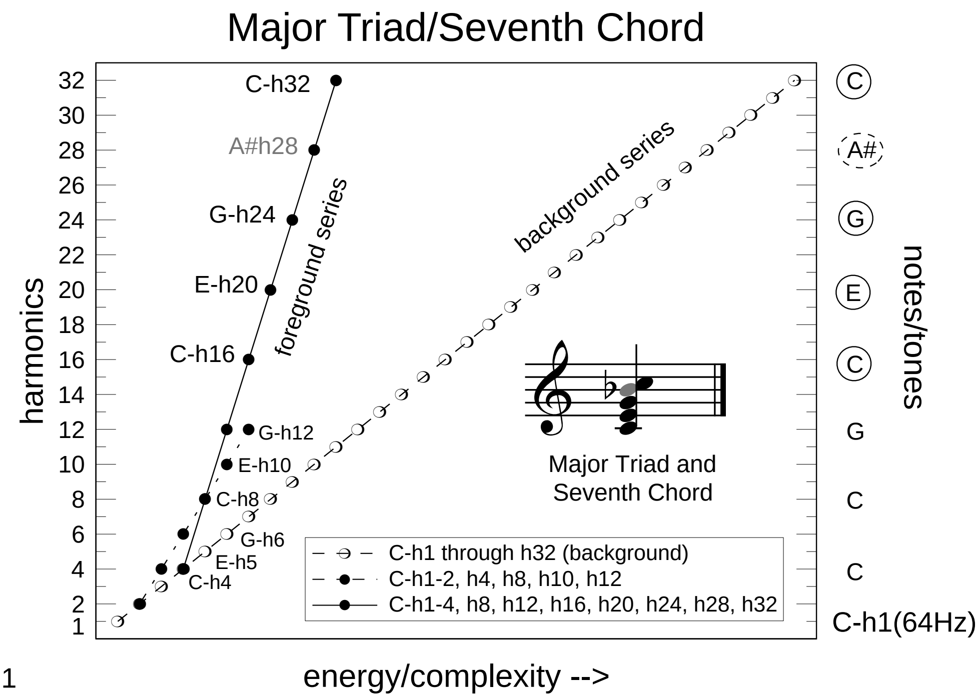


Figure 10.1

The lowest position at which the major triad and seventh chord, as configurations of partials, fit into an underlying background harmonic series is: C-h4, E-h5, G-h6 (and A#h7 for the seventh chord). Although not shown in black dots in Figure 10.1, to avoid confusion, these lowest positions for the major triad are labelled against their corresponding 'white' dots on the background series. In this position the foreground series lies on top of the background series: they are synonymous – one and the same series sharing one and the same fundamental root, C-h1 (64Hz). In addition to this base position, the major triad and seventh's configuration naturally fit as octave transpositions. The transposition of the foreground series built on C-h2 (128Hz), thus C-h8, E-h10 and G-h12, is illustrated for the major triad; the transposition built on C-h4 (256Hz), drawn with a continuous line, shows the dominant-seventh chord – C-h16, E-h20, G-h24 and A#h28. Overall, one of the transpositions is the more likely arrangement rather than the base position, given that, in the context of surrounding chord progressions, such transpositions would link more naturally with similar levels of nesting in chords with base (lowest possible) positions set higher within their background series (e.g. the chord of the added sixth – see below). However, though most often found in transposition, the major triad and seventh chords do not thereby differentiate themselves from their underlying background series, by virtue of possessing the same fundamental root 'C' – whether C-h1 (64Hz) of the first example, C-h2 (128Hz) of the second example or C-h4 (256Hz) of the third example. Equally, these three examples could be described as foreground series stepping in groups of one, groups of two and groups of four. This is an important point to grasp because, crucially, many other (but not all) chord types – that is foreground structures – do differentiate themselves from their underlying background series, by stepping in groups other than powers of two (octaves).

For example, if the foreground series is set to steps of three background ratios, groups of three rather than groups of two or four (Figure 10.2), this results in the formation of a G-major triad/seventh chord – the dominant-seventh – built upon the same C-based fundamental series as used above in Figure 10.1. Another example would be the use of groups of five, which bring us to the topic of the minor triad.

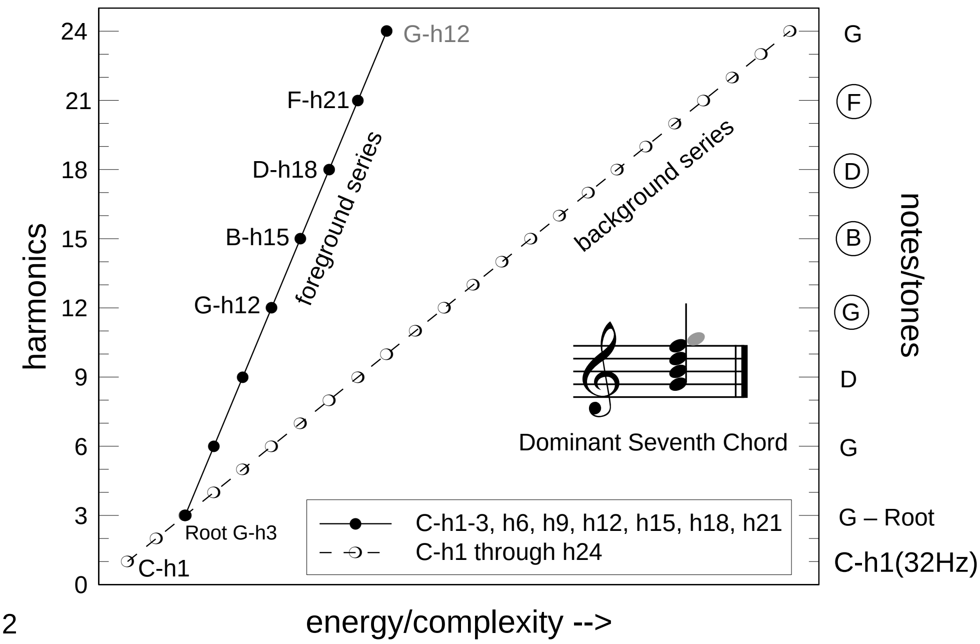


Figure 10.2

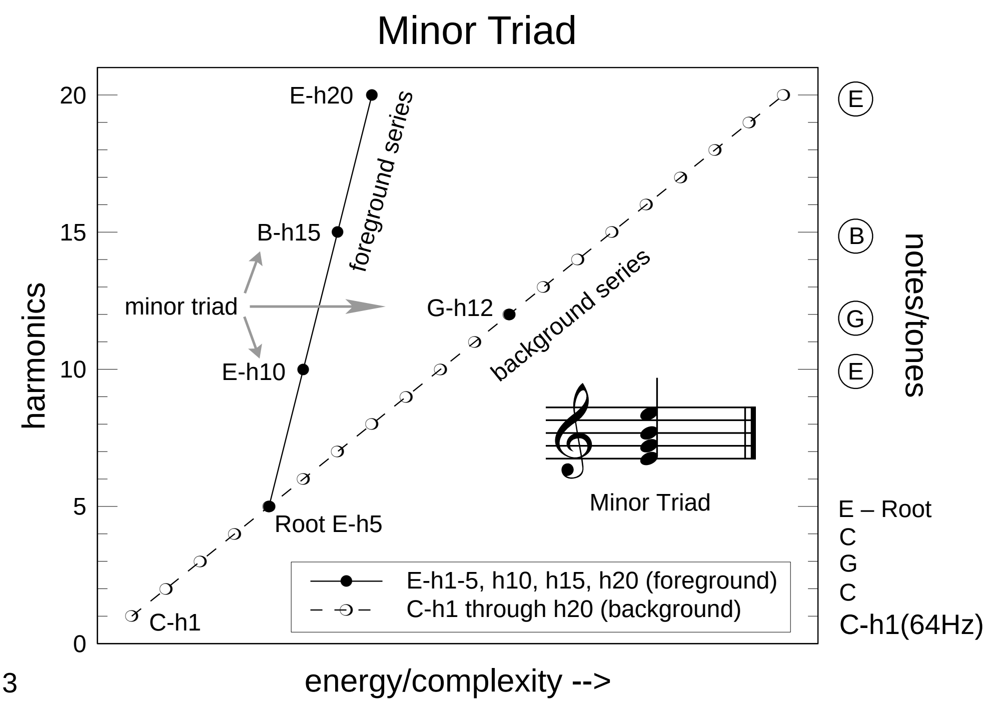


Figure 10.3

The ‘natural minor triad’, that is the *configuration of partials* within the harmonic series which match the arrangement of notes within the minor triad, is found at a much higher position than the ‘natural major triad’. The lowest or base position of the minor triad is at E-h10, G-h12 and B-h15. Lying an octave higher in the series, the harmonics matching the minor triad are necessarily spread out (i.e. are not contiguous, having other harmonics between them), so as to achieve a similar separation of resonant positions on the ear’s sound detecting membrane as the major triad. In this position within the harmonic series the structural tones of the triad, the root E and the fifth B, share a common denominator – five. It is this common ground, harmonics grouped in fives, which allows the ear to ‘paraphrase’ the relatively complex underlying relationships of (potentially) fifteen or twenty oscillators (h1 through h15/h20) into the more manageable and stable foreground series built on the perceived root E-h5. Fundamentally, nature tends to seek out the ground-state or lowest energy configuration for an entity – like the pencil balanced on end will tend to fall onto its side but not, contrariwise, jump up onto its end very often! Paraphrasing the extended series maintains the essential information content of the system (the mutable number value) but expresses it in a more efficient form. However, the ear cannot ignore the objective tone G, which forms the interval of a minor third within the triad. The tone G-h12 doesn't fit into the foreground series but remains lodged in the background series. This is significant: if G were to have a position within the foreground series, there would be little to maintain the context of a background structure – in mutable number theory at least, and maybe aural cognition too. But with G awkwardly refusing to fit into the foreground, the processes of aural cognition might be forced to take account of its undeniable objective existence by integrating it into a larger picture of two related harmonic series – a foreground series constructed in groups of five background harmonics. Essentially, the foreground series nests within the background and amounts to a physical or oscillatory analogue of mathematical factorisation. It is this stable, abbreviated, foreground structure which is identified with the harmonic 'meaning' or identity of the chord, while the awkward minor-third, lodged in the background series, interjects, as a non-dissonant coloring agent, within the harmonic gist of what is perceived to be a series built on E. Notice that the abbreviated foreground series (built on, and including, the first five oscillators in the background series) manages to reach up to B-h15 or E-h20 with roughly half the notional complexity and energy required by the background single series system – i.e. the rightward extent registered on the horizontal axis. The energy difference is neatly encapsulated by the angles of the full and dashed lines in Figure 10.3.

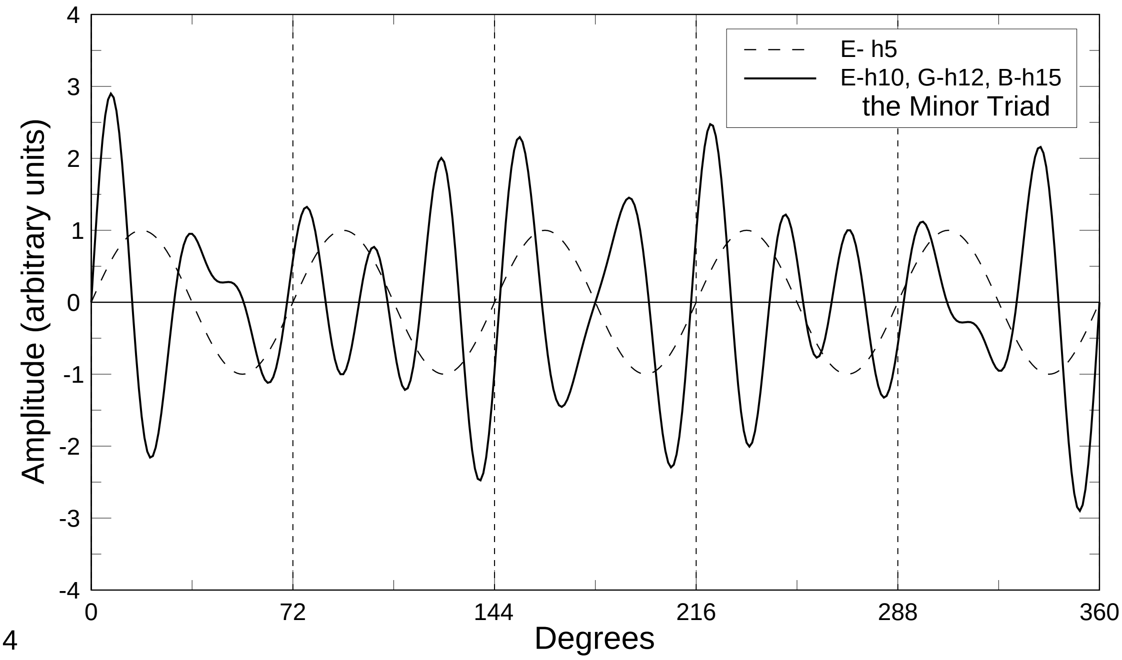


Figure 10.4

An alternative view of the whole number frequency relationships of the objective notes of the minor triad can be obtained by plotting the interference pattern that the chord would form. In Figure 10.4 above, this is done for the (E) minor triad, assuming equal amplitudes and uniform phase for each constituent tone. Though admittedly constructed under rather contrived circumstances, the interference pattern of the tones of the minor triad, when combined in this manner, articulate fairly closely the frequency h5, that is E-h5 the ‘root’ of the foreground series.

Interestingly, there is another resonance which also melds with the interference pattern of the minor triad, that of fifteen’s other factor – h3. The resonance G-h3 might provide an alternative foundation upon which a minor triad could be constructed. Although the ratio of the interior interval of a minor-third, G to A# (6:7), is rather narrower than the ‘natural’ minor-third – E to G 10:12 (6:7.2) – the equally tempered scale sharpens the minor-seventh interval considerably (i.e. the harmonics C-h4 to A#-h7 equal 1.75 of an octave, compared to the equal-tempered C to A# which is 1.78).

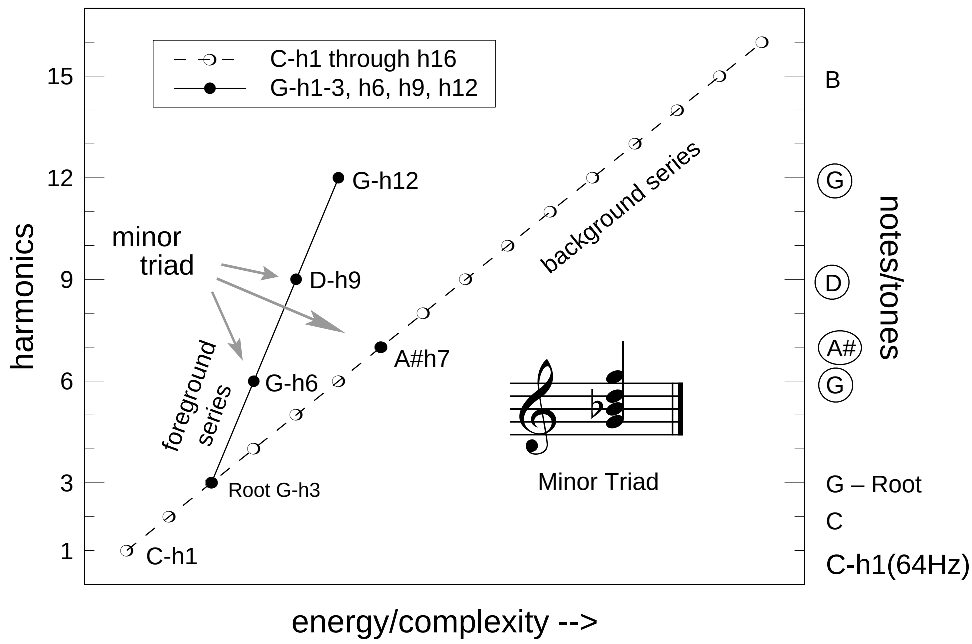


Figure 10.5

This arrangement is perhaps not the ear’s first or natural choice for the minor triad ‘in isolation’. However, given the right sequence of chords approaching it, the ear might well construe a minor triad’s *configuration of partials* as representing a foreground pattern based on G-h3. For example see Figure 6.14 where the first inversion minor supertonic chord (iib-V-I) forms aggregations in groups of threes.

Diminished Triad

As mentioned above, the lowest position of the foreground major triad series is synonymous with its own background series; they are one and same series (i.e. C-h4, E-h5, G-h6). In such a case, the aggregation of the foreground series is (somewhat theoretically) in groups of one, but can also be groups of two, of four, of eight, etc.; any power of two multiplication doesn’t change the identity of the chord – as illustrated in the major triad graph – Figure 10.1. However, the disposition of a foreground series lying on top of its background can also occur without a root note emerging from amongst the objective tones. This would particularly (but not exclusively) occur where the integer relationships in the *configuration of partials* were those of different prime numbers. Thus without any common denominator – excepting one, and possibly octave shifting powers of two – there is no basis for an abbreviated foreground structure. Put another way, the background series is the chord’s ground state configuration.

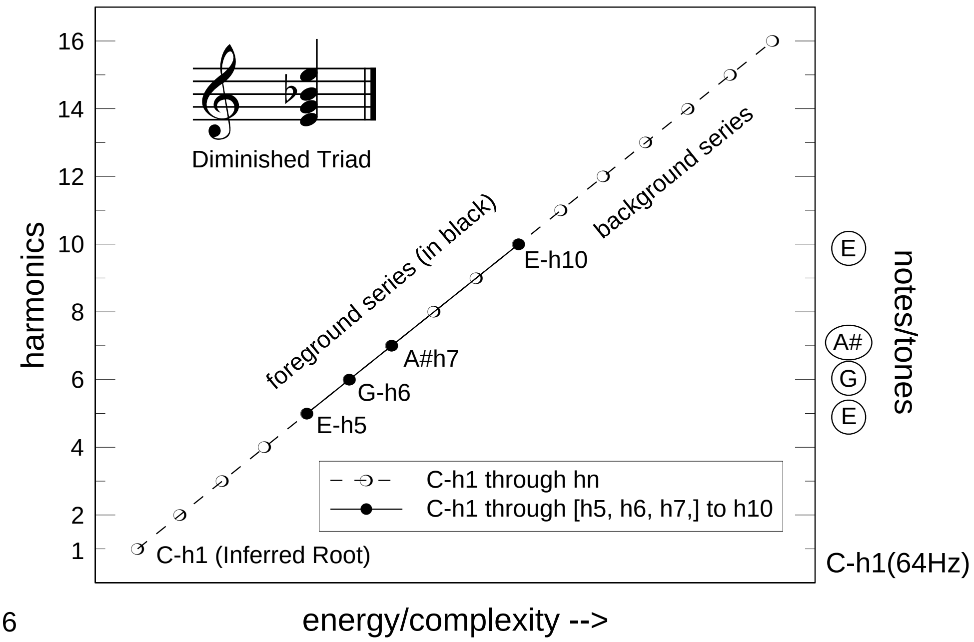


Figure 10.6

This is the situation of the diminished triad (and the augmented triad below), where the configuration of partials yields a lowest position within the (C-based) harmonic series of E-h5, G-h6 and A#h7 – Figure 10.6. Although G-h6 can be reduced to 3×2, this is still no help because no common ground can be found between the primes: two, three, five and seven. There is no common denominator upon which to build in the relationships of the diminished chord, and so, no one tone that can acquire precedence over the other tones of the chord, either as the ‘head’ of a fundamental ‘bundle’ as in the case of E-h5 in the preceding minor triad, or as an octave transposition of such a fundamental grouping.

However, whilst the notes of the diminished triad cannot agree upon one of themselves taking on the role of root, they are not so far removed from their common fundamental (C-h1) as to be totally rootless. The chord has often been described as the dominant-seventh missing its root tone, and even in the absence of the ‘missing’ root tone C-hn, the diminished triad may function as a dominant-seventh, resolving to the tonic chord – though with a somewhat reduced sense of explicit root movement. The most significant factor determining a diminished triad’s (or other similar configuration’s) ‘rootedness’ – taken in isolation from its surrounding chords – lies in it being sufficiently close to an apparent fundamental tone so as not to imply a foreground series of such complexity, that the capacity of the ear to fathom its structure, is overwhelmed. The boundary appears to be around nine harmonics, and not all chord types lie within this limit.

Augmented Triad

The situation of the augmented triad is broadly similar to that of the diminished triad. There is again no common denominator (above one) to produce an orientation towards a particular root amongst the objective tones, thus resulting in a default ‘root’ of C-h1 – the common fundamental tone. However, in the augmented triad’s case the relationships of the configuration of partials – A#h7, D-h9 and F#h11 – are set somewhat higher in the overtone series compared to the diminished triad. This is significant, as the ear finds a chord/series of h1 through h11 too complex to unravel down to its root. In contrast to the diminished triad, the augmented triad seems to the ear entirely rootless, more precisely indeterminate: too extended (complex) for the ear and processes of aural cognition to interpret, and relationally, too intractable to summarise into any meaningful abbreviated form.

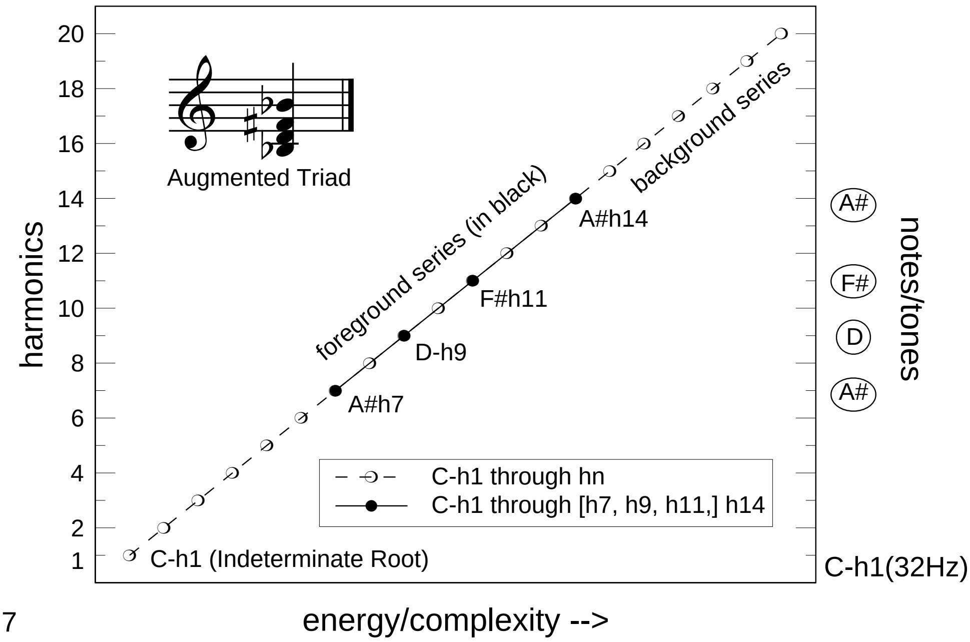


Figure 10.7

Minor minor-seventh Chord and Chord of the Added Sixth

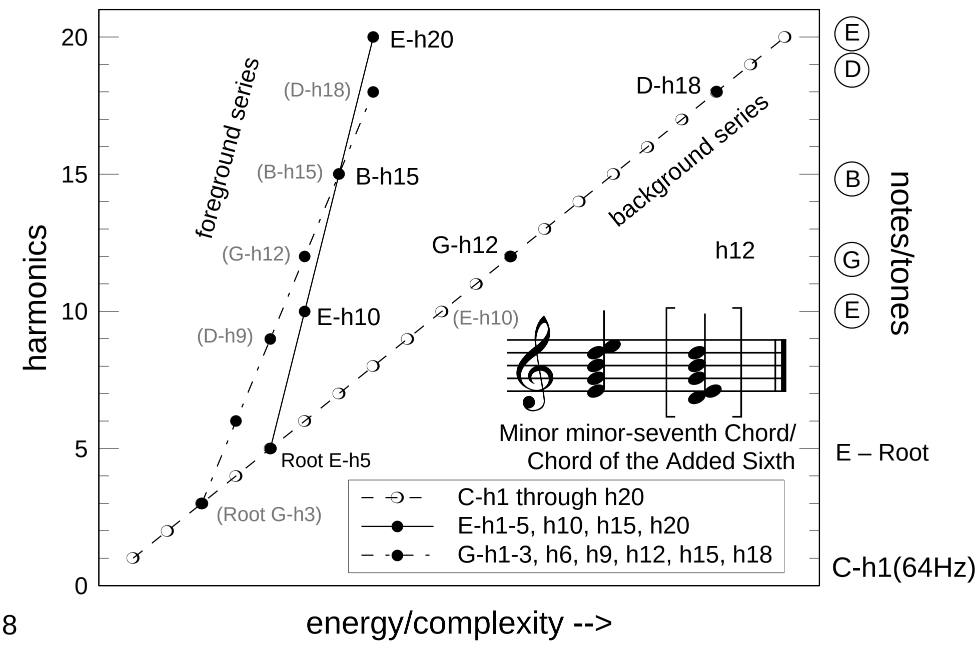


Figure 10.8

The minor minor-seventh chord, in the less dissonant combination of a minor triad plus minor-seventh, makes a more harmonious arrangement than the minor triad combined with the major-seventh, but one that still has hints of instability. In this case the instability arises not from the intractability of its internal ratios but from there being another viable configuration of aggregated harmonics (3×5); and one that is only slightly more energetic than the ground state minor-seventh chord (5×3), namely a major triad with added major-sixth: the chord of the added sixth. Indeed, the chord of the added sixth is often interpreted as a first inversion minor minor-seventh chord – a relationship described by the theorist and composer Rameau as ‘double employment’. The weakness of the minor minor-seventh configuration would appear to stem from there being an equal number of background ‘coloring’ tones counterbalancing the two structural foreground tones (E-h10 and B-h12); in addition to which the coloring tones (G-h12 and D-h18) form an interval of a perfect fifth too. Depending upon the context of the progression and disposition of notes in the chord, the balance within the chord might be shifted so as to favor the fifth, G-h12 to D-h18, being inferred by the ear as the structural interval, with a major-third supplied by B-h15. Under such circumstances, E-h10, formerly the root, is then relegated to a background coloring role. This configuration of the chord of the added sixth is illustrated by the dot-dash line in Figure 10.8: a foreground series built on the root G-h3.

Diminished Seventh Chord

The diminished seventh chord, extends the diminished triad with the addition of yet another minor-third to the stack. By this addition a symmetrical chord constructed entirely from intervals of a minor-third is produced, pushing the level of complexity beyond that of the diminished triad. And, like the augmented and diminished triads, the diminished seventh chord contains a configuration of ratios with no meaningful common denominator above one. (The configuration of F#-h11, A-h13, C-h16 and D#-h19, set at a slightly higher position within the harmonic series, illustrates this somewhat more clearly, with three prime number ratios and sixteen – 2^4 – finding no common ground.)

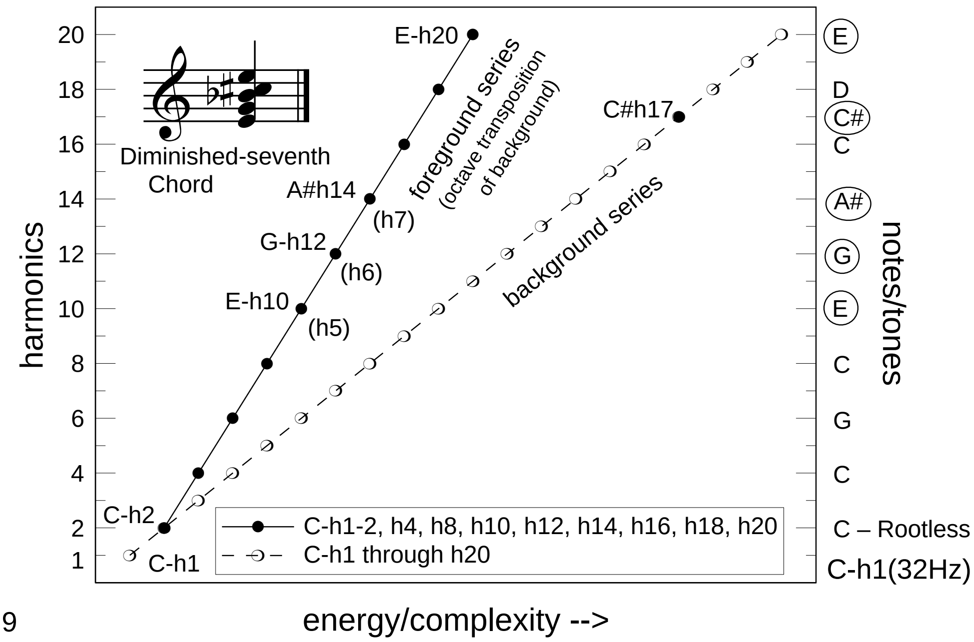


Figure 10.9

Returning to the lowest position, E-h10, G-h12, A#h14, C#h17, illustrated above, though the configuration of partials presented in the graph contains a hint of a common factor of two, indicative of a dominant ninth type chord founded on ‘root C-h2’, the tone C#h17 lies too far above an abbreviated foreground’s logical h5, h6, h7 (of C-h2) to sustain this reading. Such a ‘shorthand’ series would consist of nine or ten (foreground) harmonics to reach C#h17 (C-h2 through D-h18 or E-h20) which is relationally too intractable for the ear to grasp the root – C-h2. Effectively, the combination of ratios produces a similar outcome to the rootless F#, A, C, D# configuration lying along the background series, but perhaps with a doubling-up of background series, the full background built on C-h1 and an octave transposition of it on C-h2 (illustrated in the graph below). And in any case, such power of two groupings or octave transpositions – as also shown in the first graph for the major triad – produce no 'new' root to differentiate the foreground from the background series. *It is series stepping in prime number groupings other than two that have the property of creating a new root,* that is to say, differentiating the foreground from the background.

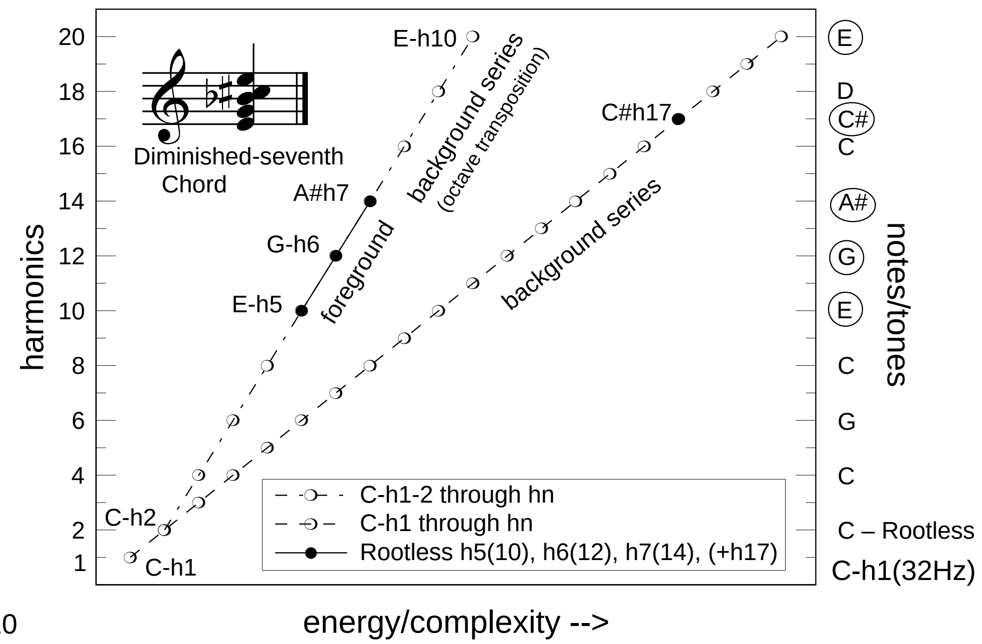


Figure 10.10

Dominant Ninth Chord

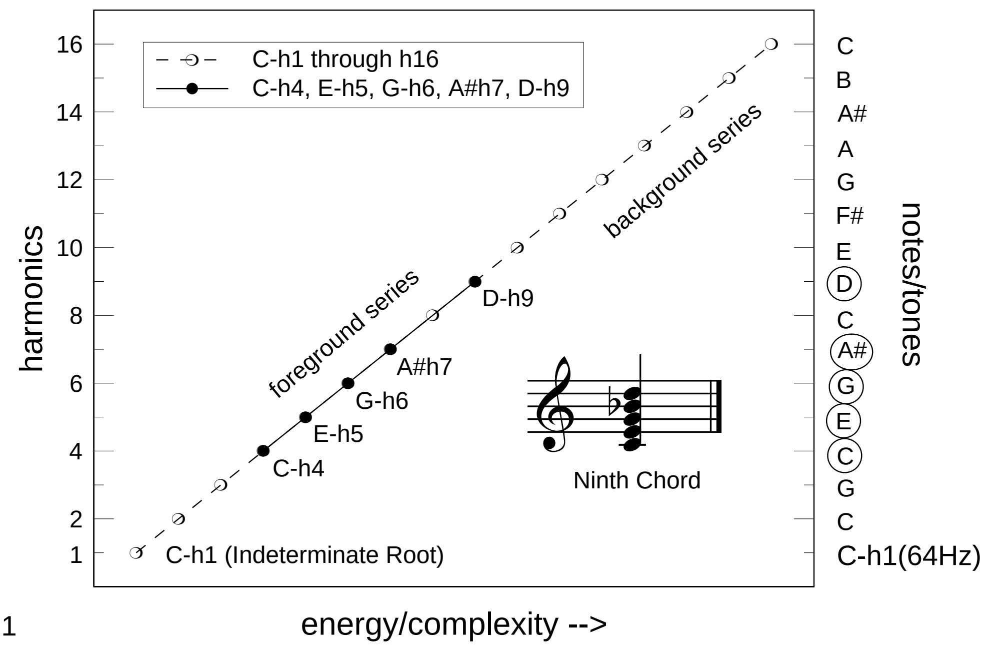


Figure 10.11

The dominant ninth chord is essentially a dominant seventh chord with an added major-ninth, and usually functions in a dominant capacity. The ninth chord's configuration of partials, though having two notes sharing the factor 2 and two sharing the factor 3, has no pre-eminent orientation written into its relationships. And, given the distribution of ratios, finds no 'shorthand' arrangement more economical than its lowest position (illustrated in Figure 10.11) lying along the background series. In this position, with the first nine ratios of the harmonic series required to encompass the objective notes, the ear struggles to perceive a clear root.

Underlying the amorphous character of the ninth chord is what appears to be a conflict between two roots: C-h1 and G-h3. If the chord is interpreted as a single series (i.e. foreground over background – as shown in Figure 10.11) the root is a rather hazy C-h1 and the function becomes that of a dominant ninth chord. However, a single series of nine oscillators can also be abbreviated to the more efficient structure of two series containing three oscillators nesting three more (3×3 = 9); but this attractive configuration produces a foreground root of G-h3 – Figure 10.12. It would appear the ear has difficulty in choosing between these two interpretations. The ninth chord lies on the edge of tonal definition: not quite complex enough to force an unequivocal two series structure with a perceived root of G-h3, yet too complex to be unambiguously recognised by the ear as a single series, with root C-h1.

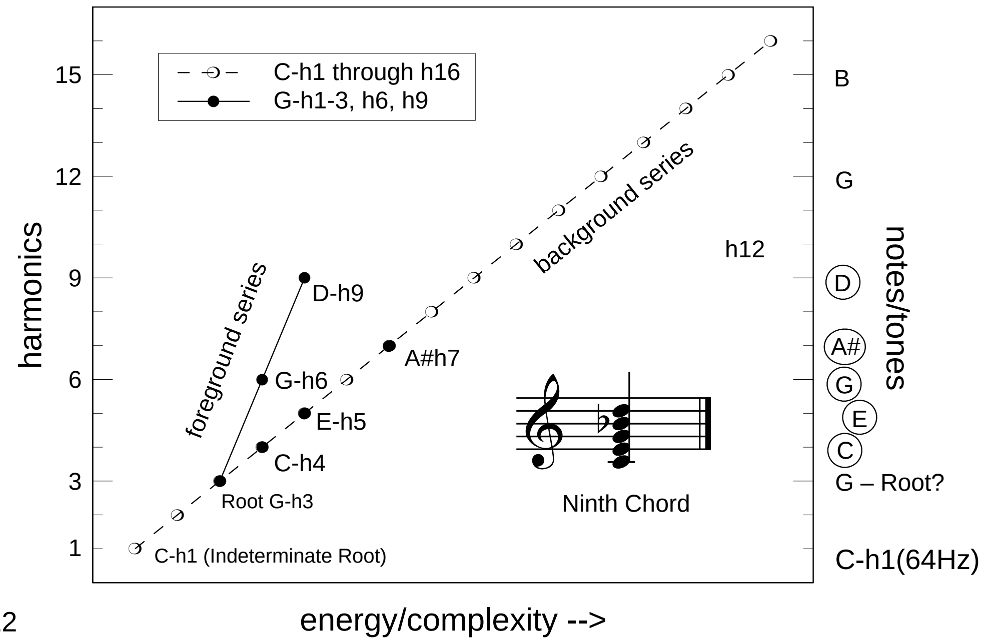


Figure 10.12

Half-diminished Seventh Chord and Minor Chord of the Added Sixth

The half-diminished seventh chord, a diminished triad with an added minor-seventh interval, is closely related to the ninth chord, in that it can be perceived as a dominant ninth chord lacking its root. This is analogous to the situation between the diminished triad and the dominant-seventh chord. As with the ninth chord, the configuration of partials places the half-diminished seventh chord on the boundary of 'rootedness'. And in a similar manner to the minor minor-seventh chord above, it has a partner, the beautifully expressive minor chord of the added sixth.

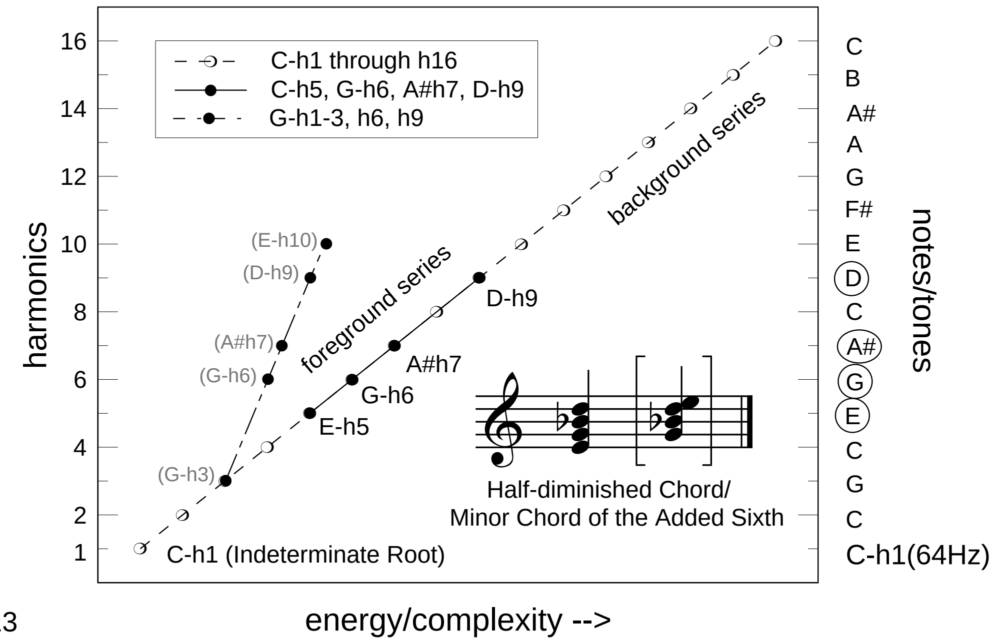


Figure 10.13

Extended, Altered and Added Tone Chords

Extended chords are really just the logical continuation of the 'third-stacking' which produced the triads and seventh chords. Adding another (major) third produces the dominant ninth, another (minor) third the dominant eleventh and yet another (major) third the dominant thirteenth. Naturally the steady accumulation of notes in these extended chords tends to obscure their clarity of rootedness, and thereby the sureness of tonal structure – a characteristic much exploited by composers of the chromatic school of the early twentieth century, e.g. Max Reger. On occasions some of the interior tones are omitted from extended chords, and like the half-diminished seventh they generally function, if they function at all, in a weakly dominant capacity.

The lack of tonal clarity introduced by extended chords may be rendered even more opaque by inflecting some of their internal tones to produce *altered chords*. The range of possibilities is considerable and again this resource has not gone unnoticed by composers. In particular, extended and altered chords have been extensively used by jazz musicians, where these rather rootless complex chords provide a 'fuzzy' harmonic texture conducive to free and unfettered improvisation. The wonderful genre of jazz might be viewed as something of an amalgam of western tonality with a lively heterophonic African heritage.

Finally, there are *added tone chords*, where one or more tones are simply 'glued-on' to a triad, principally at intervals of a second (or ninth), fourth (or eleventh) or sixth. All these additional tones are dissonant to a greater or lesser degree. The most significant are the chords of the added sixth, introduced above (Figures 10.8 and 10.13) and further discussed below in the section *Chords of the Augmented Sixth*.

SIXTH CHORDS

Chords of the Augmented Sixth

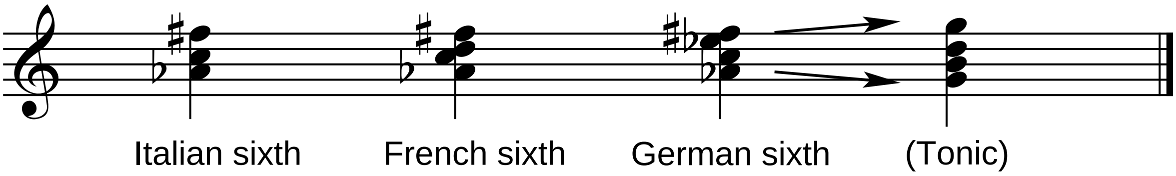


Figure 10.14 The three distinct forms of the chord of the augmented sixth (IV^{6+}), with resolution to the dominant chord (V) on right hand side.

There are four chords of the augmented sixth but only three distinct *configuration of partials* – as the ‘doubly augmented fourth’ form (6#-4#-3 chord) is a re-spelling of the German sixth. The three distinct forms have been termed: Italian sixth (6#-[3]), French sixth (6#-4-3) and German sixth (6#-5-3) – Figure 10.14. All forms have the characteristic augmented sixth interval (Aflat – F# in the key of C) which R.F. Goldman (1968) points out effectively gives the chord two ‘leading tones’ into a dominant resolution: an upward moving of F# to G combined with a downward driving Aflat to G. The arrows and right-hand chord in Figure 10.14 illustrate this forceful resolution.

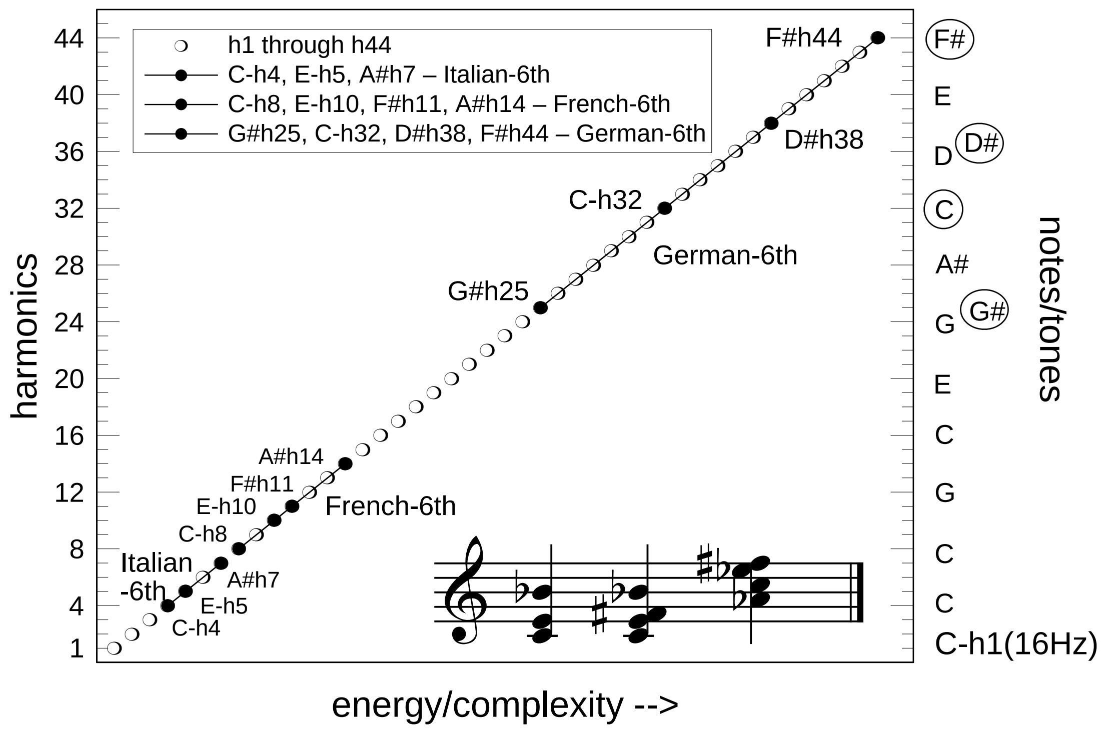


Figure 10.15 Chords of the augmented-sixth. The Italian and French forms are set at their lowest configuration of partials, while the German sixth is illustrated at the normal level at which the chords might be found, relative to their normal surrounding chord progressions.

On the one hand the German sixth chord could be interpreted as a major minor-seventh chord, in the ‘wrong’ key! Indeed, the German sixth illustrated in the key of C (Figure 10.14) would make a perfectly respectable dominant-seventh in the key of Dflat, though naturally spelt with a Gflat rather than F#. Thus the question arises as to why the German sixth doesn’t act like a firmly ‘rooted’ major minor-seventh chord, naturally seeking a 3:4 exchange to its ‘tonic’ chord of Dflat-major. The answer appears to lie in the context: the key of C. Surrounded by relationships indexed from C-h1, the augmented-sixth chord’s ratios are those of a complex and distant relationship: G#h25, C-h32, D#38 and F#44; even though the internal structure of the chord could be reduced to those of the far simpler *enharmonic* major minor-seventh chord in the key Dflat (G#h4, C-h5, D#h6, F#h7). However, the relationships derive their augmented-sixth status not from their internal structure but from their esoteric context within the key of C and vis-a-vis their position in relation to C-h1, the fundamental tone of the key.

In Figure 10.15 the Italian sixth and French sixth are shown in the lowest position of their *configuration of partials*. Equally the German sixth would translate to the position of the Italian sixth – which is a German sixth missing one or two internal intervals. However, in practice, in the flow of normal chord progressions, all types would be found in positions similar to that shown for the German sixth. And though set lying along the background series at G#h25, C-h32, [D-h36], D#h38 and F#h44, another possible configuration would be to place three of the objective tones in a separate foreground series built on C-h2, with only G#h25 lodged in the background – illustrated in Figure 10.16.

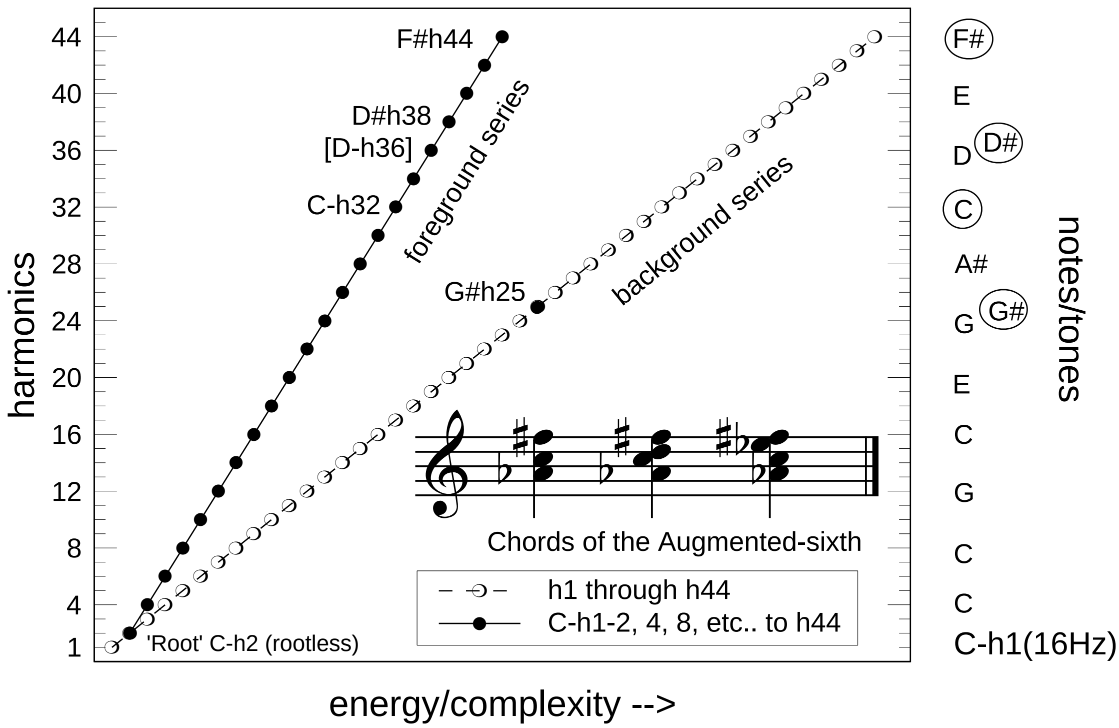


Figure 10.16 The chords of the augmented-sixth illustrated forming foreground and background series.

The illustrated arrangement of notes in Figure 10.16 does not change the harmonic ratios and though there is a degree of simplification in the relationships, relative to Ch-2, the ‘shorthand’ foreground ratios are still intractably complex, logically: C-h16, D#h19, F#h22 (of fundamental C-h2), and G#h25 (of C-h1) unchanged.

R.F. Goldman also describes the chords of the augmented-sixth as essentially rootless, though mentioning the possibility of the French sixth being interpreted as a major minor-seventh chord on the supertonic with diminished fifth (II^{5-}) and, that overall, the chords operate as dominants to the dominant (V of V). The MOS view is somewhat different, but results in the same essentially rootless outcome. That is, the chords do have a root: by virtue of lying along a series – whether the series is the background series built on C-h1, a foreground series built on Ch-2 or perhaps some *other arrangement*. However, because of their extreme distance from the fundamental, they form impossibly complex entities – extended harmonic series of potentially 22 or 44 oscillators – far beyond the ear and aural cognition's capacity to interpret and place within the basic relationships of the key. Thus a rootless, or indeterminate, impression emerges from the internal relationships of the chord of the augmented-sixth.

Other Arrangements

While holding to the conclusion that the ear is somewhat bamboozled by the complexity of the augmented-sixth's relationships, relative to its background, it might still be possible that a more sophisticated and economical interpretation of the ratios is arrived at – but one that remains essentially rootless. The crucial ratio is G#h25 which is not only the logical fifth harmonic of a nested series stepping in groups of five background ratios, but also, potentially, the fundamental of a 'super-foreground' series stepping in units of five groups of groups-of-five! This would yield the foreground series G#h25, G#h50, D#h75, etc., marked as an almost vertical dotted line in Figure 10.17. Whether this five by five feature might lend extra weight to the objective tone G#h25 is uncertain, but a possibility. Indeed, perhaps a packing up of ratios into units of 'five fives' may be one of the ways that chords rooted on flattened scale degrees (i.e. IIflat and VIflat) are integrated into the overall relationships of a key.

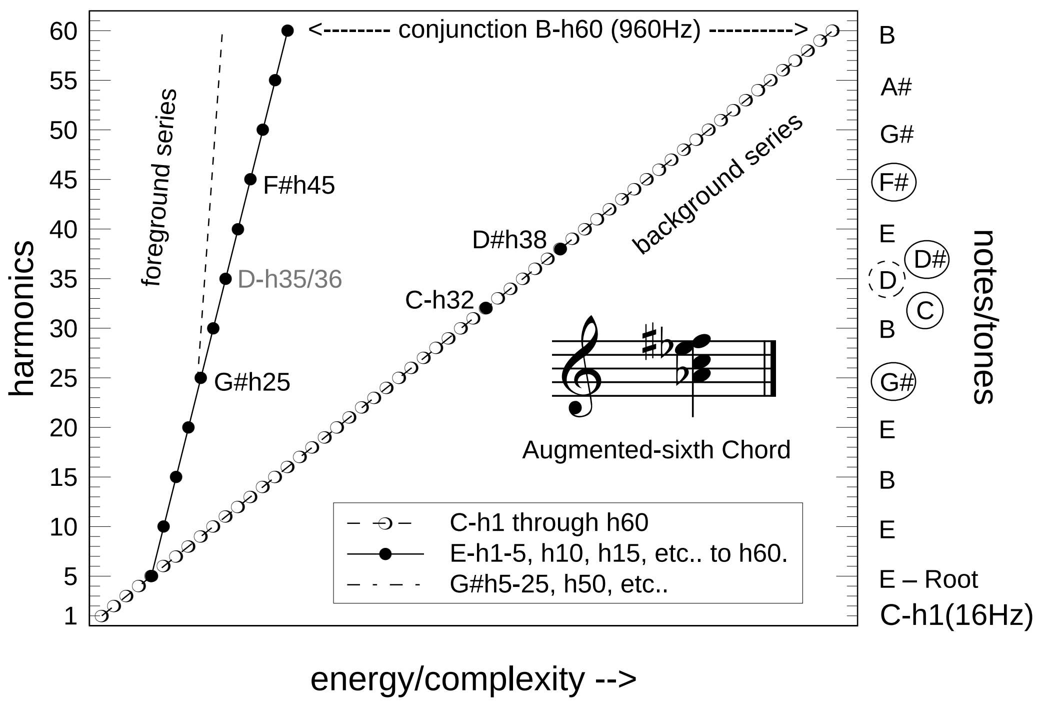


Figure 10.17 The chords of the augmented-sixth illustrated forming foreground series built on E-h5. Potentially, G#h25 is the 'root' or fundamental tone of a 'super-foreground' series stepping in units of twenty-five ratios – but lacking any extension beyond h25 (dashed near-vertical line).

Leaving aside the super-foreground series, there is certainly support for a foreground series built on E-h5, in that, as well as G#h25 being its logical h5, the note F#h44 could be construed as F#h45 by a tolerant ear (i.e. F#h45 the logical h9 of E-h5). This foreground series built on E-h5, illustrated in Figure 10.17, forms a relatively extended and complex structure itself, equating to a chord of the ninth, built on E-h5, with two additional objective tones lodged in the background series – C-h32 and D#h38. (In the case of the French sixth, D-h36 would probably be adjusted by the ear to a rough and ready D-h35, leaving only C-h32 in the background.)

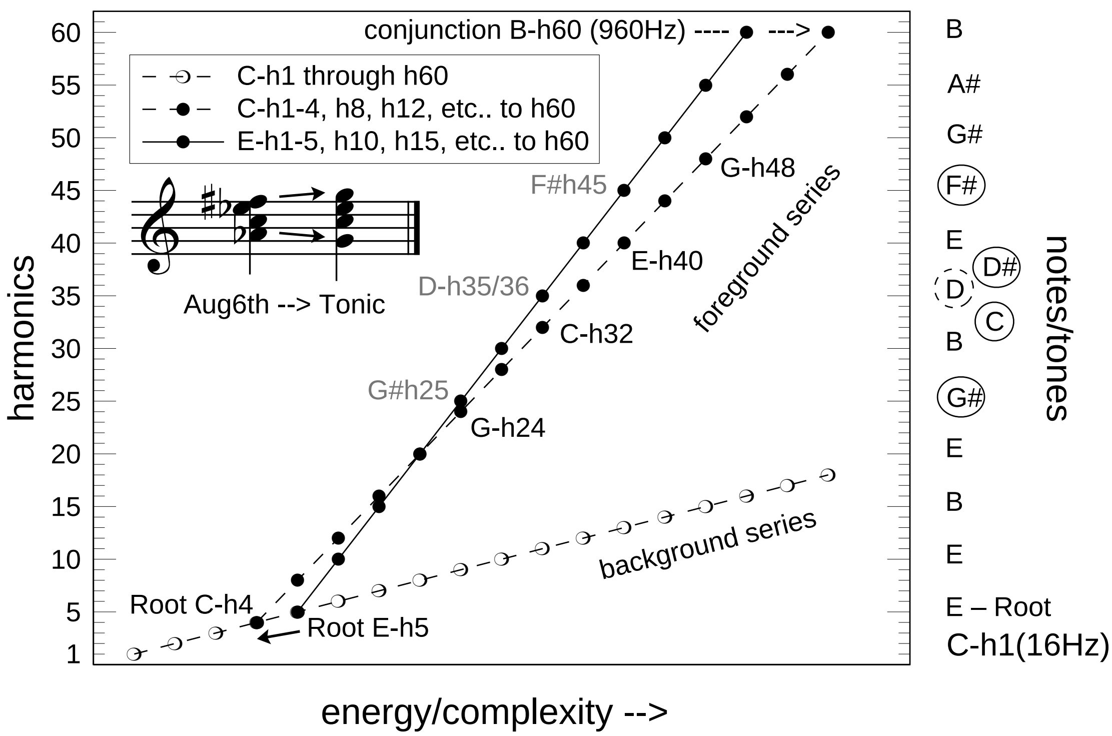


Figure 10.18 The modulation exchange between augmented-sixth chord's foreground series built on E-h5 and the tonic chord's foreground series built on C-h4. This exchange absorbs rather than releases energy, but the second inversion tonic (6-4) chord produced almost invariably relaxes to a root position dominant (5-3) chord, Figure 10.19.

The various augmented-sixth configurations of partials/ratios, put out a number of different, competing, signals: firstly, an unrealised (super)foreground series built on G#h25; secondly, a difficult to fathom, extended foreground series of nine ratios built on E-h5 (but with no supporting objective tone E in the chord); and thirdly, support for the background series built on C-h1 in the form of the objective tone C-h32. The complexity of these three factors taken together probably account for the augmented-sixth chord's impenetrable and rootless character – though subtly different from other chords lying along their background series – as the double leading tones, F# ascending and Aflat descending, produce a highly unstable chord with a strong directional pull toward the dominant chord.

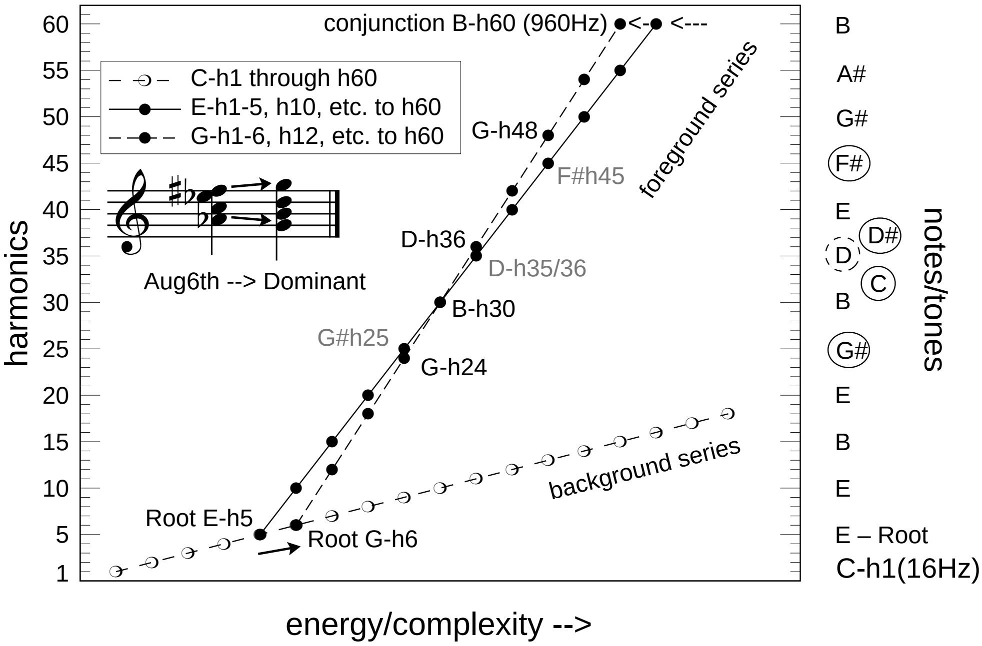


Figure 10.19 The modulation exchange between augmented-sixth chord's foreground series built on E-h5 and the dominant chord's foreground series built on G-h6. As indicated by the anticlockwise rotation of the foreground series, energy and complexity decline in this exchange.

There are two natural resolutions for the augmented-sixth chord: to a second inversion tonic chord (Figure 10.18) which subsequently relaxes to the dominant, or a resolution straight to the dominant chord in root position, illustrated in Figure 10.19. Normally, once the dominant has been reached by either route, the next and final step will be to the tonic chord.

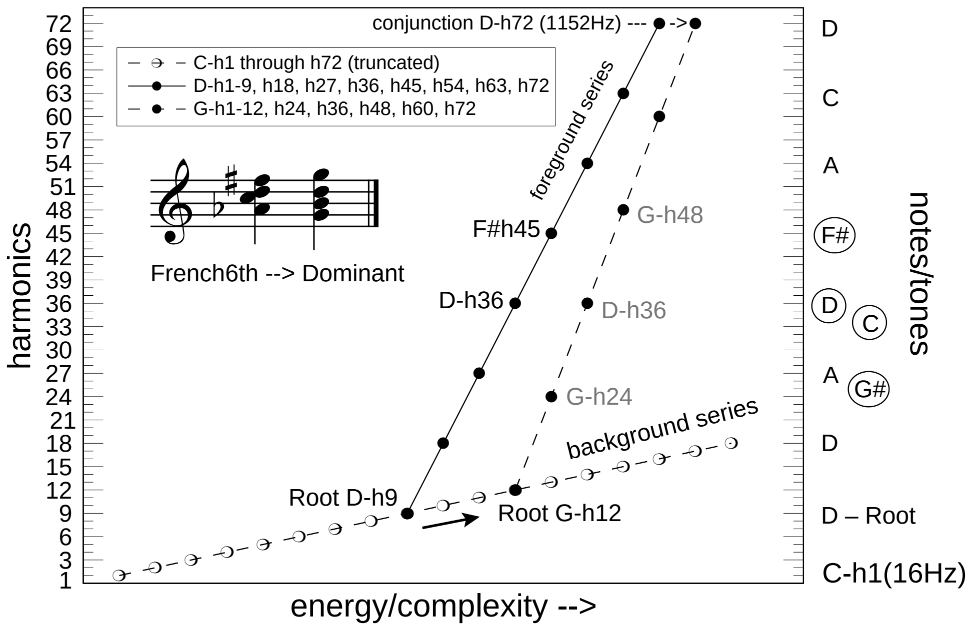


Figure 10.20 The French augmented-sixth chord interpreted as II^7 resolving to the dominant.

It is possible to argue that the French sixth can be interpreted as a major-seventh chords, with lowered fifth, built on the second degree: an altered II^7 chord. While yet to be convinced by this view myself, Figures 12.20 and 12.21 illustrate how the MOS model might render such an arrangement. The difficulty with this interpretation is, apart from not evincing a strong sense of a D root, that the addition of D-h36 would, if interpreted as a makeshift h35, strengthen rather than weaken or change the hold of a foreground series stepping in groups of five. A II^7 chord from the MOS perspective would have foreground groupings of nine.

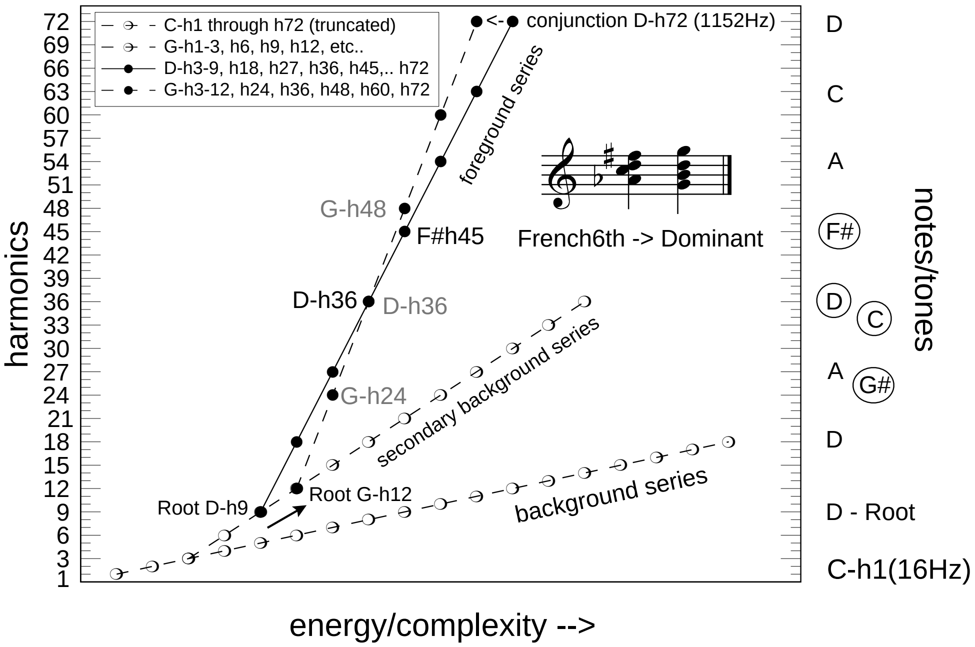
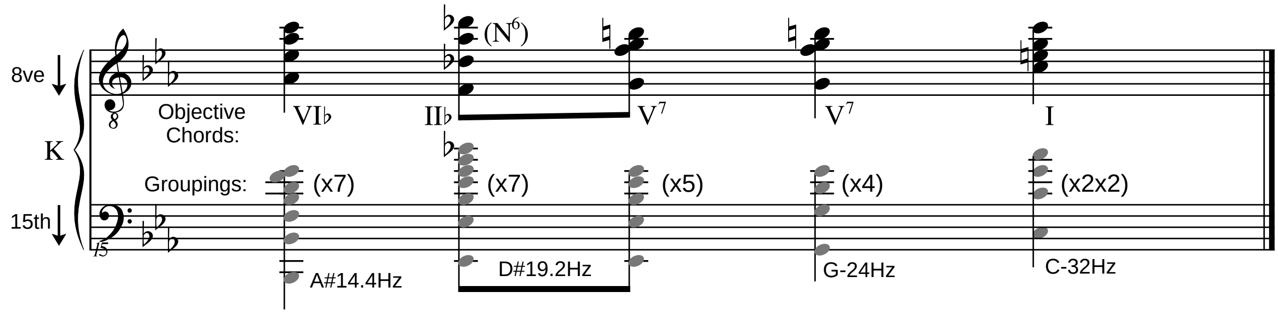


Figure 10.21 An alternative configuration of the French augmented-sixth chord interpreted as II^7 resolving to the dominant.

Chord of the Neapolitan Sixth

Up to this point chords embedded in foreground series formed from groups of background ratios stepping in twos and threes have been identified with the major triad/dominant seventh chord, and groups of five with the minor triad. Latterly, in discussing the chords of the augmented sixth, a range of aggregates of background ratios have appeared: four, five, six, nine and twelve. Next we examine the effect of groupings of seven – the Neapolitan chord, N^6.

The characteristic chord of the Neapolitan sixth, built on a lowered second degree of the scale and resolving to the dominant chord, is normally found in a first inversion configuration (the second chord in Figure 10.22), a configuration that gives some hint of connection with the typical cadential progression: ii/II -> V -> I, where chord ii or II, on the unflattened second degree, also often appears in its first inversion. Although the Neapolitan sixth doesn’t always appear in first inversion, it does take this form in a great many instances, and sufficiently so for it to be an excuse for consideration under the heading of *chord type* as well as chord progression.



G#806.4(h56)*>	G#806.4(h42)*	G#806.4(h42)	G-768(h32)*>	G-768(h24)**
G-777.6(h54)	-787.2(h41)	-787.2(h41)	-744(h31)	-736(h23)
-748.8(h52)	G-768.0(h40)	G-768.0(h40)*>	F#720(h30)	F#704(h22)*
F#720.0(h50)	-748.8(h39)	-748.8(h39)	-696(h29)	F-672(h21)
-705.6(h49)*	F#729.6(h38)	F#729.6(h38)	F-672(h28)*	E-640(h20)**
F-691.2(h48)	-710.4(h37)	-710.4(h37)	-648(h27)	D#608(h19)
-662.4(h46)	-691.2(h36)	-691.2(h36)	E-624(h26)	D-576(h18)*
E-633.6(h44)	F-672.0(h35)*>	F-672.0(h35)*	D#600(h25)	C#544(h17)
-619.2(h43)	E-652.8(h34)	E-652.8(h34)	D-576(h24)*	C-512(h16)**
D#604.8(h42)*	-633.6(h33)	-633.6(h33)	-552(h23)	B-480(h15)
-590.4(h41)	D#614.4(h32)	D#614.4(h32)	C#528(h22)	A#448(h14)*
D-576.0(h40)	-595.2(h31)	-595.2(h31)	C-504(h21)	A-416(h13)
-561.6(h39)	D-576.0(h30)---	D-576.0(h30)*---	B-480(h20)*	G-384(h12)**
C#547.2(h38)	-556.8(h29)	-556.8(h29)	A#456(h19)	F#352(h11)
-532.8(h37)	C#537.6(h28)*	C#537.6(h28)	A-432(h18)	E-320(h10)*
C-518.4(h36)	C-518.4(h27)	C-518.4(h27)	G#408(h17)	D-288 (h9)
-504.0(h35)*	-499.2(h26)	-499.2(h26)	G-384(h16)*	C-256 (h8)**
B-489.6(h34)	B-480.0(h25)---	B-480.0(h25)*---	F#360(h15)	A#224 (h7)
-475.2(h33)	A#460.8(h24)	A#460.8(h24)	F-336(h14)	G-192 (h6)*
A#460.8(h32)	-441.6(h23)	-441.6(h23)	E-312(h13)	E-160 (h5)
-446.4(h31)	A-442.4(h22)	A-442.4(h22)	D-288(h12)*	C-128 (h4)**R
A-432.0(h30)	G#403.2(h21)*	G#403.2(h21)	C#264(h11)	G- 96 (h3)
-417.6(h29)	G-384.0(h20)---	G-384.0(h20)*---	B-240(h10)	C- 64 (h2)*
G#403.2(h28)*	F#364.8(h19)	F#364.8(h19)	A-216 (h9)	C- 32 (h1)
-388.8(h27)	F-345.6(h18)	F-345.6(h18)	G-192 (h8)*	
G-374.4(h26)	E-326.4(h17)	E-326.4(h17)	F-168 (h7)	
F#360.0(h25)	D#307.2(h16)	D#307.2(h16)	D-144 (h6)	
F-345.6(h24)	D-288.0(h15)---	D-288.0(h15)*---	B-120 (h5)	
-331.2(h23)	C#268.8(h14)*	C#268.8(h14)	G- 96 (h4)*R	
E-316.8(h22)	C-249.6(h13)	C-249.6(h13)	D- 72 (h3)	
D#302.4(h21)*	A#230.4(h12)	A#230.4(h12)	G- 48 (h2)	
D-288.0(h20)	A-211.2(h11)	A-211.2(h11)	G- 24 (h1)	
C#273.6(h19)	G-192.0(h10)---	G-192.0(h10)*---		
C-259.2(h18)	F-172.8 (h9)	F-172.8 (h9)		
B-244.8(h17)	D#153.6 (h8)	D#153.6 (h8)		
A#230.4(h16)	C#134.4 (h7)*R	C#134.4 (h7)		
A-216.0(h15)	A#115.2 (h6)	A#115.2 (h6)		
G#201.6(h14)*	G- 96.0 (h5)---	G- 96.0 (h5)*R--		
G-187.2(h13)	D# 76.8 (h4)	D# 76.8 (h4)		
F-172.8(h12)	A# 57.6 (h3)	A# 57.6 (h3)		
E-158.4(h11)	D# 38.4 (h2)	D# 38.4 (h2)		
D-144.0(h10)	D# 19.2 (h1)	D# 19.2 (h1)		
C-129.6 (h9)				
A#115.2 (h8)				
G#100.8 (h7)*R				
F- 86.4 (h6)				
D- 72.0 (h5)				
A# 57.6 (h4)				
F- 43.2 (h3)				
A# 28.8 (h2)				
A# 14.4 (h1)				

(8 groups of 4) -3:4-> (6 groups of 4 [2x2])

(8 groups of 5) -4:5-> (8 groups of 4)

(5 groups of 7) -1:1-> (7 groups of 5)

(8 x four) -----3:4-----> (8 x three)

(8 groups of 7) -3:4-> (6 groups of 7)

(8 x five) ------4:5------> (8 x four)

(14 x four) ------3:4--> (14 x three) --------------->

KEY

*	Aggregated Series	◯	Objective Notes	hn	Logical Pitch	Foreground (Aggregated) Series in Black
**	2nd. Aggregated Layer	⬚	Conjunction	R	perceived 'Root' Tone	Background (Nested) Series in Grey
						Figures in Hertz: C-256.0 Hz (middle C)

Figure 10.22 The chord of the Neapolitan sixth, set within its surrounding chordal context.

Indeed, the MOS perspective of the internal workings of both chords, that is minor triad and seventh chord on the unflattened second degree (ii and ii^7) and the major triad and seventh chord on the flattened second degree of the scale (Dflat in the key of C), indicates a similarity in underlying mechanisms – a similar flexibility of structure – leading in both instances to two viable internal arrangements. In the case of the minor chord on the unflattened second degree, the alternate forms are those of minor triad/minor-seventh chord (normally ii in first inversion) and a chord of the added sixth (iv^{+6}) in root position. Thus the chords usually hold in common the same bass note. However, with the Neapolitan sixth, the two arrangements take the form of different major triads (flatII and V, with or without sevenths), with the two chords usually built on different bass tones (F and G in Figure 10.22). In the key of C, these two alternative configurations yield the Dflat-major and G-major chords. Significantly, the Neapolitan sixth chord (Dflat-major in first inversion) is normally followed by the dominant chord (G) or less often the tonic (C). Thus, the alternate (and somewhat more energetic) form within the Neapolitan sixth chord – the G-major configuration – links seamlessly to a following (identical) dominant chord, or (near, 3:2 related) tonic chord. The trick of the Neapolitan progression lies in the ear's ability to 'compute' this reconfiguration, under the influence of the objective progression to the dominant or tonic chords, and thereby smooth the path across the awkward intervals of an augmented fourth from roots Dflat to G, or minor second, from root Dflat to C, ratios of 17:24 or 17:16 respectively.

Quite frequently the Neapolitan relationship is transferred along a cycle of lowered scale degrees before breaking the sequence of perfect fourths on flattened roots (e.g. Eflat-> Aflat-> Dflat-> G-> C or Eflat-> Aflat-> D-> G-> C) and so to illustrate this tendency, the first chord in Figure 10.22 is that of Aflat.

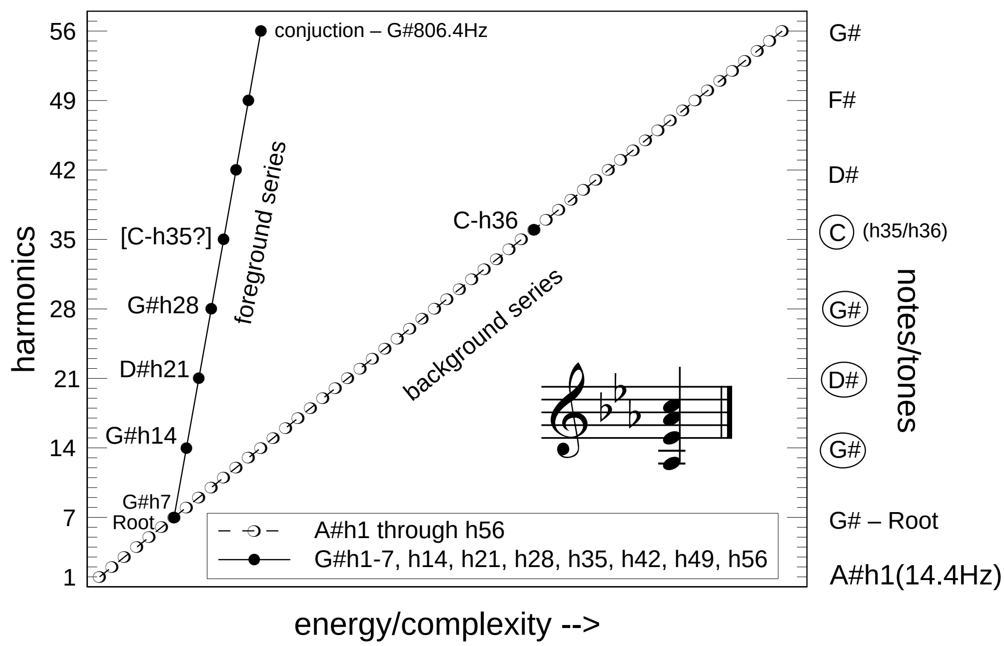


Figure 10.23 The first chord in the sequence of the Neapolitan sixth, Aflat major.

The chord of Aflat built on a foreground structure of groups-of-seven background harmonics, raises the question of whether or not the major-third interval formed by the tone C is interpreted by the ear as h36 a background ratio or h35 in the foreground series. In Figure 10.23 it has been placed in the background – the strict interpretation – though I suspect the ear and processes of aural cognition massage the objective data into the much less complex arrangement of all tones set in the foreground series, with C interpreted as a ‘dirty’ h35. Either interpretation is sustainable, but such an adjustment of the objective data will not work for the second chord of our example, where the ‘major-third’ tone is at the bottom of a first inversion configuration, F-h9 in Figure 10.24, is too low for a foreground slot (the major-third only emerges higher up at h35).

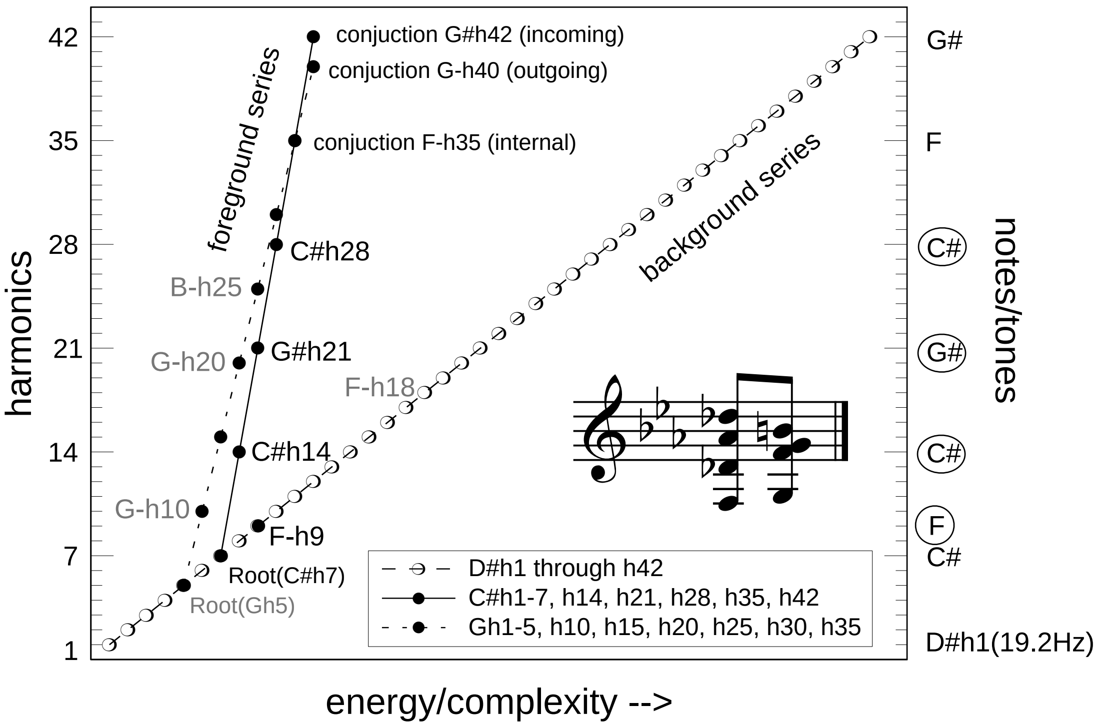


Figure 10.24 The second chord in the sequence of the Neapolitan sixth, Dflat major; also illustrated is an alternative arrangement of foreground tones (dot-dash line) built on h5 of the background series which yields a G-major chord.

The chord of Dflat-major is the Neapolitan sixth chord: a first inversion major chord on the lowered second scale step, arrayed with a minor-sixth between its lowest tone F, and the root tone Dflat. The objective notes, in this configuration, require a complex extended harmonic series built on the fundamental D#h1(19.2Hz) to include all tones as whole numbered ratios. Notwithstanding this, acting under the influence of the law of increasing entropy, the second law of thermodynamics, the ear is induced to ‘summarise’ such an overly complex arrangement of one extended series, into a much simpler foreground series built on C#h7 (Dflat) that encapsulates the principal ‘harmonic gist’ of the objective chord. While also not entirely forgetting the objective tone F-h9, which cannot fit into the foreground series but remains, and maintains, along with C#h7 the ‘root’ tone of the foreground series, the shadowy viability of the more complex and profound background – out of which the foreground series has grown. The continuous black line in Figure 10.24 delineates the foreground Dflat harmonic series, running from (nested) fundamental C#h7 (logical foreground h1) up to G#h42 (Aflat) where both foreground and background series conjoin with the preceding chord/series. The conjunctions are marked by dotted line boxes in Figure 10.22, and the mechanism of the modulation algorithm of symmetrical exchange is detailed elsewhere.

The heart of the Neapolitan exchange lies in there being another alternative configuration of foreground ratios built on G-h5 of the existing background series, which conjoin with the ‘Dflat’ foreground series at F-h35. This alternative ‘G-major’ series is marked by a dot-dash line in Figure 10.24, with the notes of the third chord in our example progression shown in grey lettering. The dot-dash line intersects the Dflat-major series at an ‘internal’ conjunction (F-h35) and because the line is tilted a little more in a clockwise direction about the conjunction, it represents a slightly more energetic arrangement than that of the ‘Dflat’ series.

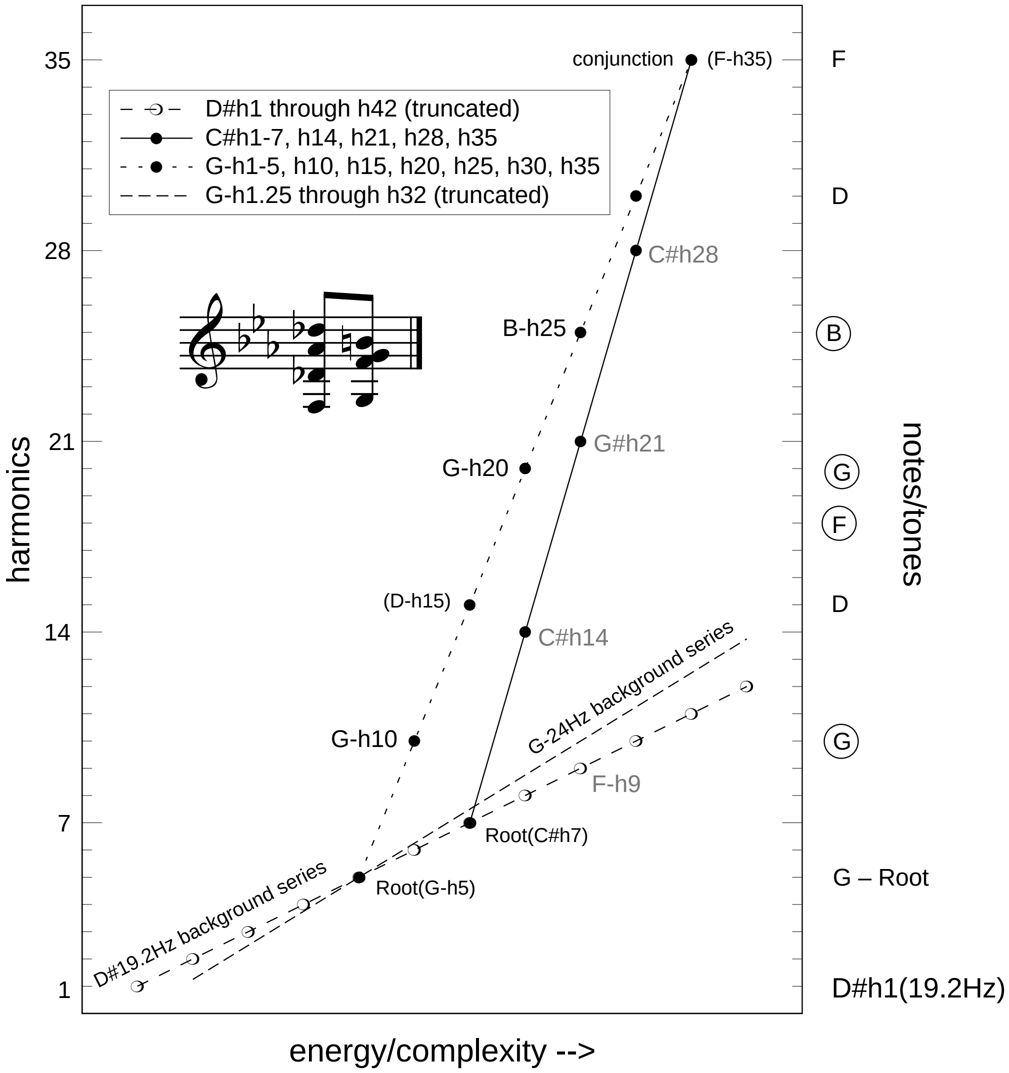


Figure 10.25 Enlarged second chord in the sequence of the Neapolitan sixth, Dflat-major, with alternate G-major arrangement. The dotted line running through G-h5 marks the position of a background series built on G-24Hz – the background series of the next (third) chord in the sequence.

By homing in on the area of contact between the foreground and background series (Figure 10.25) it is easier to make out the salient details. In particular, the line of the background series of the third chord in our example progression, G-major seventh (V^7), has been added to the graph. This dashed line of the background series built on G-24Hz intersects the background series of the second (Dflat-major) chord at G-96Hz (h5 of D#19.2Hz and h4 of G-24Hz). Also changed in Figure 10.25, the Dflat-major objective notes are now in grey while the G-major notes are in black, illustrating the step toward the third chord of the example progression.

Effectively, as the objective chords step from Dflat to G, the ear attempts to graft the new pattern of tones on to the existing background series built on D#19.2Hz. This it can do through a modulation exchange linked at the internal conjunction at F-h35. The five ratios of the Dflat-major foreground series based on C#h7 are exchanged for seven ratios of the G-major foreground series based on G-h5 (both h7 and h5 are on the D#19.2Hz background series). An important feature of this foreground modulation is that it is ‘uphill’; the exchange absorbs rather than releases ‘energy’, though the two arrangements contain equal numbers of harmonics, the former arrangement has overall lower frequencies, implying proportionally a lower energy level – given that all other factors remain constant. However, once the system reaches this configuration, another more fundamental exchange becomes possible: a modulation exchange between a background series of five ratios built on D#19.2Hz for four ratios built on G-24Hz. And in contrast, this low-level exchange releases significantly more energy and complexity than the foreground exchange absorbed, thus overall the double exchange represents a net release of energy and through this double exchange the system manages to reach the configuration shown in Figure 10.26.

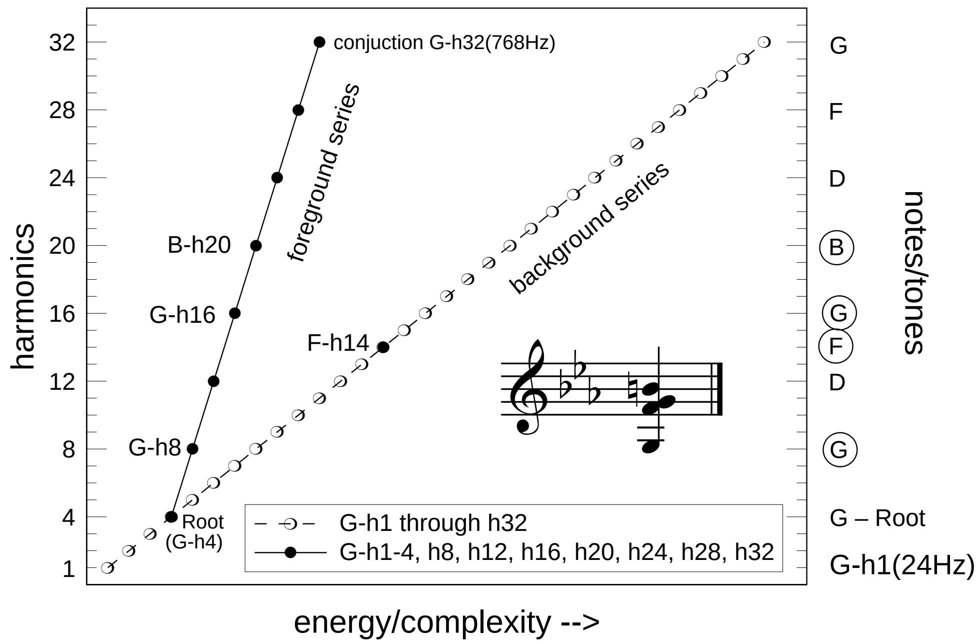


Figure 10.26 The third chord in the sequence of the Neapolitan sixth, G-seventh.

In the somewhat rarer situation of the Neapolitan sixth chord resolving directly to the tonic, a second dashed line running through G-96Hz, might be drawn on Figure 10.25, with an even more anticlockwise tilt than the G-24Hz background series (even less energetic). This second dashed line would be the background series of C-32Hz, the final chord in our example progression. In the case of a direct resolution to the tonic, the dominant arrangement built on G-96Hz is not actualised, merely travelled through, an intermediate step in what is in effect a triple computation to reach the configuration illustrated in Figure 10.27 (more or less).

After all the complexities of chords two and three in our example, the final chord is not to be outshone, it has its own story to tell, a tale of nesting one background series within another. Although a straightforward background series built on C-32Hz and foreground series based on C-64Hz would suffice to describe all the objective notes (the two white dot lines in Figure 10.27), it is in the nature of systems to seek out their lowest level of complexity and least energetic configuration – their ground states. And there is a most powerful modulation exchange to hand for the system to achieve this state: the dupla 1:2 modulation.

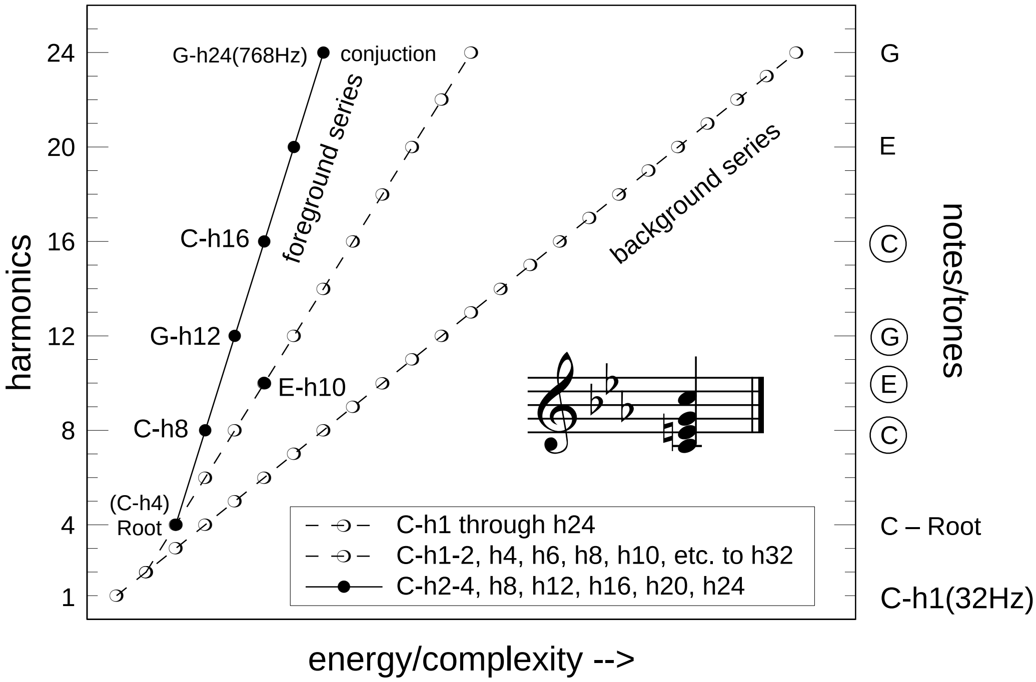


Figure 10.27 The fourth and final chord in the sequence of the Neapolitan sixth, C-major.

With all of the objective notes of the C-major chord lying along the series based on C-64Hz, there is only this fundamental maintaining the viability of the original background series built on C-32Hz – originally constructed in the modulation step from the previous G-seventh chord. The system can 'let go' of any ratios beyond this point (C-h2) without losing any of its objective notes. This is very attractive, as the dupla 1:2 modulation exchange, notionally, releases a large amount of energy and complexity. However, once the system reaches a single series arrangement built on the tightly knotted C-h1-2, a further release of energy and complexity beckons, in the by now familiar form of a 'summarising' of the principal harmonic gist of the (relatively) extended single series based on C-64Hz. Thus a new 'shorthand' foreground series could emerge on C-128Hz, to economically carry the bulk of the information contained in the objective notes, while the viability of the C-64Hz (now background) series is maintained by the presence of E-h10 and the foreground root tone, C-h4. This arrangement is broadly similar to that described for the minor triad (first section of this chapter) with E-h10 of the C-major chord playing the same role as G-h12 in the minor triad.

CHORD PROGRESSION

A *succession* of chords, any collection of randomly chosen chords played one after the other, generally does not produce a wholly satisfactory tonal composition. Music in the tonal era, and its continuation up to the present day, exhibits a particular and restricted choice of sequences of chords that strike the ear and aural cognition as being somehow 'right' – commensurable chord *progressions* rather than incommensurable and incoherent chordal successions. The question is: what is it about chord progressions that makes them 'right'?

For the most part, in this document, the focus has been on each chord's *configuration of partials* (i.e. chord structure or type) set at their lowest positions within the harmonic series. Potentially, however, there is an inexhaustible supply of higher positions in the overtone series where the various chord types could also find a comfortable home. And considered as a formal mathematical scheme, the MOS model of nested harmonic series can be extended both in an upward and a downward direction without limit so as to accommodate any configuration of partials imaginable. In practice though, the limits of hearing place upper, and lower, boundaries for the range of sound to which the MOS model can be applied – when considered in some degree as modelling aural cognition. Well within the practical upper range of the ear lies a band of frequencies where the more complex chord structures, in their base positions, can combine with upward transpositions of the less complex chords (chords with base positions lower down the overtone series, e.g. the major triad), so as to form a more or less continuous 'stream' of contiguous harmonic exchanges linked together by common values (conjunctions), expressed in the form of different mutable number digit sequences. A worked example of this *flow of harmonic computation* is given in Chapter 12 and Example S – the first prelude from the *Well-tempered Clavier* by J.S. Bach – and other examples are listed in the Contents.

The ear's most discriminating frequency range lies around 2 kilohertz – the top octave of the piano keyboard – and in the octave below and above this range the ear is nearly as good. However, all of this band of greatest sensitivity lies above the written pitch range of normal music-making! The ear lavishes its resources not upon the fundamental pitches of the 'music' so much as on the higher resolvable overtones. The reason largely for this is the direction-finding component of aural processing – the last evolved and predominant feature of the mechanism. The directional processors must match up finely graduated time differences (circa one thousandth to one ten-thousandth of a second) between the frequency patterns reaching each ear. Achieving this resolution has dictated the ear's range of sensitivity both for frequency and volume. As described by Beament[1], the processors in the auditory pathway package all the inputs with a common time difference (between left and right ears) into a single sensation, a sound perception of pitch/tone color, combined with the attribute of direction of source. (Left right differences in the volume of upper range frequencies also play a lesser but not insignificant role in determining directionality.)

Upon apprehending a chord progression (or chord succession), as with any natural sound, the hearing mechanism discovers a common time difference between the arrays of harmonics entering each ear (or a common 'no time difference' for sound coming from straight ahead or behind), and packages up these 'time-stamped' frequency patterns as a single sensation, which is passed on to conscious perception. The determination of direction will chiefly be reliant upon matching transient frequency patterns up to the 2 kilohertz range, and, in commensurable chord progressions, the unconcious processors will find amongst these patterns one or more 'conjunction pitches' that remain constant against the shifting background induced by the flow of notes in a tonal composition. These beacons of stability are the unchanging mutable number values which link together the chord progressions of intelligible tonal music. In contrast, incommensurable chord successions will have no common frequencies with the pertinent range to connect them together. Whether the recognition of this commensurability arises within unconscious aural processes and is passed on to the conscious mind as an attribute of the sound sensation or, is recognised at a higher level is not critical to the argument. (Though I favor the automatic low-level processors because the recognition of commensurability/incommensurablity appears to be instantaneous and unmediated.) However, like word recognition, which presumably does occur at a higher mental level, the different relative patterns of changing and unchanging pitch exhibited by the various tonal chord progressions will be consciously recognisable and memorable, and recognisable at any frequency level (i.e. in any key and at any pitch).

Rather more questionable than the accessibility of the upper levels of nested overtone series is where the practical limits, at the bottom end of these extended series, lie. Some of the harmonic series illustrated in this chapter reach down to C-32Hz, close to the threshold of audible sound, and elsewhere in the MOS documents, frequencies near to the ‘heart at rest’ rhythm of circa 1 hertz are touched on. This is beyond the the audible range of the ear, though electrical rhythms in this low frequency range in the brain do exist. However, the question of to what extent human perception can reach down to such infra-sound levels may well not actually be a critical issue in that the processes of musical cognition might conjure or construe frequencies beyond the limits of actual hearing from the objective sound stimuli. That is, by processing music well within the ear’s range low frequency relationships are synthesised from, or simply implied by, the patterns found in the audible sound. With aural cognition operating rather like a lemur living in the canopy of a tropical forest, leaping from branch to branch and tree to tree, the shadowed ground no more than a hazy distant reality, occasionally glimpsed. Similarly aural cognition may fly from chord to chord in the upper reaches of a forest of mutable number digit sequences, with only a faint perception of an absoulte fundamental pitch arising. Indeed, one could summarise the whole MOS model approach to tonal music in terms of the factorisation of a harmonic series, in which the top-most one or two factors – ‘the sunlit canopy’ – captures the harmonic gist.

Of course, these ideas are highly speculative: Although great advances have been made in neuroscience, the full story of musical cognition has yet to be unravelled. However, notwithstanding the very real danger that experiment and observation will in future reveal the processes of aural cognition to be entirely unrelated to the MOS model presented in these documents, the mathematical consistency of the scheme provides an alternative and rigorous arena of validation, and one independent of the difficulties presented by the convolutions of human sensibility.

Although by no means an exhaustive catalog of all chord types, hopefully, this chapter’s collection of basic types, set at their lowest positions within the overtone series, is enough to at least demonstrate how the mechanism of nesting one harmonic series within another, can provide a useful approach to elucidating the nature and attributes of chords, and chord progressions, in tonally organised music. Perhaps in time more chord types will be added to this chapter, existing entries amplified, and no doubt, some of the rasher assertions, ‘clarified’.

METERS

Although chords and meters inhabit distinctly different frequency domains they are linked together in the MOS approach by their shared periodic nature. This common character running through the basic materials of tonal music provides the foundation upon which a single mathematical model of both harmony and meter may be constructed – the modulating oscillatory systems model of mutable numbers. And though the structure of musical sound (i.e. mechanical longitudinal waves) and metrical pulses (i.e. periodic amplitude fluctuations and other nuances carried within those waves) differ in their elementary architecture in the objective raw stimuli, eventually, in labyrinth of aural cognition and the theatre of the mind they are all subsumed within an encoding of nerve impulses – the precise mechanisms of which are at present not fully understood. Notwithstanding, the common denominator of the neural processing of aural information, and its instantiation into the experience of music, does lend some hope to the notion of a unified period-based mental system of musical apprehension. However, of course, while the former –

the MOS model – lies open to inspection the latter mental processes are shrouded and unclear. Yet whether or not the MOS model of mutable numbers makes any connection with, or strikes any parallel to, the processes of aural cognition, it hopefully remains a useful tool for music analysis by virtue of its wide ranging scope and mathematical rigour.

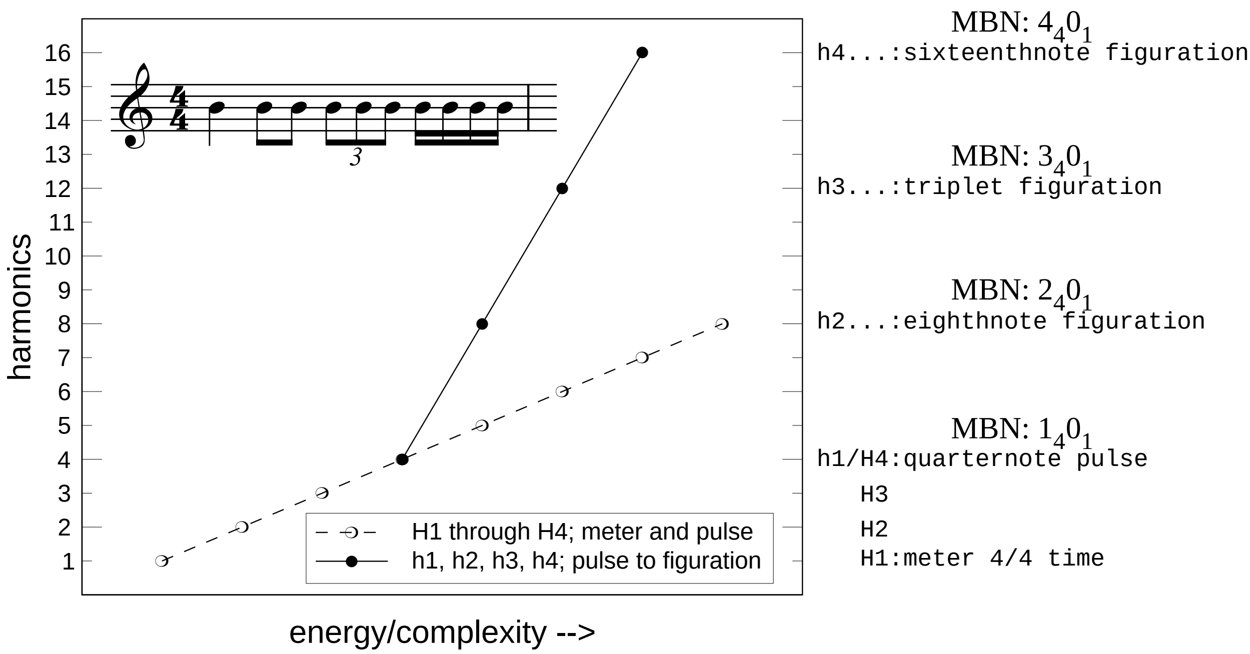


Figure 10.27 Simple quadruple meter, 4/4 time (common time).

Presented in Figures 27 through 32 are six of the most frequently used meters, expressed in the form of modulating oscillatory systems and labelled with mutable base numbers. All of these meters create simple nested systems consisting of two, three or four pulses nesting similarly low numbers of nested elements. That is, basic nesting/nested pairs of harmonic series that create low value mutable numbers.

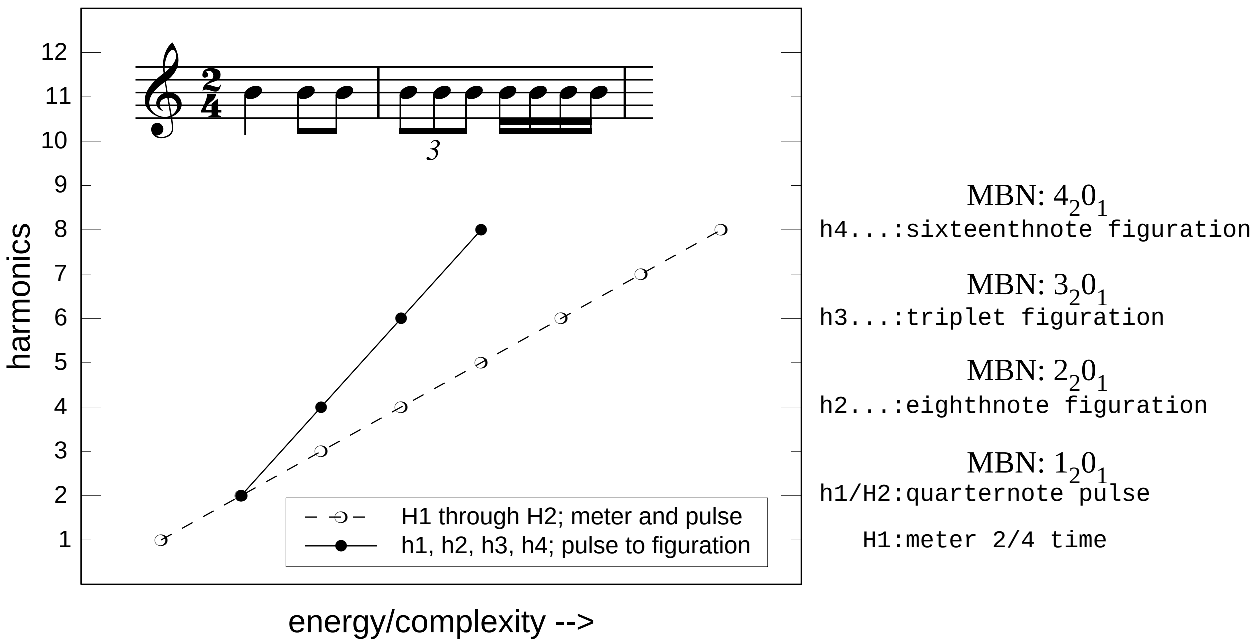


Figure 10.28 Simple duple meter, 2/4 time

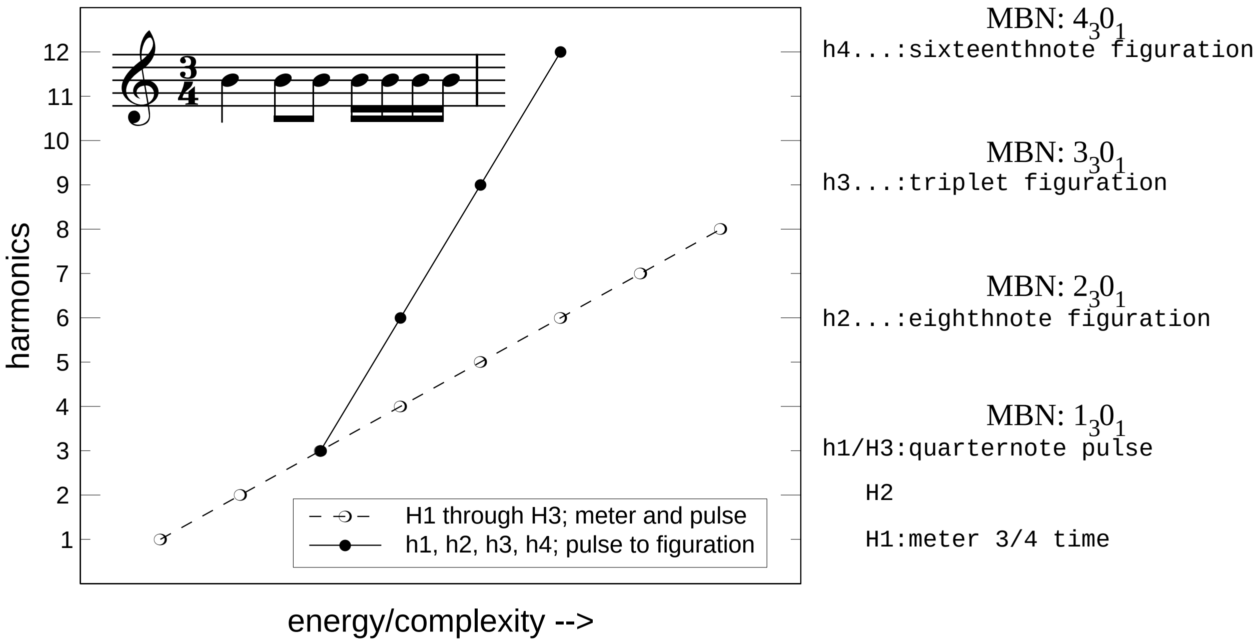


Figure 10.29 Simple triple meter, 3/4 time.

The Figures themselves are pretty much self-explanatory and can be usefully compared with Figures 9.8 and 9.10 in the preceding chapter, as well as with the graphs for different chord types above. However, one point that bears further discussion is the labelling of the meters themselves and the varying degrees of figuration within them with a range of mutable numbers. After defining the unit in the first column '0_1' the next column base gives the pulse of the meter: duple, triple or quadruple. This is the fundamental nesting series, which in Figure 10.27 is written MBN: $1_4 0_1$ or 4_1 if there is no subdivision of the pulse. In any but the most simple pieces of music it is likely that some degree of figuration or subdivision of the pulse occurs, and to accommodate these rhythmic features, column digits greater than one may be added to the basic pulse to reflect the degree of rhythmic articulation in the composition or particular section.

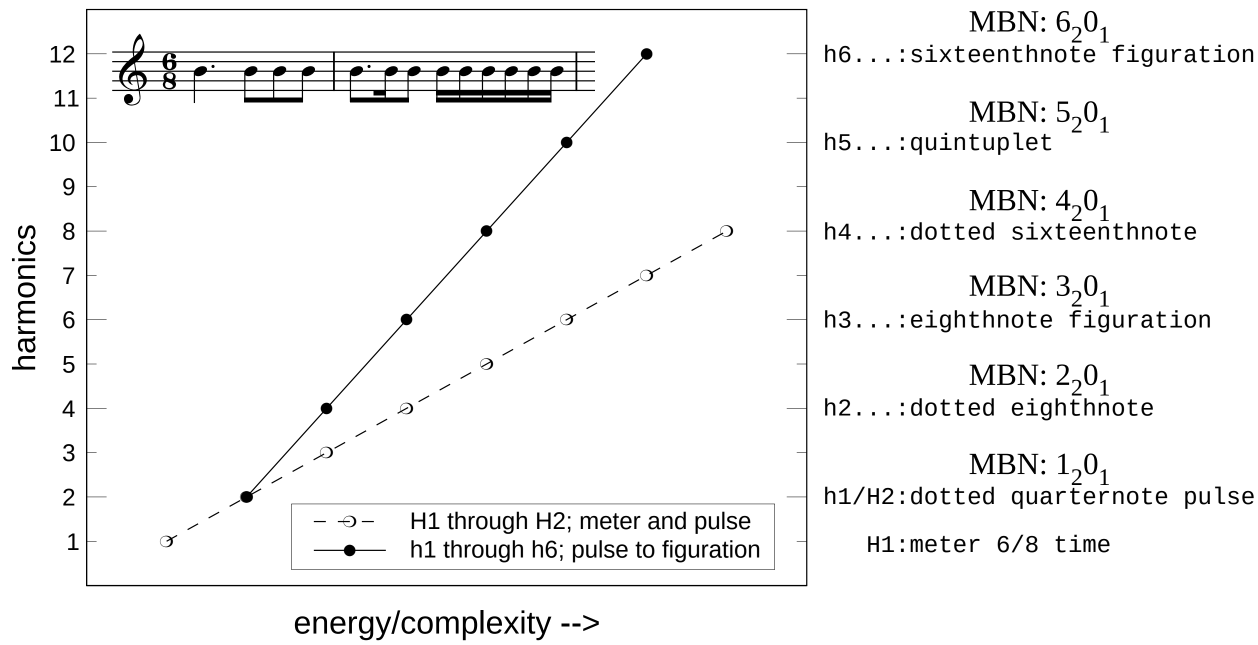


Figure 10.30 Compound duple meter, 6/8 time.

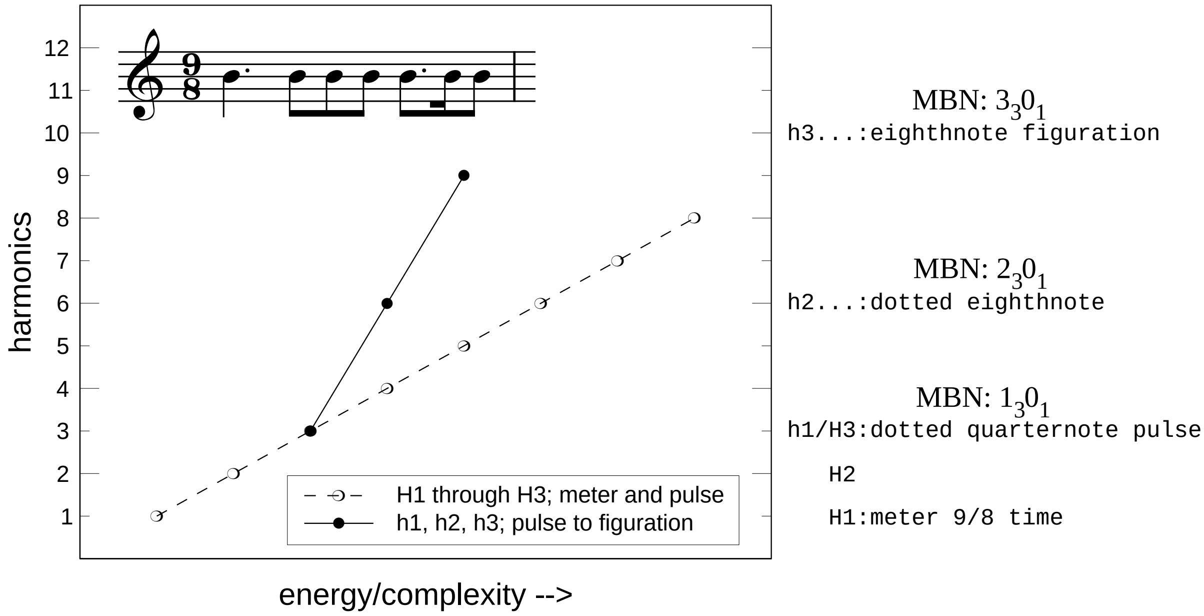


Figure 10.31 Compound triple meter, 9/8 time.

Taking Example K (first movement of Beethoven's Waldstein Piano Sonata) as an illustration of mutable numbers as expressions of meter, pulse and figuration. The meter is 4/4 common time throughout and the movement opens with a very downright eighthnote figuration (MBN: $2_4 0_1$) which extends to measure 12. After one measure of semibreve closure with pause (MBN: 4_1), the figuration onward from measure 14 is dominated by sixteenthnotes (MBN: $4_4 0_1$) up to measure 30, where it reverts to eighthnotes (MBN: $2_4 0_1$) for the four measures 31 through 34. From measure 35 to 41 the figuration is in halfnotes and quarternotes (MBN: $1_4 0_1$) before, at measure 42, Beethoven breaks into sixteen measures of triplets (MBN: $3_4 0_1$), taking the piece through to measure 57 where sixteenthnote figuration emerges again (MBN: $4_4 0_1$). And so the sonata proceeds, locked into the one 4/4 meter but constantly moving back and forth amongst the range of digit sequences allowed by the basic pulse.

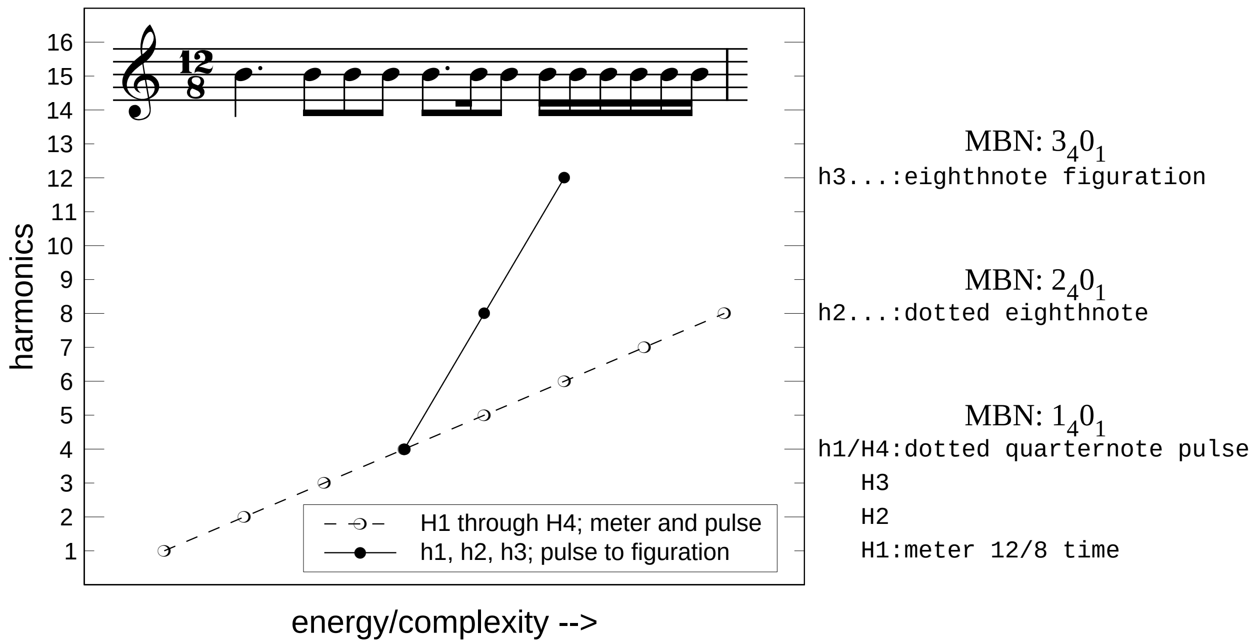


Figure 10.32 Compound quadruple meter, 12/8 time.

The range of values produced by the 4/4 meter (in the first movement of the Waldstein sonata) by means of changes in the level of figuration is not great, in decimal they are: 4, 8, 12 and 16 (plus somewhat theoretically 1 in measure 13 and 2 in measure 40). However, meter is not as limited in the range of values it can access as it would seem from these numbers – explored in the next section.

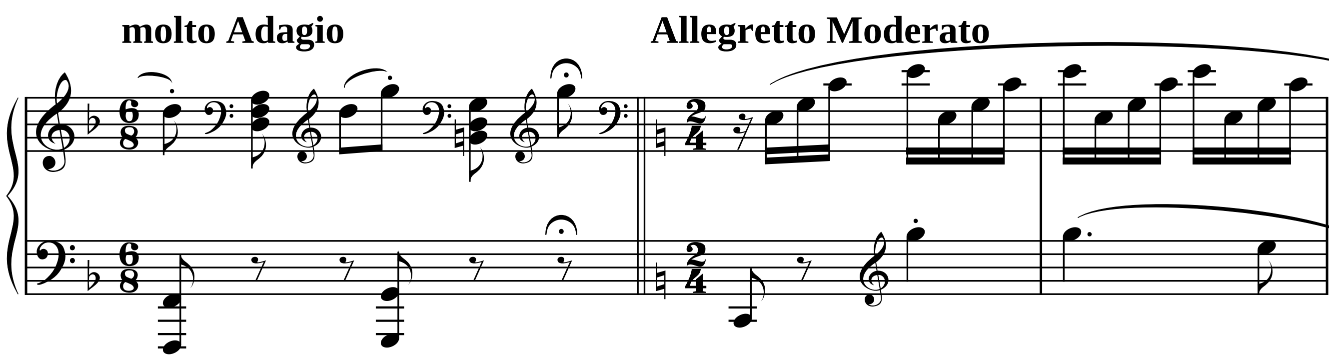
In contrast to mutable numbers in the domain of pitch/timbre (i.e. single unconnected nested harmonic series), definitive values have be ascribed to individual isolated meters without the necessity of an exchange between two different meters delineating a conjunction – and therefore also a value. The reason for this is that meters (i.e. the set of metrical/durational relationships) are constructed by the composers/performers with the cardinality of the system of nested harmonic series explicitly given by the degree of subdivision or figuration in the passage, section or composition as a whole. That is, a particular highest value relationship is specified in the score by means of the finest subdivision of duration – this defines a value in relation to the meter, the unit. Nature on the other hand, in the production of harmonic spectra in musical sound, generates series (i.e. the set of overtone relationships) with a theoretically unlimited cardinality. Thus as nature defines no value or cutoff point itself in the more intense and extensive realm of pitch, it is necessary for the algorithm of symmetrical exchange to find a conjunction between two harmonic series (i.e. chord progression) before any value can be unambiguously established. Indeed, amongst the plethora of overtones nature provides and the ear apprehends, the algorithm of symmetrical exchange in isolating one conjunction, in fact, defines a whole harmonic series of conjunctions stretching as far as human hearing ranges, and beyond. Notwithstanding, in the domain of pitch/timbre a single mutable number value is generally ascribed to the first conjunction lying above the frequency of the highest written note, though strictly accompanying this notated value there is also an associated range of integer related conjunction values. While it is open to debate as to which conjunction value(s) best encapsulates the progression between two nested systems – fundamental conjunction, first conjunction above highest note, some other conjunction or all conjunctions together – the choice of *first conjunction above highest written note* preserves a uniformity of structure and application between metrical and harmonic modulating oscillatory systems, by effectively drawing an equivelence between note-pitch and temporal subdivision.

Meter and Tempo

The big question yet to be addressed, and indeed, a question I have shied away from up till now, is that of tempo variation: How can the smooth fluctuations of tempo (both on the small and the large scale), so essential to the expressive qualities of much of the tonal canon, be accommodated within a model built on the discrete whole number ratios of the harmonic series? As illustrated above, the MOS model handles the face value fluctuations of figuration within a meter and the face value change from one meter to another (Figures 6.11 and 6.12) easily, as these are represented in the score by simple discrete numerical relationships, but this is only half of the story of duration. The other half is tempo variation.

The answer turns out to be simple and not entirely new: tempo variation is accommodated in the MOS model as a change in the size of the fundamental period (H1) of the metrical system. These changes are taken to have the character of a smooth continuum of Malzel Metronome values and are similar to the small fluctuations in the fundamental unit of factor format mutable numbers – only grossly larger. Whereas the small fluctuations in the unit of the MOS model's treatment of harmonic progression stem from a flexing in the values of relationships in complex schemes of nesting, which go largely unnoticed by the human ear; in contrast, the gross changes in the unit value involved in tempo variations are easily apprehended by aural cognition, and indeed are intended to be highly noticeable – highly expressive. Essentially, large changes in the fundamental unit value of a metrical MOS have the effect of projecting the rather limited relationships of a meter over a substantial range of absolute values, i.e. Malzel Metronome values. Indeed, one way of looking at tempo variation is to consider it as a rigid translation[2] of the ratios of the particular meter over the absolute values of a number-line expressed in metronome values (see also Figure 14.11). For example, to produce a ritardando the fundamental unit (e.g. 4/4 time measure = 15MM) may gradually be scaled in size (i.e. quarternote = 120MM to 60MM), and accelerando vice versa.

Remarkably, the upshot of expressive tempo variation – large smooth changes in the unit value of the meter – is that despite the apparently limited relational resources of metrical modulating oscillatory systems, they are able to traverse a similar range of values to their rather more sophisticated counterparts in the discretely structured domain of pitch/timbre! That is, through tempo variation – expanding or contracting the absolute duration of the fundamental unit – a relatively small set of relationships (typically less than twelve ratios) can move across the entire temporal range from the gentlest molto Largo to the most vigorous Prestissimo, and back again. In the example Figure 10.33, the temporal range is not quite this extreme, and here whole number values have been chosen so as to easily match-up the rows across the sesquialtera 2:3 exchange; whereas under normal performance conditions the overall 4:1 tempo variation shown here is most likely to be some real value such as 3.734...:1, that the player feels intuitively, guided by the impulse of the moment.




```
Introduzione: molto Adagio   ♪= 60MM  --1:4-->  ♪= 240MM   Rondo: Allegretto Moderato
                                                h8   :960MM :thirtysecondnote figuration
                                                h4   :480MM :sixteenthnote figuration
                                                h3   :360MM :triplet figuration
thirtysecondnote figuration: 240MM:h12          h2   :240MM :eighthnote figuration
   sixteenthnote figuration: 120MM: h6          h1/H2:120MM :quarternote pulse
      eighthnote figuration:  60MM: h3   2:3       H1: 60MM :meter 2/4 time
          dotted eighthnote:  40MM: h2
   dotted quarternote pulse:  20MM: h1/H2
            meter 6/8 time:   10MM:    H1    1:6

        6/8 meter = (2 groups of 3) ——2:3—> (2 groups of 2) = 2/4 meter
```

Figure 10.33 Extract from Piano Sonata (Waldstein) Op.53, by Ludwig van Beethoven, illustrating the effect of a metrical sesquialtera 2:3 exchange between the 6/8 meter Introduzione and 2/4 meter Rondo, combined with a 1:6 tempo variation; which produces an overall 1:4 diminution in the duration of the written notes (i.e. 2/3 × 6 = 4).

In passing from the Introduzione to the Rondo with a change from a slow 6/8 to a moderately quick 2/4 meter, two different metrical events take place simultaneously: an *exchange of meter* and a *change in tempo*. These are two distinctly different processes: the former strictly governed by the modulation algorithm of symmetrical exchange with all the careful accounting that that implies; and the latter, applied as it were by hand, a change arbitrarily imposed upon the system from outside – and a change technically invisible in terms of the internal relationships. Yet of course to us, looking in upon the 'little world' of a tonal composition, changes in tempo are strikingly obvious and highly effective – which our emotional response often confirms. *By transmitting a message of quantitative change, as with the computations of mutable numbers in the domains of pitch/timbre, tempo variations add a second leg to the number processing that appears to be integral to our apprehension and experience of tonal music.*

Finally, just for clarity, taking the metrical exchange separately (i.e. stripping out the tempo variation so as to leave constant durational values across the double barline) produces a familiar sesquialtera 2:3 exchange, as detailed in Figure 10.34.

```
              Introduzione     ♪= 60MM  ———>  ♪= 60MM   Rondo

   eighthnote figuration:  60MM: h3            h2       :60MM :eighthnote figuration
       dotted eighthnote:  40MM: h2      2:3   h1/h2    :30MM :quarternote pulse
dotted quarternote pulse:  20MM: h1/h2     2:3    h1/H3 :15MM :meter 2/4 time
          meter 6/8 time:  10MM:    h1/H2            H2 :10MM :
      fundamental series:   5MM:       H1            H1 : 5MM :fundamental series
```

Aggregated Series: 6/8 pulse = (1 group of 3) ———2:3——> (1 group of 2) = 2/4 pulse

Nested Series: 6/8 meter = (2 × three) ———2:3——> (2 × two) = 2/4 meter

Figure 10.34 The metrical sesquialtera 2:3 exchange viewed in isolation from any tempo variation at a steady metronome rate. Notice that in the absence of any tempo variation three levels of nesting are required – top level aggregated series, middle level nested series and bottom level fundamental series – as is similarly the case for normal chord progression in the domain of pitch/timbre.

Notes

1. Beament, J., *How We Hear Music* (The Boydell Press, Woodbridge, Suffolk, UK, 2005), Chapter10 – 'A Sense of Direction'. Here I rely on, more or less in toto, and reproduce in part, James Beament's model of hearing.

2. Here the term 'translation' is used in the vein of symmetry operations from mathematical group theory, explored in Chapters 13 and 14.