7

Nested Harmonic Series

THE FORMAT OF COMPUTATION

The introduction described how, a while ago, I wrote a group of AWK scripts to examine and search music data which had been rendered into a plain text description of the composition in question. This textual description was derived from digital scores via the MIDI format – Musical Instrument Digital Interface. However, the limitations of this method stimulated a search for a less cumbersome way of accessing the information. I began to think about expressing a piece of music entirely as ratios of one given frequency and amplitude, as this might offer a better way to capture and expose relationships and underlying structures within the music data. So I started to look at matrices of harmonic series – the systematic nesting of the harmonic series within itself – and was intrigued by the patterns and relationships that emerged. In this chapter we shall examine some of these fascinating structures and connections.

Nested Patterns

Nesting is a concept made familiar by a computer's directory structure and often likened to Russian dolls packed one inside another. Many of the simple cellular automata discussed in Chapter 5 produce nested patterns of triangles, and often, equally simple procedures generate the related patterns of *fractals* – which we shall look at later in this chapter. A key feature of these systems is the generation of *structure* through recursive processes, leading to a build up of the same pattern over many different scales – nesting. Rule18's output of many layers of nested triangles is a good example – Figure 5.1. A matrix of harmonic series shares this nested character, plus *mobile features* which thread their way through the pattern.

The script *th.awk*, which can be found in the scripts Chapter 19 directory, will produce a matrix or table of nested harmonic series similar to the *Table of Harmonic Series* illustrated in Figures 7.3 and 7.11 (often abbreviated to THS in these documents). The THS is equivalent to the *Sieve of Eratosthenes* – the original procedure discovered in ancient times for identifying prime numbers from among the list of natural numbers, by means of systematic division. Running AWK scripts is described in Chapter 5. The script has defaults, but may be given instructions attached to the run command so as to override them through the use of the '-v variable=value' option. The width in characters and number of generations can be specified by '-v width=yourvalue' and '-v generations=yourvalue'. The defaults are width=100 and generations=300. Figure 7.1 contains a fragment of the output produced by the *th.awk* script.

The 'arrowhead' feature in the fragment below reveals something of the structure that arises within a matrix of nested harmonic series, where each individual 'note' of the fundamental series generates its own 'child' harmonic series – offspring which look rather like chords or harmonies in music. (The file *bigArrow* in the examples directory (Chapter 19) provides a stunning, though remote, example at position 1:6,498,159,880,212,000 where the nested harmonics of H1 through H36 all come together.) In the Table of Harmonic Series, illustrated in Figures 7.3, 7.11 and 7.12 lesser arrowheads can be decerned at 1:12, 1:24, 1:36, 1:48 and 1:60. For those who don't wish to run AWK themselves, there is another file in Chapter 19, *th_out*, which contains the pattern of the THS extrapolated to one thousand generations.

```
BXX  X                                                    X.354
M   X                                                     ..355
BX X                                                      ..356
M X   X         X   X                             X       ..357
BX                                                        ..358
M                                                         ..359
BXXXXX XXX X  X  X X   X     X     X   X    X              X360
M                 X                                       ..361
BX                                                        ..362
M X       X                     X                         ..363
BX X  X     XX           X X                       X      ..364
M   X                                                     ..365
BXX  X                                                      X
```

Figure 7.1 Fragment from the file *th_out*, produced by running *th.awk* script, circa generation 360.

What the *th.awk* script does is to chart all the 'child' harmonic series which could arise from a fundamental 'parent' series, in ascending whole number order, like playing the *universal chord of the harmonic series* described in Chapter 1 and Chapter 9 (see Figure 9.1) on the piano and writing out the 'child' harmonic series that emanate from each note (i.e. the frequencies of timbre). The rule or algorithm to do this, built into the script, might be called the *standing wave* rule and could not be simpler: Divide the generation number (harmonic/row) by the fundamental frequency (column number) and if it fits exactly (forms a wave node) print X, otherwise leave the position blank. This was the algorithm employed by Eratosthenes in ancient times to find prime numbers. Figure 7.2 shows the first eight frequencies of a harmonic series with each nested series that emanates from them as horizontal rows. Note the formation of arrowheads centered on columns containing a G-h3, G-h6 and G-h12.

Eratosthenes of Cyrene (circa 276–195 BC) worked at the Library in Alexandria and, around 200 BC, produced the first known table of prime numbers and a method for deriving them by division, now called the Sieve(s) of Eratosthenes. He was also the first person to measure the circumference of the Earth, calculated by measuring the shadow cast by the Sun at the summer solstice in upper and lower Egypt and applying Euclidian geometry to the result. In later life he was head librarian of the great institution of ancient learning and an instructor in the royal household of the Ptolemys. It is reported that at the age of eighty, after losing his sight, he fasted until death.

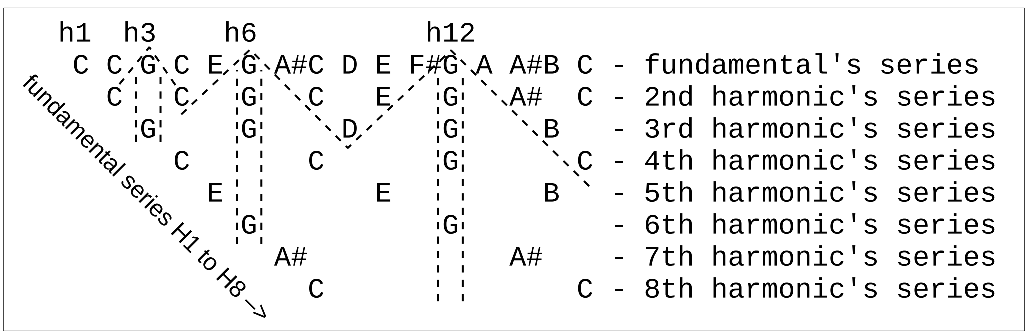

Figure 7.2 Parent series with child harmonic series built on the fundamental frequencies H1 to H8.

The eight harmonic series shown above are just the first of an unending theoretical sequence of nested series, all of which use no other harmonic ratios than those contained within the base series and so are sub-sets or aspects of the fundamental series. All of these nested sub-series are of the same structure as the fundamental series in which they nest, though built on ever higher (nested) fundamental frequencies.

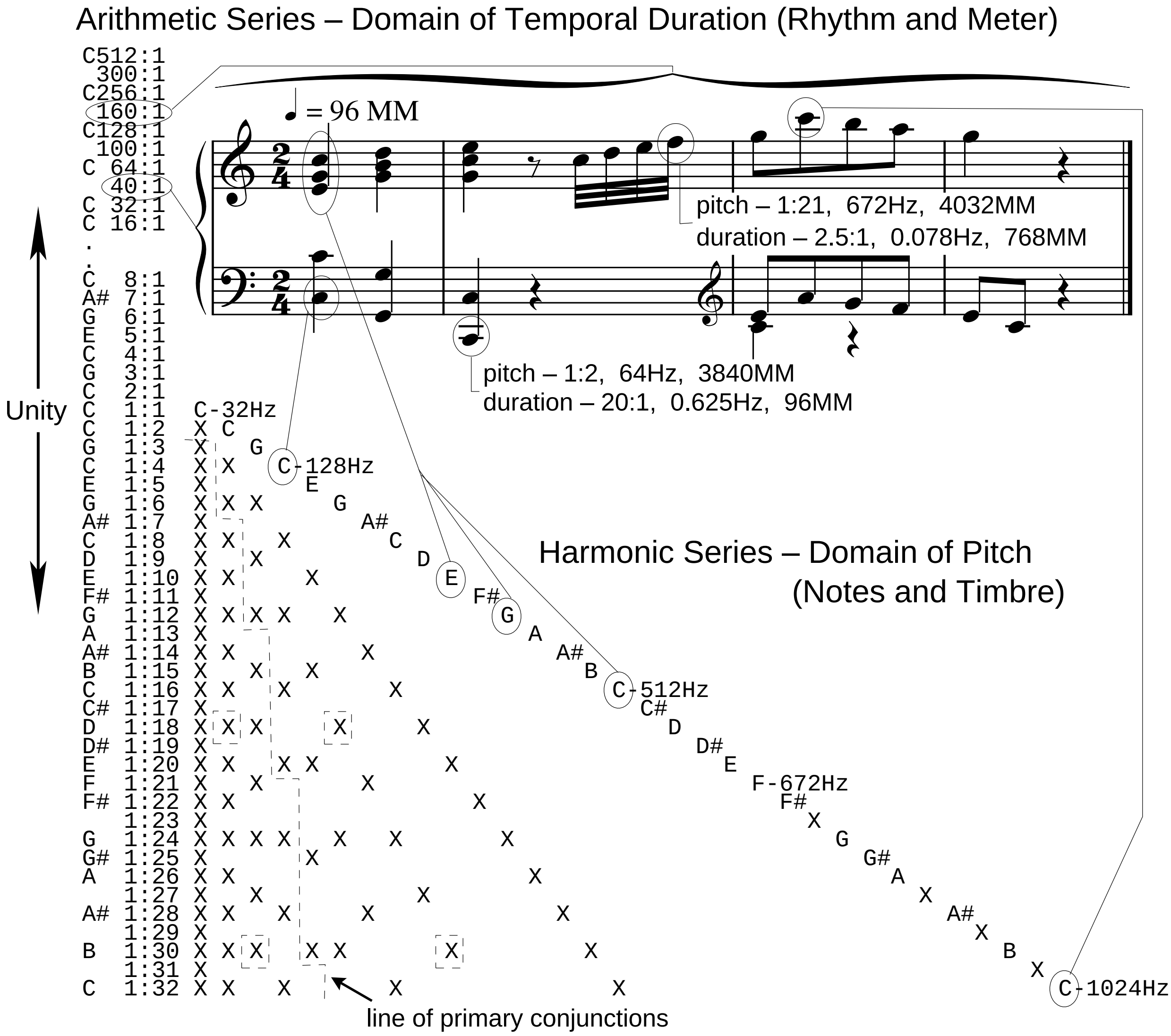

Figure 7.3 The expanded Table of Nested Harmonic Series, with the periods of note frequencies circled (bottom of figure), extended out into the arithmetic domain beyond unity (1:1) with note durations circled (top of figure).

Introducing the THS Format

In this chapter I would like to introduce the *Table of (Nested) Harmonic Series* (abbr: THS) and lightly touch on a few of its features. As mentioned earlier, when I began looking for an alternative format to search music data, the idea of a matrix of harmonic series seemed promising; and so I constructed the THS to explore such a data structure. The arrowhead patterns that emerged puzzled and intrigued me. Later a mobile pattern of harmonic *conjunctions* between columns caught my eye (Figure 7.3 dashed line). This much less conspicuous feature, weaving a path through the columns and rows, turned out to be the crucial element *as it could be linked with chord progressions found in tonal music*. Indeed, the dashed line in Figure 7.3 threads it way through every possible basic chord progression available to composers, that is, the whole numbered proportions: 1:2, 2:3, 3:4, ... to n:n+1. (This mobile feature is also discussed in Chapter 9.) Here are some examples of the conjunctions that lie along the dashed line in Figures 7.3 and 7.10, check them out.

```
Row   Column                   Column
1:2   C-h1 (2nd harmonic)  = C-h2 (1st harmonic)
1:6   C-h2 (3rd harmonic)  = G-h3 (2nd harmonic)
1:12  G-h3 (4th harmonic)  = C-h4 (3rd harmonic)
1:20  C-h4 (5th harmonic)  = E-h5 (4th harmonic)
etc.
```

This holds for multiples too,

```
1:4   C-h1 (4th harmonic)  = C-h2 (2nd harmonic)
1:12  C-h2 (6th harmonic)  = G-h3 (4th harmonic)
1:24  G-h3 (8th harmonic)  = C-h4 (6th harmonic)
etc.
```

and the conjunctions can be extended laterally away from the dashed line (dashed line boxes).

```
1:18  C-h2  (9th harmonic) = G-h6  (3rd harmonic)
1:30  G-h3 (10th harmonic) = E-h10 (3rd harmonic)
```

And interestingly in Figure 7.3, in the conjunction at row 1:24 between columns G-h3 and C-h4, if all the column harmonics (i.e. Xs) are taken together, one has the chord exchange of the pivotal dominant-seventh to tonic full cadence – GGDGBDFG –> CCGCEG – the fulcrum of key change in tonal music.

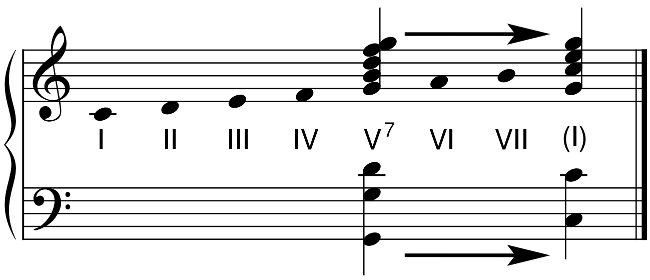

Figure 7.4 Dominant-seventh chord resolving to the tonic major chord, a full cadence in the configuration of two harmonic series built on columns G-h3 and C-h4 in Figure 7.3 (i.e. the two columns of Xs up to row 1:24).

Patterns are usually indicative of some underlying structure or mechanism, and, assuming they run deeper than merely a product of tabulation, which the alignment with natural chords suggests, what, I wondered, might they signify? My attempt to answer this question is set down in these documents, and of course, it is entirely possible that I haven't found the right answer or that there is nothing to be found at all! However, throughout, my instinct has been that there was something interesting here. But what? Arrowheads and conjunctions in a system of nested harmonic series are the principal leads I followed up.

Now to introduce the table in more detail, I will describe a little of how it was originally intended to be an exploration of a ratio-based music data format – a project that has been side-lined by the unexpected emergence of these patterns of arrowheads and conjunctions. Down the left-hand side of the THS is the backbone of the table (Figure 7.3), a sequence of whole number ratios radiating from the central unit ratio 1:1. These ratios link together the information in the first chord from the Haydn Piano Sonata: the notes (C 1:4, C 1:8, E 1:10, G 1:12, C 1:16), the tempo (40:1) and the timbre – nested harmonic series, columns headed by the note letters C at row 1:4, C at row 1:8, E at row 1:10, G at row 1:12 and C at row 1:16).

Given middle C at 256Hz, the lowest pattern of frequencies that can match the opening chord (Haydn's Piano Sonata No. 2, Hob. XVI/7) is generated by a harmonic series with a fundamental tone of

32Hz, three octaves below middle C. This core value, 32Hz, is literally the ‘Key’ value, the tonal centre, i.e. C major. And in this scheme it remains constant until a modulation to another key occurs. One way of understanding this is to think of a natural trumpet or horn which derives all its notes from one harmonic series, that of its fundamental frequency. As long as the fundamental is low enough (the trumpet long enough) any natural tone could in theory be played and all tones are related more or less closely to that fundamental frequency. Any combination of notes could be specified in this approach by the fundamental dropping down to a point where all chromatic tones become available. Also, it should be noted that the nature (and intention) of the format is that these notes will be truly ‘just’ in intonation, the notes shifting frequency to reflect any change of tonal centre, so as to be equally ‘just’ in all keys. And even within the ratios of a single tonal center there will be different perspectives. For example, the series constructed on A#H7 sees C-H63(h9) in contrast to the series built on C-H8 which views the same note as C-H64(h8), or the two series emanating from G-H3 and D-H9 which see A-H27, F#H45 and C-H63 in contrast to the ‘tonic’ series built on C-H2 which views these three crucial notes as A-H26, F#H44 and C-H64. Such discrepancies hint at the relational fracture that lies between tonal centers. However, by relying entirely on ratios to define the music data, the format remains utterly faithful to the proportions of the harmonic series, both within and between tonal centers – with the shifting relational patterns of tonal centers charting the larger scale structures of key. This is somewhat analogous to the consequences of relativity theory: any and all ‘relational frames of reference’ are placed on an equal footing, no one view of a note’s frequency is privileged over others. There is no fixed, absolute grid.

Tempo

With the root value (1:1) at 32Hz, the tempo, the period of metrical duration (2/4) is set to be forty times the period of the root value frequency – 32 cycles per second. Thus a measure of 2/4 will have a duration of 1/32 × 40 = 1.25 seconds which yields metronome tempo of quarternote = 96MM (MM = 60/(1.25/2). The alignment of tempo with a ratio of the tonal centre, C-1:1 the ‘home’ key, is entirely arbitrary, any number along the arithmetic series which rises up from the fundamental 32Hz could be specified so as to produce whatever tempo is desired, that is, within the limits of the discrete units of the system. Equally the tempo can change at any point by adjusting this ratio.

Parent Series (Notes) and Child Series (Timbre)

From the left-hand side of Figure 7.3, looking to the right of the column of ratios, starting from 1:1, the first column of Xs represents the harmonic series of the fundamental tone, C-32Hz in this case, marked as a continuous column of Xs. Every nested series, i.e the remaining columns to the right of the continuous column of Xs, represents a limited selection from these ratios of the fundamental series. The right-most diagonal arm duplicates the parent column but gives the note letter names A, A#, B, C, etc. or ‘X’ for non-note ratios, of the fundamental series.

Within the fundamental or ‘parent’ harmonic series built on C-32Hz (the first column) are nested an unending sequence of ‘child’ series. The same pattern of relationships and proportions is repeatably nested within the fundamental series at intervals of 2, 3, 4, 5, etc. based on the 2nd harmonic, 3rd harmonic, 4th harmonic, 5th harmonic, respectively. These form the remaining columns and they are the method by which the patterns of timbre of the individual notes/instruments are integrated into the fundamental series. An individual selection of these child ratios (with amplitudes) would record the tone color aspect of the composition.

Longer Waves

A measure of the example piece in Figure 7.3 has the relationship (period) of 40:1 with the core fundamental frequency of C-32Hz – in a sense the measure's duration is a low-frequency sympathetic resonance of the central reference tone. This period could be viewed as an 'out-nesting', an inverted or arithmetic resonance. Inside this measure of 2/4 there are two beats, the first stronger and the second weaker, the relationship of these quarternote beats to the out-nested fundamental of the measure is that of 1:2, a familiar pattern – a temporal octave relationship. And in the second measure the much more sprightly thirtysecondnotes reveal a shortcoming in the format: it has a problem expressing the combination of low pitch and short durations in whole number ratios of a single core C-1:1. This problem of *graininess* around the central 1:1 ratio could be addressed by lowering the fundamental or introducing two foci related by a simple integer ratio (e.g. 1:2 or 1:4), one for notes and timbre and a second for duration, with each set at a suitable relational distance so as to eliminate the problem from each particular frequency domain. However, it represents an unwelcome loss of simplicity and elegance.

In any extended piece of music there are larger relationships to the level of one measure, those of sections, movements, etc. from chunks up to the whole piece. In the table above, Figure 7.3, there is just a four-measure snippet. So for this small chunk, at the level of the whole extract (160:1) another out-nested resonance can be found which has a relationship with a single measure of 4:1, i.e. 160:40. Although the tiny extract only shows one level of out-nesting, these relationships are widely documented in works of musical analysis. Durational relationships of (thematic) musical phrases frequently display simple patterns such as 1:1, 2:1, 3:1, 4:1, denoted as AB, ABA, AABA, etc. with expositions, developments, recapitulation and the like, forming larger units out these phrases (of generally rather more arbitrary periods) leading eventually up to the period of the whole piece – the ultimate period. That is, the fundamental period of the whole composition, the subdivisions of which, both large and small, in principle, can be thought of as forming the entire musical edifice. Thus through many levels of temporal duration an inverted pattern of out-nested arithmetic ratios can be constructed which mirrors that of the nested harmonics of the frequency domain of notes and timbre.

One can visualise the relationships of out-nested arithmetic series and nested harmonic series as forming an 'hourglass-like' structure.

```
       9           X   9:1 Nested Arithmetic Series
         8       X   X 8:1   Temporal Domain <16Hz
           7           7:1    -rhythm
             6     X X 6:1    -meter
               5       5:1    -duration
                 4   X 4:1
                   3   3:1
                     2 2:1
                       1:1
Nested Harmonic Series 1:2 2
  Audible Domain >32Hz 1:3   3
   -notes              1:4 X   4
   -harmony            1:5       5
   -timbre             1:6 X X     6
                       1:7           7
                       1:8 X   X       8
```

Figure 7.5 'Hourglass' structure of out-nested arithmetic series and nested harmonic series.

And so in the extended THS format of Figures 7.3/5, music's domains of audible sound and temporal duration are united in a symmetric mirrored structure of arithmetic and harmonic ratios. Yet for economy only one wing of the THS is necessary, as the table can be read as frequency: C-h1, C-h2, G-h3, C-h4, E-h5, G-h6, etc. or as wavelength: C-a1, C-a2, F-a3, C-a4, G#a5, F-a6 etc. yielding either interpretation. (The letter 'a' for 'arithmonic' is used in place of the customary Greek symbol for wavelength, lambda.)

The above is largely a description of how I first envisaged using the Table of Nested Harmonic Series as a music data format. However, gradually, as I looked at the table and thought about the patterns of relationship, and in particular studied the mobile pattern of conjunctions between adjacent nested harmonic series, I began to realise that embedded within the table lay a *system of computation*: a means of exchanging one arrangement of nested harmonic series for another, and thereby, a method of describing the process by which one chord is commensurably succeeded by another. Which in other words is to say, a systematic explanation of the core organisational principle in tonal music: *harmonic progression*.

PATTERNS IN THE TABLE OF HARMONIC SERIES

One way of looking at the harmonic series is as a sequence of self-similar patterns, with each succeeding level of the pattern consisting of a lower, middle and upper tone. This produces an interlocking chain of even-odd-even relative frequencies (i.e. h4–h5–h6) with the upper tone in one level becoming the lower tone in the next. These successive 'trines' form a self-similar nested pattern within an expansion of the positive natural powers of two – the octaves from h2 – with each successive octave yielding a doubling of the number of patterns: 1, 2, 4, 8, 16, etc.

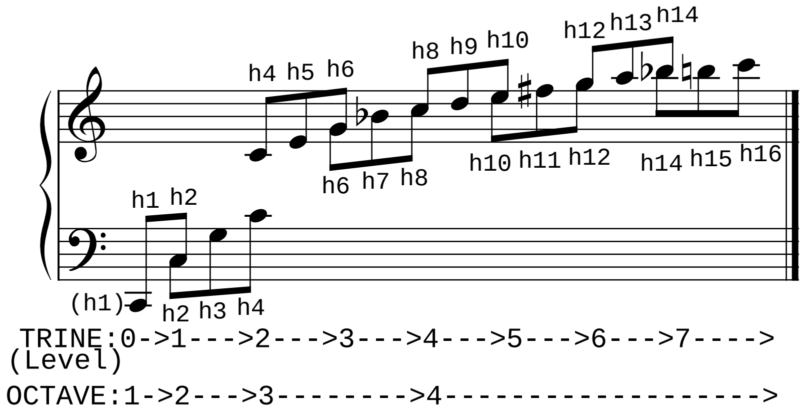

Figure 7.6 Trines up to h16 with (relative) even upper and lower boundary tones and odd median[1] tones.

It is interesting that music's most characterful sonorities are associated with prime number vibrational patterns, and so all of these intervallic sonorities are odd-numbered patterns, excepting two when it forms a prime vibrational pattern in the first octave of the harmonic series (i.e. where it divides the fundamental). Such sonorities are the raw expression of number in sound, the concrete audible sensation of magnitude directly felt: that is, the sensation of number unmediated by the higher modes of abstract thought and conceptual observation normally employed in the pursuit of traditional mathematics. Though, of course, in playing and listening to tonal music we may also bring to bear these higher mental functions, the core experience remains sensual. Thus of the primes, h1 is associated with the interval of a unison, h2 the octave, h3 the interval of a perfect fifth and h5 the major-third. The true sonority of h7 is somewhat obscured by equal temperament 'bending' it to accommodate the minor-third (G to A# ratio approx. 5:6) rather than sounding the alloy of major-second and minor-third ratio 6:7, nearly a quartertone flatter (21.4% of a semitone). And more generally in the various scales the foundational sonorities of the harmonic series are somewhat cramped by the straight-jacket of equal temperament.

The characterful intervals, the most memorable sensations, each introduce a novel oscillatory ‘shape’ – a new auditory symmetry. One, the fundamental C-h1, though technically not a prime number in abstract mathematics, is however completely different from silence; and the number two, C-h2 the octave interval, breaks new oscillatory ground for it is sensibly different from the unison, though we take it for granted as it adds no new partials to C-h1. The third harmonic, G-h3, produces clearly a new ‘shaped’ sound, the very memorable ‘empty’ fifth sensation; however, in contrast, the quality of the perfect fourth sensation (G-h3 to C-h4) is basically no different from that of the fifth. This is because h4 doesn’t introduce a novel irreducible oscillatory pattern, four – double twoness – is not a prime number. Five is a prime number, and its associated interval, the ‘sweet’ major-third (C-h4 to E-h5) is probably the most characterful oscillatory pattern of all to our ears. The major-third is arguably the iconic sound of western musical art in the tonal era. (As was the fifth to the preceding era: the epoch of organum.) Again in contrast G-h6, the minor-third, introduces no new oscillatory shape – six is an amalgam of two and three. This derived quality is demonstrated in the major triad (CEG) where the superposition of a minor-third (E-G) over a major-third (C-E) still results in a ‘major’ sensation – the major triad. If the whole series is taken into account – h1, 2, 3, 4, 5, 6 – the same derivative phenomenon is present, the chords CCG and CCGC produce much the same sensation, as does CCGCE and CCGCEG. The minor-third does not smother the major-third sensation when placed over it in a triad, yet the major-third predominates in a chord containing the fourth, fifth and octave; and, the minor-seventh practically smothers the lot of them in the dominant-seventh chord – CCGCEGA#. The minor-seventh, crowned by A#h7, is again a prime vibrational pattern. Beyond h7 in the harmonic series, h9 produces another derived sonority (3×3), which is attested by the ninth chord’s dominant function and character; and, the utilitarian nature of the major-second interval ratio 8:9. However, further on a unique pattern within the ear’s reach is created by F#h11 – the devil in music – the memorably strident augmented-fourth C-F#, where the third harmonic of the lower tone falls a dissonant half-step from the upper tone’s second harmonic. Interestingly it could be argued, that the next prime pattern, the major-sixth interval from C-h8 to A-h13, introduces a characterful minor-third sonority by the backdoor of inversion – the ratio 13:16 (i.e. 2×8) is only marginally wider than 5:6. The topic of the minor mode and ‘dualism’ – a mirrored, arithmetic component in tonal music – is touched on lightly below and in Chapter 9, and more fully explored in Chapters 11 and 14. As for the minor-third in its triad (E-h10, G-h12, B-h15), though this chord strikes the ear as surely being a characterful sonority, upon closer investigation, it reveals itself to be the characterful sonority of the perfect fifth (E-h2 to B-h3), enclosing a non-dissonant coloring tone (G-h2.4). Finally, at C#h17 a unique and characterful minor-second sonority appears (16:17), just intonations use the more utilitarian 15:16 proportion and the equal-tempered minor-second lies closer to ratio 17:18. However, these intervals are too narrow for the ear to sensibly appreciate, frequency ratios closer than 5:6 destructively interfere with each other on the ear’s detector membrane producing disorderly nerve pulses – dissonant sensations.

Out of this upward march through the harmonic series it is possible to detect a certain historical progression – a sequence of harmonic revolutions in western music. Beginning with melodic plainsong (h1 and h2); in the later middle ages the fifth and fourth (h3 and h4) were accepted as consonant in the revolution of organum, then the major and minor thirds (h5 and h6) in the tertian revolution 1500–1900, followed by the brief revolution of the chromatic style 1900–30 (h7 and h9) when sevenths and ninths were herded into the fold of consonance before the final atonal revolution broke upon western music.

The Fractal Connection

The self-similar trine patterns illustrated in Figure 7.6, which double in each ascending octave of the harmonic series, share much in common with the structure of fractals and find an interesting parallel in the well known Mandelbrot Set. In this example the lesser nodules of the main fractal pattern take on an odd number characteristic, almost a graphical resonance of threeness, fiveness, seveness, etc. Fractals are rather similar to cellular automata in that they are usually generated by the repeated application of a simple rule or formula and feedback upon themselves. Unfortunately the illustration Figure 7.7 isn't fine grained enough to show the filigree threads which branch by twos, threes, fives and so on, around the child nodules. However, an enhanced picture of the detail of the fiveness pattern is shown in Figure 7.8.

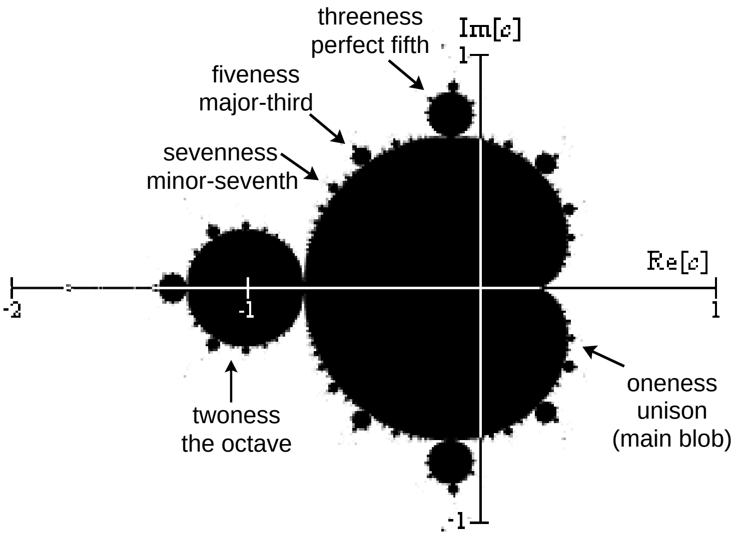

Figure 7.7 The Mandelbrot Set.

The arrangement of the nodules is of particular interest. The main blob itself has one filigree extension, which runs along the x-axis. The largest nodule attached to it, with filigree branches dividing by two – the octave relationship – lies over this single extension also on the x-axis. However, the next largest nodule(s), with a three-branching filigree pattern, are as far removed as they can be (while still being attached to the main blob) from the twoness nodule and the x-axis orientation. They point pretty well 90 degrees away from the horizontal. From the outlying positions of the threeness nodules an unending succession of ever smaller nodules, extend down toward the twoness nodule, with filigree patterns of five branches, seven branches, nine branches, etc.

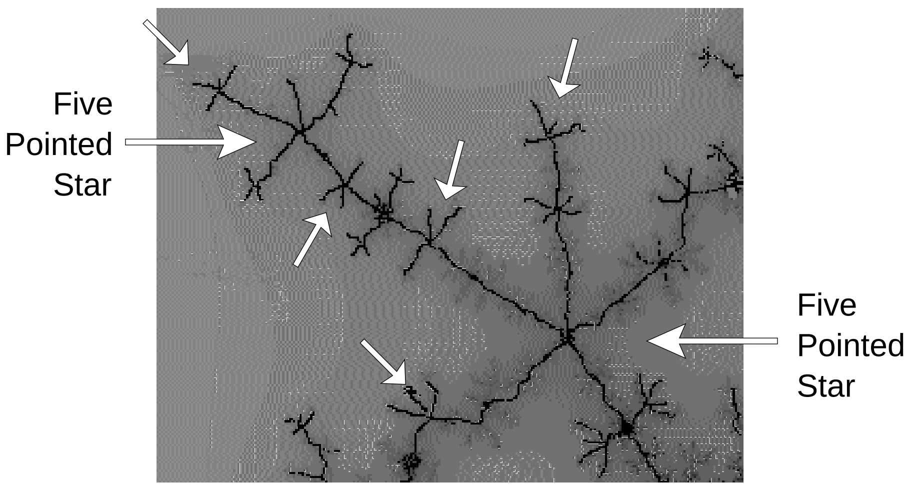

Figure 7.8 Enhanced close-up of the filigree pattern surrounding the fiveness nodule.

Quite why the Mandelbrot fractal takes this particular form I do not know, but I suspect there might be a deep underlying connection with self-similarity and the processes of iteration and feedback – characteristics also found in cellular automata and seen in the harmonic series above. (Though the harmonic series itself is not, strictly, a fractal entity.) Whatever the connection is, the apparent message of this fractal's geometry is that *threeness* is the point of most *not-twoness* or 'anti-twoness' in a system based on whole numbers. The principle of threeness contains the most not-twoness that it is possible to have (in whole numbers) and the higher odd numbers: five, seven, eleven, thirteen, etc. contain declining amounts of 'anti-twoness'.

Benoit Mandelbrot was born on the 20th November 1924, in Wasaw, Poland, into a family with scholarly and academic connections. In 1936 the family moved to France hoping to escape the looming threat of war and Nazi anti-Semitism. The young Mandelbrot's studies in Paris were interrupted at the beginning of the Second World War when the family moved again to south-central France, and could only formally resume his education in 1944. From 1945 to 1947 he studied mathematics at the Ecole Polytechnique in Paris and from 1947 to 1949 aeronautics at the California Institute of Technology, followed by a PhD in mathematics at the University of Paris in 1952. In 1955 Benoit Mandelbrot married Aliette Kagan, and after brief stays in Geneva and Lille, they moved to the United States where Mandelbrot worked for IBM in a research facility. Although working on a wide range of pure and applied mathematical problems over the many years he was with IBM, it is for his pioneering work on fractals that he is most widely known (*Fractals: Form, Chance and Dimension*, 1975/77 and *The Fractal Geometry of Nature*, 1982). A crucial factor in the success of Mandelbrot's research was the use of computers to create images of the fractal structures that he was studying. The power to compute the effect of myriad reiterations of recursive formulae in graphical form (rather like cellular automata rules and output) galvanized scientific and public opinion. A central feature of interest in his study of fractals, has focused on their ability to describe and model the outcomes of unpredictable, and perhaps even chaotic natural systems, generated and governed by feedback processes. Benoit Mandelbrot remains an inspirational figure in mathematics; laden with many honours he died in October 2010.

There is I suspect, something special or especially fundamental about the octave relationship between h1 and h2: the first structure to emerge from the undivided, undifferentiated wholeness of unity. In the graph Figure 7.9, the Metric Two 'knot' (h1, 2) at the base of an oscillatory system is drawn as a full line. Once established, a Metric of Two has a strong affinity with even-numbered harmonics by virtue of their compatible metrical accent at the half-period – 180 degrees – as well as of course, the full period. While the accent of the fourth harmonic, the dash-dot line, agrees at the half-period (and all other even harmonics too), the third (and all odd-numbered) harmonics reach the midpoint in direct opposition. Adding the third harmonic to an M2 (h1, 2) structure would force the system to change radically to M6 (h1, 2, 3) – a significant lowering of entropy. But adding h4 merely confirms the existing configuration by 'nested doubling' – h1, 2, 4 – a Metric Two nesting another Metric Two within itself, amounting to a compound Metric Four. This would still lower the entropy of the system, but not by as much as adding h3 and creating a Metric Six – all other factors being equal. This preference produces a distinction, a two-valuedness, between the odd-numbered and even-numbered harmonics. The even-numbered harmonics of a system with a Metric Two knot at its foot are preferred over the less compatible odd-numbered partials.

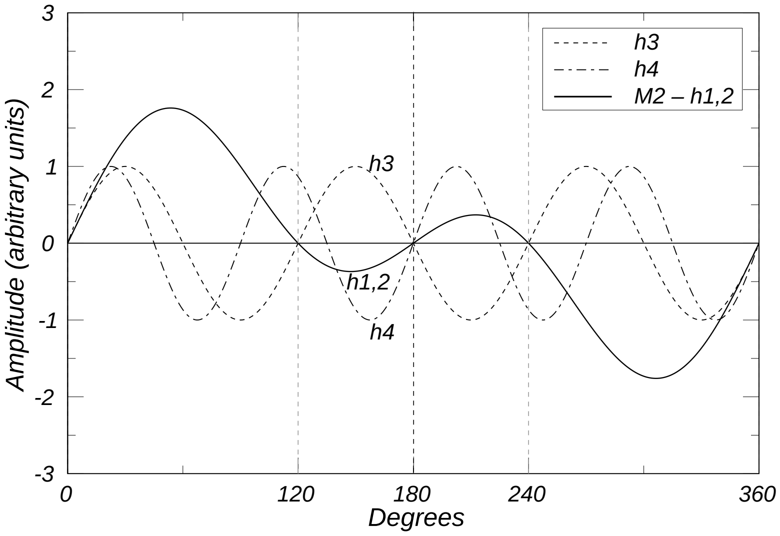

Figure 7.9 A Metric Two system (full line) with h3 and h4; h3 is in counter-phase at the half-period – 180 degrees.

The essential point is to think in metrical terms. Metrically h3 is the most incompatible frequency for a Metric Two system to swallow, h5 is slightly more digestible and h7 a little more so than h5. The method by which the principle of threeness first enters an oscillatory system is by the back door, through the 'two faces' of six – factors 2×3. The factor two makes six acceptable, digestible to a system built on a Metric of Two, but six carries a sting in its tail – the factor three.

A Metric Two system (h1, 2) grows by adding h4, *not h3* which is less compatible, to become Metric Two nesting Metric Two (Metric Four, overall, by Euler's LCM principle); next h6 is added, *not h5*, taking the system to a Metric Two nesting a Metric Six (Metric Twelve overall). Now, with this configuration, the system stands with its head, its leading edge, at a conjunction in the THS – row 1:6, columns C-h2 and G-h3 – a point of exchange:

Metric Two nesting Metric Six –> Metric Six nesting Metric Two
H1, H2 nesting h4, h6 –> H1, H2, H3 nesting h6

It is by means of this exchange of configurations between column two and column three in the THS that threeness insinuates itself into the fundamental level of a Metric Two system. Expressed in mutable base numbers both these configurations represent digit sequences which equate to the number six. Thus:

Mutable Base Number Six
Factor Format: 1x2x3 = 1x3x2
Subscript Format: MBN $3_2\,0_1 = 2_3\,0_1$

Some Detailed Patterns in the THS

The lines drawn on the Table of Harmonic Series (Figure 7.10) illustrate some of the patterns that emerge from nesting the harmonic series within itself. It is important to remember that patterns revealed by the table are 'inverted out', so to speak, and are in reality all packed ever more tightly inside the period of the fundamental h1. Here in this narrow corner, nested harmonic series combine, where present, to produce an interference pattern of ever increasing complexity. However, as the arrowheads show, no matter how many nested series (columns) are included in a system and no matter how many harmonics (rows) of those nested series are introduced, a predominant pattern of multiples of twelve will emerge – given that all or a representative sample of harmonics, with coherent amplitudes, are present.

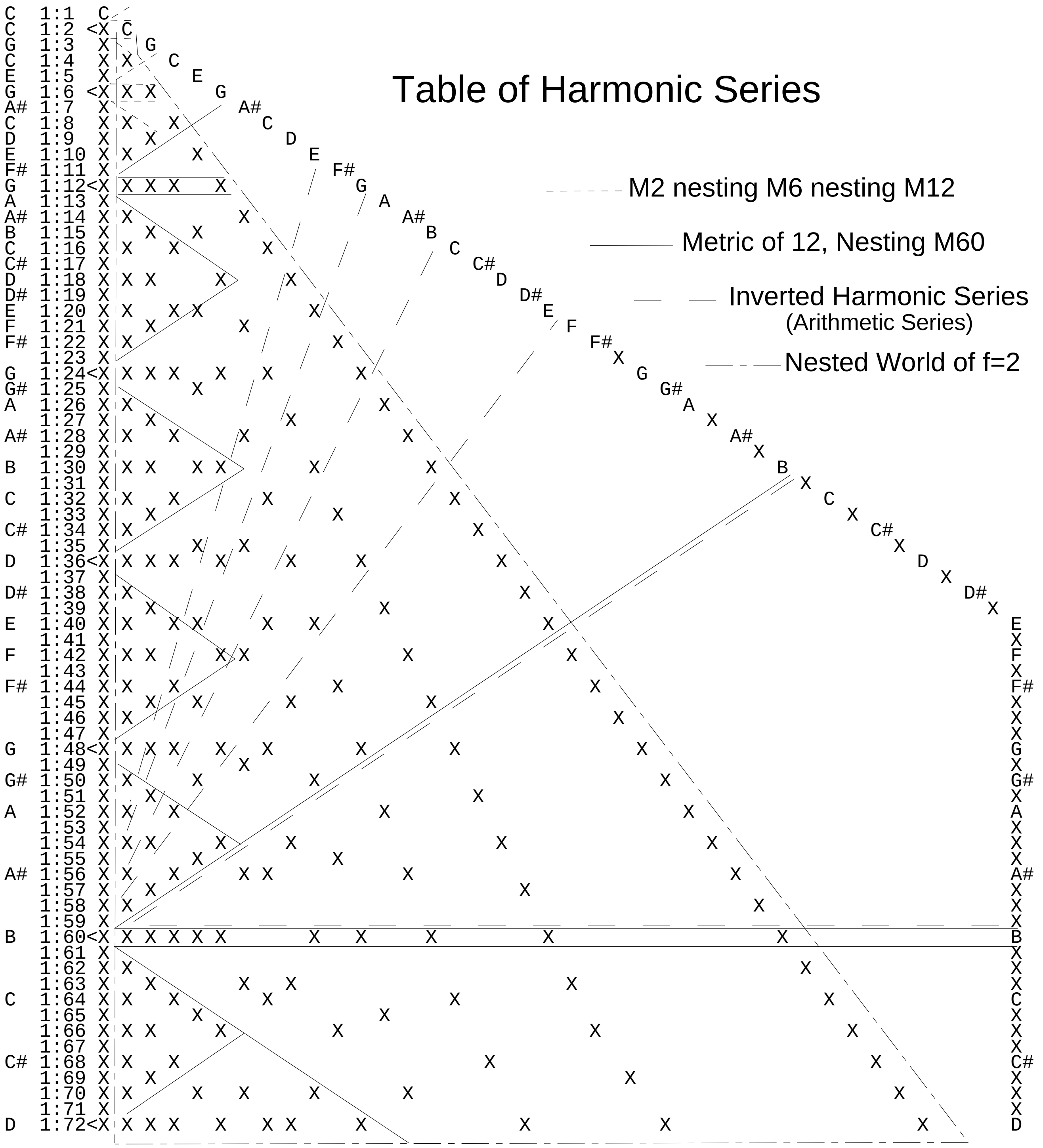

Figure 7.10 Arrowhead patterns of Metrics nested within each other. Metric Sixty nested in Metric Twelve nested in Metric Six nested in Metric Two – all nested in metrical unity, Metric One.

Prominence of Twelve

Before constructing the table of harmonic series I had expected, if anything at all, that it would show a primary association with the octave relationship of powers-of-two – 2, 4, 8, 16, etc. However, as is made clearly visible by the *arrowheads* pointing to the multiples of twelve at 1:12, 1:24, and in particular 1:60, the table orients itself toward the number twelve, with the multiples of five and powers of two vying for second and third place. Wherever you look in the table, a multiple of the twelve ratio has more 'hits' – more harmonics, more Xs – than the multiple of two further on down the list and often the same holds for multiples of five/six, i.e. 1:30 versus 1:32. It would seem that twelve has a natural precedence in the system. Twelve, the lowest common multiple of 1, 2, 3, 4, 6, appears to emerge as the natural choice for a metrical scheme – the best compromise, at a level of organization above that of individual units. Something of this 'theme of twelve' has already been touched on regarding number systems, in Chapter 2 with the twelve-note scale and twelve key centers; and, at the end of Chapter 3 the low entropy of the fecund numbers that combine powers of two with a factor of three: 6, 12, 24, 48, 96, etc. In the table Figure 7.11 below, the first six columns of the THS have been extended out to the 1:200th row, illustrating in effect, an oscillatory system of nested harmonic series built on the fundamental ratios of 1:2:3:4:5:6.

```
C   CCGCEG h1         X X    51           X      101          X      151
C   XX     2          XX X   52           XXX  X 102          XX X   152
G   X X    3          X      53           X      103          X X    153
C   XX X   4          XXX  X 54           XX X   104          XX     154
E   X   X  5          X   X  55           X X X  105          X   X  155
G   XXX  X 6          XX X   56           XX     106       e >XXXX X 156
    X      7          X X    57           X      107          X      157
(c) XX X   8          XX     58        a->XXXX X 108          XX     158
    X X    9          X      59           X      109          X X    159
    XX  X  10      b->XXXXXX 60           XX  X  110          XX XX  160
    X      11         X      61           X X    111          X      161
 g->XXXX X 12         XX     62           XX X   112          XXX  X 162
    X      13         X X    63           X      113          X      163
    XX     14     (c) XX X   64           XXX  X 114          XX X   164
    X X X  15         X   X  65           X   X  115          X X X  165
(c) XX X   16         XXX  X 66           XX X   116          XX     166
    X      17         X      67           X X    117          X      167
    XXX  X 18         XX X   68           XX     118       f->XXXX X 168
    X      19         X X    69           X      119          X      169
    XX XX  20         XX  X  70        b->XXXXXX 120          XX  X  170
    X X    21         X      71           X      121          X X    171
    XX     22      d->XXXX X 72           XX     122          XX X   172
    X      23         X      73           X X    123          X      173
 g->XXXX X 24         XX     74           XX X   124          XXX  X 174
    X   X  25         X X X  75           X   X  125          X   X  175
    XX     26         XX X   76           XXX  X 126          XX X   176
    X X    27         X      77           X      127          X X    177
    XX X   28         XXX  X 78       (c) XX X   128          XX     178
    X      29         X      79           X X    129          X      179
    XXX XX 30         XX XX  80           XX  X  130       f#>XXXXXX 180
    X      31         X X    81           X      131          X      181
(c) XX X   32         XX     82        c#>XXXX X 132          XX     182
    X X    33         X      83           X      133          X X    183
    XX     34      f->XXXX X 84           XX     134          XX X   184
    X   X  35         X   X  85           X X X  135          X   X  185
 d->XXXX X 36         XX     86           XX X   136          XXX  X 186
    X      37         X X    87           X      137          X      187
    XX     38         XX X   88           XXX  X 138          XX X   188
    X X    39         X      89           X      139          X X    189
    XX XX  40         XXX XX 90           XX XX  140          XX  X  190
    X      41         X      91           X X    141          X      191
    XXX  X 42         XX X   92           XX     142       g->XXXX X 192
    X      43         X X    93           X      143          X      193
    XX X   44         XX     94        d->XXXX X 144          XX     194
    X X X  45         X   X  95           X   X  145          X X X  195
    XX     46      g->XXXX X 96           XX     146          XX X   196
    X      47         X      97           X X    147          X      197
 g->XXXX X 48         XX     98           XX X   148          XXX  X 198
    X      49         X X    99           X      149          X      199
    XX  X  50         XX XX  100          XXX XX 150          XX XX  200
```

Figure 7.11 The first six ratios of the THS extended to 1:200, with the 'arrowhead' nested series built on G-h12 shown in lower case and the 'tonic' octaves in brackets. Table of Harmonic Series to 1:200

Equally significant as the emergence of a pattern of twelve are the frequencies that this pattern is actually selecting from h12 upward, in steps of twelve: G, G, D, G, B, D, F, G, etc., the harmonic series built on G, the family of G-h3 – not C-h2, h1's nearest 'blood' relation. In essence an oscillatory system built on C is picking out from its own harmonic spectrum a self-similar nested pattern of vibration built on G-h3, the dominant. Without any outside intervention, the harmonic series, when bound systematically in a scheme of self-nesting, tends to focus on a relationship built on G-1:3, :6, :12, :24, :36, :48, etc. And while all schemes of uniform steps will pick out nested harmonic series – e.g. steps of 1:2, :4, :6, :8, :10,

:12 produce CCGCEG; steps of 1:5, :10, :15, :20, :25, :30 produce EEBEG#B; yet still, the most prominent naturally occurring pattern is in steps of twelve: GGDGBDF... One might perhaps interpret this as a ‘harmonic field’, with the strongest lines of resonant force, the multiples of twelve. The effect of this is to lay out resonant markers, sweet points or dips in the relational landscape, which might influence the future evolution of the system. And, as discussed elsewhere, there is a slight shifting of these relational markers by the dominant of the dominant – the D^7-major chord in the key of C – undermining the coordinates of the present tonic and thus tending to establishing the different topography of a new key.

Taking the table out beyond the sixth fundamental tone and its harmonics (i.e. Figure 7.11) makes little difference to the pattern, as while the columns h7, h11, h13, h17, etc. tend to disrupt the pattern, so h8, h9, h10, h12, h16, tend to confirm the pattern of twelve to a greater or lesser degree. The end result of extra harmonics is broadly neutral with a slight bias towards twelve in any normal distribution of harmonics and amplitudes. Basically, in a system of bound harmonics, the numbers/ratios 1, 2, 3, 4 conspire together to rule the roost with a repeating pattern of twelve. The only threats to their hegemony (apart from not being present in the system) comes from amplitude and perhaps scale. Amplitude, if applied in large enough doses to ratios running counter to the pattern of twelve, could always overturn the settled order. And there is also the factor of scale or granularity to bear in mind. The size of the THS above is, in a sense, focusing on the patterns of twelve because it is the first one to emerge. Much larger charts would appear to favour patterns of sixty, and after that: 420, 840, 2520, 27720, etc.; while a narrower view would pick out patterns of six or two. However, there is a particular association between the patterns of twelve and sixty at the foundational level which might uniquely mark these two Metrics as having a special status.

Inverted ‘Arithmetic’ Series

One of the most striking features of the Table of Harmonic Series is the sequence of ‘radial arms’ (formed by diagonal lines of Xs) emanating from the point of column one, row one. From an angle of 45 degrees, each radial arm in turn increases its angle, the whole pattern gradually approaching the vertical. Running counter to this dominating pattern are what might be termed ‘contra radial arms’, made up of far fewer Xs, running in the opposite direction and emanating from column one, row sixty (long dash lines in Figure 7.10). The contra arms are like a faint and ghostly echo of the principal pattern in the table. If you examine where these contra arms strike the edge of the pattern, i.e. the principal radial arm inclined at 45 degrees, the note values touched in descending order are: B, B, E, B, G and E, with an intervallic spacing matching the ascending harmonic series (octave, fifth, fourth, major-third and minor-third).

Long before I had ever heard of Arthur von Oettingen and harmonic dualism, this feature of the THS alerted me to a possible arithmetic component in tonal music. To begin with I was unsure what role these inverted relationships played, if any, and for some time I tried to integrate the feature as a ‘bridge’ carrying and perhaps transforming energy and information between successive harmonic steps. That is, attempting to develop it as an explanation and explication of tonal chord progression. But this approach proved redundant as the modulation algorithm coped perfectly well with exchanges between harmonic series without any need for ‘arithmetic hocus-pocus’. Gradually, however, I began to realise that the bottom end of this inverted series looked for all intents and purposes like the minor triad, and the extension of the series to eight tones (i.e. B, B, E, B, G, E, C# and B) like the minor chord of the added sixth. Thus the inversion of the harmonic series of the common major triad (h1 through h6) produced a common minor triad and the inversion of the harmonic series of the dominant-seventh type chord (h1 through h8) produced the minor chord of the added sixth. Playing these two cadences over (Figure 7.12), the familiar and powerful V^7–I and its delicately beautiful reflection iv^6–i, I wondered what logic, what scheme, links these two progressions? As I sought more information on these inverted relationships, reading new texts and following references to other sources, I soon came upon the work of A. v. Oettingen and realised what a lamentable hole existed in my knowledge! (This topic is pursued in Chapters 11&14.)

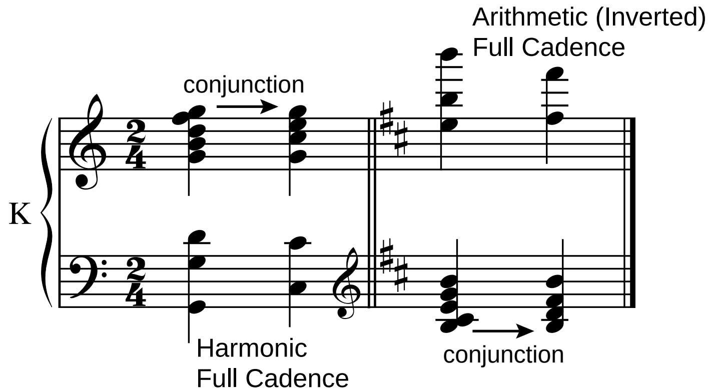

Figure 7.12 The familiar full cadence and its harmonic inversion derived from the contra-radial arms in the THS.

Nested World of Frequency ‘2’

Another pattern that is manifest in Figure 7.10 is that of a nested ‘table of harmonic series’ built on frequency ‘2’ (enclosed within a dot-dashed line) and delineated by the second of the principal radial arms. Indeed, each of the ‘radial arms’ extending from the first column at row 1:2, 1:3, 1:4, etc. delineates successive nested systems – but only the first system based on 1:2 is marked in the figure. Each one of these nested systems built in succession on frequency 2, 3, 4, 5, etc. produces a complete mirror of the THS at a whole numbered multiple of the original. This feature is just another expression from the nesting of harmonic series within each other.

Meter Emerges

Looking at the Figure 7.10 Table of Harmonic Series, *rotate the table 90 degrees anticlockwise,* so as to view the X nodes as a series of peaks and troughs, accented and unaccented beats, perhaps using a sheet of paper to cover all the (rotated) columns except the first column – the family of one:

```
 X X X X X X X X X X X X.....
h1 2 3 4 5 6 7 8 9...........
```

One Column – All ratios belong in the family of column one, each has equal precedence and in musical terms a meter of 1/1 would express the equal relationship between the ratios – 1:1, 2, 3, 4, 5, 6, 7, ... none more important than another. Really, a meter of 1/1 implies in a sense, no meter at all, or alternatively an infinite meter including all ratios/harmonics. Metric One represents the *two edges of meter* within which metrical units can make sense. In any real world situation a system of such perfect evenness as a Metric One would soon be disturbed so that it fell into a cycle of some form. It is only when there is some differentiation of weight or stress in the system that meter can truly gain some traction, and, once established through a process of self-reinforcing feedback, stabilise or fix the system. This is what happened in the example of the Millennium footbridge, where the assumption of overall randomness in the pedestrians’ gait was overturned by the structure *communicating* some tiny congruence of step and building upon the faint pattern until, at the last, it was able to shake the whole structure.

```
   X   X   X   X   X   X
 X X X X X X X X X X X X....
h1 2 3 4 5 6 7 8 9..........
```

Two Columns – Now uncovering column two, the family of Two, the situation changes dramatically – the former equality is swept away, half the numbers have two entries. The even ratios gain more ‘weight’ and the poor odd numbers are left behind. Reality is so unfair! The new situation could be expressed as a 2/1 meter, in music more usually as 2/4. This is a very stable (though rather boring) situation metrically, probably the most stable meter of all. And as will be discussed in Chapter 15, a meter that perhaps finds a mirror in the electronic structure of the element helium and the other noble gases.

```
     X     X     X     X
   X   X   X   X   X   X
 X X X X X X X X X X X X....
h1 2 3 4 5 6 7 8 9..........
```

Three Columns – With the sheet moved to reveal column three the situation changes once more, the structure is now much more complicated. Looking a little way down the three columns to avoid the ‘boot-up’ of the first few rows, there are three levels of precedence. For example, 1:6 has XXX, 1:7 X, 1:8 XX, 1:9 XX, 1:10 XX, 1:11 X, and so on. In music a grouping of stresses or weights in this order would normally be expressed as a 6/8 meter or 3/4 – which of the two depending on the delicate balance of individual stresses (amplitudes). The firm metrical grip of 2/4, a square pattern of two nesting two, has been superseded by a rather ambivalent meter mixing factors two and three. There is little difference in the energy levels between these two ‘oblong’ arrangements (6/8 and 3/4), one metrical form can easily be buffeted into the other, and back again. In contrast square meters strongly resist change by virtue of their balanced oscillatory shape and the generally large energy gap to their next available rectilinear configuration – illustrated in Figure 7.13 below. This differentiation between square and oblong meters brings to the fore the all important path traced out by the mobile feature in the table of harmonic series – the dashed line of primary modulation exchanges, Figure 7.3 – which consists of an unending sequence of alternate square and oblong numbers: 1, 2, 4, 6, 9, 12, 16, 20, 25, 30, etc. It is this pattern in the table of nested harmonic series that lies at the root of the MOS model’s interpretation of harmonic progression and metrical change in tonal music via the agency of the algorithm of symmetrical exchange.

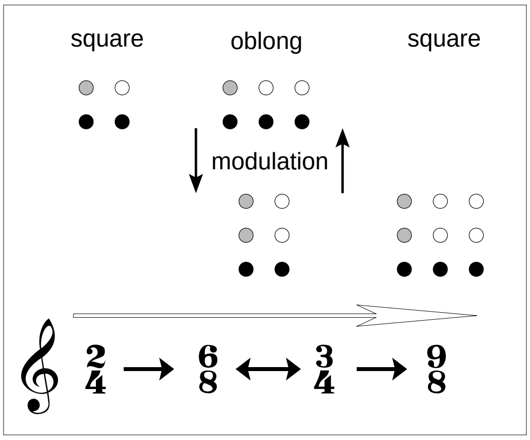

Figure 7.13 The metrical view of the dotted line mobile feature near the beginning of the THS: columns two and three in Figure 7.3 expressed as number patterns and time signatures. The square numbers (e.g. 4, 9, 16, ...) lie at intermediate positions between the ‘sideways’ column steps of the modulation exchanges.

```
       X       X       X
     X     X     X     X
   X   X   X   X   X   X
 X X X X X X X X X X X X....
h1 2 3 4 5 6 7 8 9.....12...
```

Four Columns – Moving on to four columns uncovered. The situation becomes even more unequal with the family of four added into the mix. Indeed, with each additional column, more structure/ order is entering the system. Continuing on from the same point in the table: 1:12 XXXX, 1:13 X, 1:14 XX, 1:15 XX, 1:16 XXX, 1:17 X, 1:18 XXX, 1:19 X, 1:20 XXX, 1:21 XX, 1:22 XX 1:23 X, 1:24 XXXX, and so on. Musically this pattern would normally have a 12/8 meter. As far as practical music is concerned, in metrical terms, 12/8 can encompass the widest variety of meters – 2/4, 3/4, 4/4, 6/8 and their less common larger and smaller denominators 2/2, 3/8, etc., as well as itself. It is the most flexible of compromises coping with groupings of 2, 3, 4 and 6. A Metric of Twelve represents both a highly ordered and a highly flexible scheme.

Thus we have progressed from the non-meter 1/1, an unstable equality, through a stable simple duple meter 2/4, to a compound metre 6/8 or 3/4 and on to the complex compound 12/8 meter. This is about as complex a meter as is found in normal musical use. Most examples of longer metrical units or odd meters like 5/4 and 7/4 tend to break down into some combination of their internal structural units in practice, i.e. 5/4 to 3/4 + 2/4 or vice versa. Though there are no doubt contemporary and exotic genres which use very unusual meters.

In music, longer structures are normally obtained by adding a number of metrical units together to create a phrase or group of phrases. So it appears that, beyond a measure length of twelve units, generally, phrases form the next larger unit of organisation. Internally phrases often follow a power of two progression – two measures, four measures, eight measures. And more often than not the measures length will already be a power of two (eg. 2/4 or 4/4) disguising the switch to what is in reality a ‘long wave’ meter (hypermeter) – phrases.

```
         X         X         X
       X       X       X
     X     X     X     X     X
   X   X   X   X   X   X   X
 X X X X X X X X X X X X X X X....
h1 2 3 4 5 6 7 8 9.....12....15...
```

Five Columns – Uncovering the fifth column to reveal the family of five produces a radical change, a Metric five times as broad is required to accommodate the pattern of 5 – a metrical numerator of 60 in musical terms. Five makes no compromise (like all prime numbers it is an oscillatory shape that cannot be nested at a lower level) 15/8 or 30/8 will not work because a division of two and/or four is lost. In an ordinary musical way, a meter of 60/8 or 60/16 is not a practical proposition, it would inevitably break down into some smaller grouping of units of stresses.

However, the ‘natural’ precedence of sixty in a system of nested series can clearly be seen from the table Figure 7.10, particularly at 1:60 where an arrowhead of Xs makes *the point*. This is a big leap but there is some evidence from natural systems suggesting that jumping to a Metric Sixty with five sub-divisions of twelve, six sub-groups of ten or ten sub-groups of six is feasible. One might think of the recently discovered buckminster fullerene molecule: sixty carbon atoms joined as a ‘sphere’ of interlocking pentagons and hexagons. So, while a meter of sixty units in subdivisions of twelve is a difficult proposition in music (the nearest I can think of is Brahms/Haydn Opus 56a Variations with a phrase length of five measures) nature appears to be able to handle metrical units beyond the normal reach of music.

```
           X           X           X
         X         X         X
       X       X       X       X
     X     X     X     X     X     X
   X   X   X   X   X   X   X   X   X
 X X X X X X X X X X X X X X X X X X....
h1 2 3 4 5 6 7 8 9.....12....15....18...
```

Six Columns (and beyond) – Uncovering column six (Figure 7.11) only confirms the pattern of sixty divided into five groups of twelve and at this point the system ‘closes’, in that beyond sixty the pattern formed by the first six columns in the THS repeats. There does appear to be something rather special about a Metric Sixty: a natural sweet point that combines elements of both simplicity and variety. Extensions to this natural meter occur at Metric 420 as seven is added into the system (a big jump from M60), with other stopping points at Metric 840 where eight becomes available, Metric 2520 for nine, Metric 27720 for eleven (includes 12, 14 and 15), Metric 360360 for thirteen and Metric 720720 for sixteen... and so onward.

PRIMES, FACTORS AND DIVISORS

Also illustrated by the arrangement of X nodes in the THS – Figures 7.10, 7.11 and 7.15 – is the way in which the regular resonances of two, three and four, combine with five’s awkward nature to restrict the occurrence of prime number vibrational patterns to either side of the multiples of six. The association of prime numbers with six and its multiples is interesting. It is as if 6, 12, 18, 24, 30, etc. are the points on either side of which the *numeric/metrical crust* fractures, allowing new (prime) numbers to well-up. Twin primes (excepting three and five at the start of the table) always lie on either side of a multiple of six – Figure 7.14 gray columns.

C-h32,33,34,35,36,37,38,39,40,41,42,43,44,45,46,47,48,49,50,51,52,53,54,55,56,57,58,59,60,61,62,63,64
C-h16,17,18,19,20,21,22,23,24,25,26,27,28,29,30,31,32
C-h8,9,10,11,12,13,14,15,16
C-h4,h5,h6,h7,h8
C-h2,h3,h4
C-h1,h2

Figure 7.14 The ‘fractal/automata-like’ octave generations of the harmonic series allow prime vibrational patterns (underlined) to emerge in the slots created between doubled values from the previous generation of frequencies.

In contrast with the repeating pattern of Xs established in columns two, three, four and six of Figure 7.10, the prime numbers in the table have only one entry, an X in the first column – they are not caught by Eratosthenes’ sieve. This pattern of primes can be seen by examining each of the *arrowheads and intermediate positions* at 1:12, 1:18, 1:24, 1:30, etc. from 1:1 to 1:60. The regular resonances of columns 2, 3 and 4 form the arrowheads – which prevent prime numbers occurring except on either side of a multiple of six – while the *interference* of column five Xs around the positions of multiples of six produces a regular pattern (with a period of 60) allowing or denying prime vibrational patterns the possibility of occurring. The pattern of *five denial* is, after the first period where five is prime itself, to block the first, last and two five-off central openings in each period of sixty (i.e. positions 5, 25, 35 and 55 in each successive period of sixty). Of course, not all these openings for primes to occur will actually hold prime numbers, as many will be multiples of primes from earlier in the sequence, as for example row 1:49 in the table.

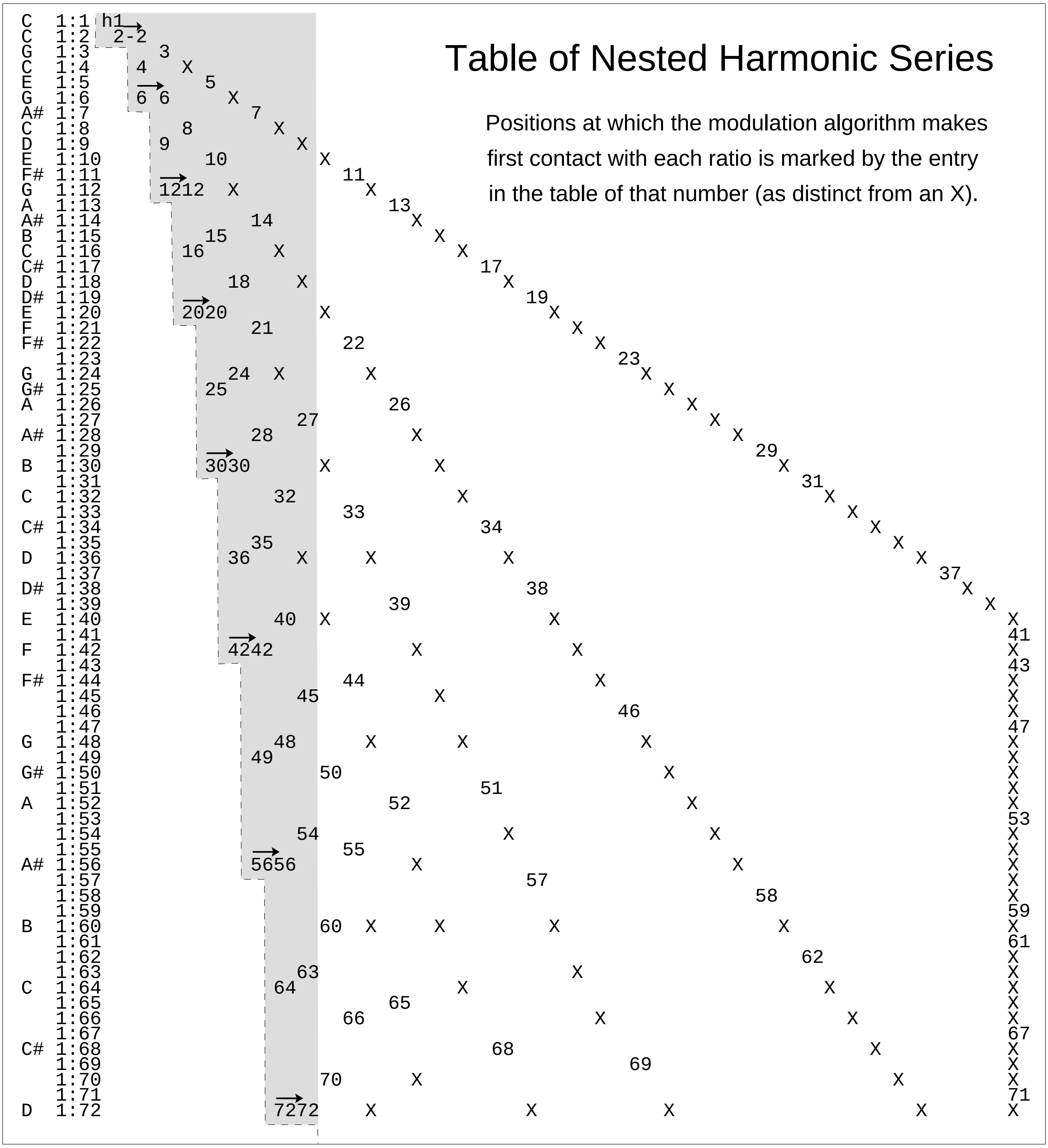

Figure 7.15 The progress of the modulation algorithm of symmetrical exchange (marked by the dashed line and gray area) brings the system into contact with new ratios either by addition or modulation. The number 71, in the bottom right-hand corner will eventually be computed by the algorithm when it reaches row 1:4970.

Another way to approach this is to look at the *columns* of h1 and h2 and h3 etc., and imagine beams of light shining out, down each column. The beam of number/column one, lights up all numbers/ratios, while the beam of number/column two catches only half – all the even numbers. The beam of number/ column three picks out one in three... but half of these have already been illuminated by the beam of column two (6, 12, 18, etc.) so they must share these vibrational patterns. However, out of the Xs in column three, most of which will have to be shared with other primes – for example 15 shared by three and five – the number three has its own private family 9, 27, 81... the powers of three. The beam shining down column four lights up no new numbers, because four is a member of two's private family. In this way the beams build up from each prime number column, gradually lighting up most, but not all the numbers higher up the pattern. The ever-diminishing but never exhausted Xs not illuminated, are the remaining prime vibrational patterns.

In Figure 7.15 the point(s) at which each ratio is introduced into a modulating oscillatory system by the succession of primary modulation exchanges, are marked by that ratio's number. Other positions where the ratio plays a role in the unfolding modulations are marked by Xs. New ratios are either attached to the top of the current harmonic series or computed through a modulation exchange. Two ratios are added between each modulation as the system grows upward to the next conjunction and primary modulation (the dashed line). Ratios added or created within the body of the system all have nested configurations (composite numbers) while those ratios created as nested fundamentals (tops of columns), by the modulation algorithm of symmetrical exchange, are prime oscillations – i.e. they do not have a nested configuration. It is on the edge of the expanding system that new, prime, vibrational patterns are created. The process casts the composite numbers (compound meters) in a different, more assertive role than normal, in that primes are 'extracted' from within compound meters, where they occur as a 'byproduct' of the prevailing pulse. For example, the composite number six enters the sequence of primary modulations first as a duple meter (i.e. 6/8 time, 2×3: H1, 2 n~ h4, 6) and only later, through the agency of a sesquialtera 2:3 modulation, does the prime oscillation, three, emerge in the fundamental series in the form of a triple meter (3/4 time, 3×2: H1, 2, 3 n~ h6).

In Figure 7.15 the gray area covers all those ratios/numbers encountered by the system up to the sesquioctava 8:9 modulation at conjunction h72. The remaining numbers shown will be found by the system in due course, as the modulations gradually step sideways through each successive column; though the system will have to reach h4970 before it computes the value h71.

Finally, thinking metrically, in terms of *oscillatory patterns* rather than *abstract numbers*, would seem to imply that a vibrational pattern, like that of four fluctuations in the form of a Metric Twelve system (h1, 2, 3, 4) should also be thought of as being a *prime* oscillatory pattern, while its more stable real world manifestation of a Metric Two nesting another Metric Two (abbreviated M2n~M2), corresponds to our non-prime, *abstract*, number four. A Metric Twelve system is physically different and distinguishable from M2n~M2. The challenge is to think in metrical terms about oscillatory structures, not so much as systems described by *abstract* mathematics, but as actually being, in and of themselves, mathematics in material form.

And, having raised the topic of mathematics, we now go on to extract from the THS two relationships which are intimately connected to the outward progress of the modulation algorithm as it steps through the columns and rows, depicted in Figure 7.15 and subsequent figures.

Bow Wave Relationships

Stephen Wolfram's book *A New Kind of Science* (Champaign, USA; 2002) contains numerous illustrations (e.g. pages 29 and 30) of cellular automata, the computational output of simple algorithms or programs. Many of these cellular automata are made up of patterns and groupings of triangles. Similarly, the Table of Harmonic Series (THS) which also can be generated by a simple program (see directory/folder CHPT19.ZIP/SCRPT/th.awk), contains patterns and groupings of triangle-like forms. *Bow waves* is the term given to them here – named for their 'V' shapes, streaming away from the 'prow' of the mobile pattern of modulation exchanges, as it threads its way through the ratios of the THS. In Figure 7.17 a few of these bow waves are illustrated. A bow wave consists of a symmetric sequence of ratios (i.e. Xs in the THS) based on a row, for example, row nine reading from right to left: 1×9, 2×8, 3×7, 4×6, 5×5, 6×4, 7×3, 8×2 and 9×1. This is the bow wave of row 1:9 and it has an association with the modulation exchange between columns five and six. Every row in the THS has its own bow wave, and the bow waves based on odd numbered rows have an interesting feature, illustrated in Figure 7.16, and applied to the THS in Figures 7.18/19.

```
X X X X X   X X X X X X   X X X X X X X   X X X X X X X X   X X X X X X X X X
X X X X X   X X X X X X   X X X X X X X   X X X X X X X X          9×1
X X X X X   X X X X X X   X X X X X X X         8×2
X X X X X   X X X X X X         7×3
X X X X X       6×4                                               X X X X
   5×5                                         X X X              X X X X
                                X X            X X X              X X X X
                 X              X X            X X X              X X X X
   0×0          1×1             2×2             3×3                 4×4
```

Figure 7.16 The terms or positions (i.e. Xs) in an intermediate bow wave (row 9, column 5 illustrated) have squares of whole numbers as remainders – working outward from the central column (row 1:25 illustrated).

The bow waves shown and discussed below might be termed *primary* bow waves – bow waves which reach the shore line of the THS, that is the left and right edges of the pattern. There are many more bow waves within the THS, most often obscured by scale and nesting, which never make landfall and so are harder to see. These other bow waves form on multiples of the primary bow waves, for example, row eighteen from one X in, again reading right to left: 2×1×9, 2×2×8, 2×3×7, 2×4×6, 2×5×5, 2×6×4, 2×7×3, 2×8×2 and 2×9×1. This multiple of the primary bow wave based on row nine, is founded on ratio/row h18. There is no end to this multiplication of bow waves as the Table of Harmonic Series expands outward, lending the pattern of the THS something of the flavor of cellular automata and fractals. Though equally, it is just another manifestation of the nesting of harmonic series within each other.

However, it is the primary bow waves that are of most interest here. As can be seen in Figure 7.17, the progress of primary modulation exchanges through the ratios of the THS (gray dashed line and horizontal black arrows) coincides with this *bow wave* effect of connections rippling outward through the pattern. These bow waves link successive conjunction and intermediate steps in the mobile pattern of exchanges to the edges of the THS. Indeed, given the dynamical interpretation of mutable numbers provided by the MOS model it might be argued that the chain of modulation exchanges is in some way the motive force lurking behind these waves of relationship and perhaps the complete structure of the THS. Each individual primary bow wave touches all columns from its point of creation to where it comes to break on the shore line. The sequences of connections follow coherent, symmetric, bifural patterns from prow to shore. The intermediate bow waves begin with an even number at their prow point, followed by pairs of odd numbers and pairs of even numbers (dashed black line). In contrast, the conjunction bow wave number pairs are even-odd and odd-even throughout (continuous black line).

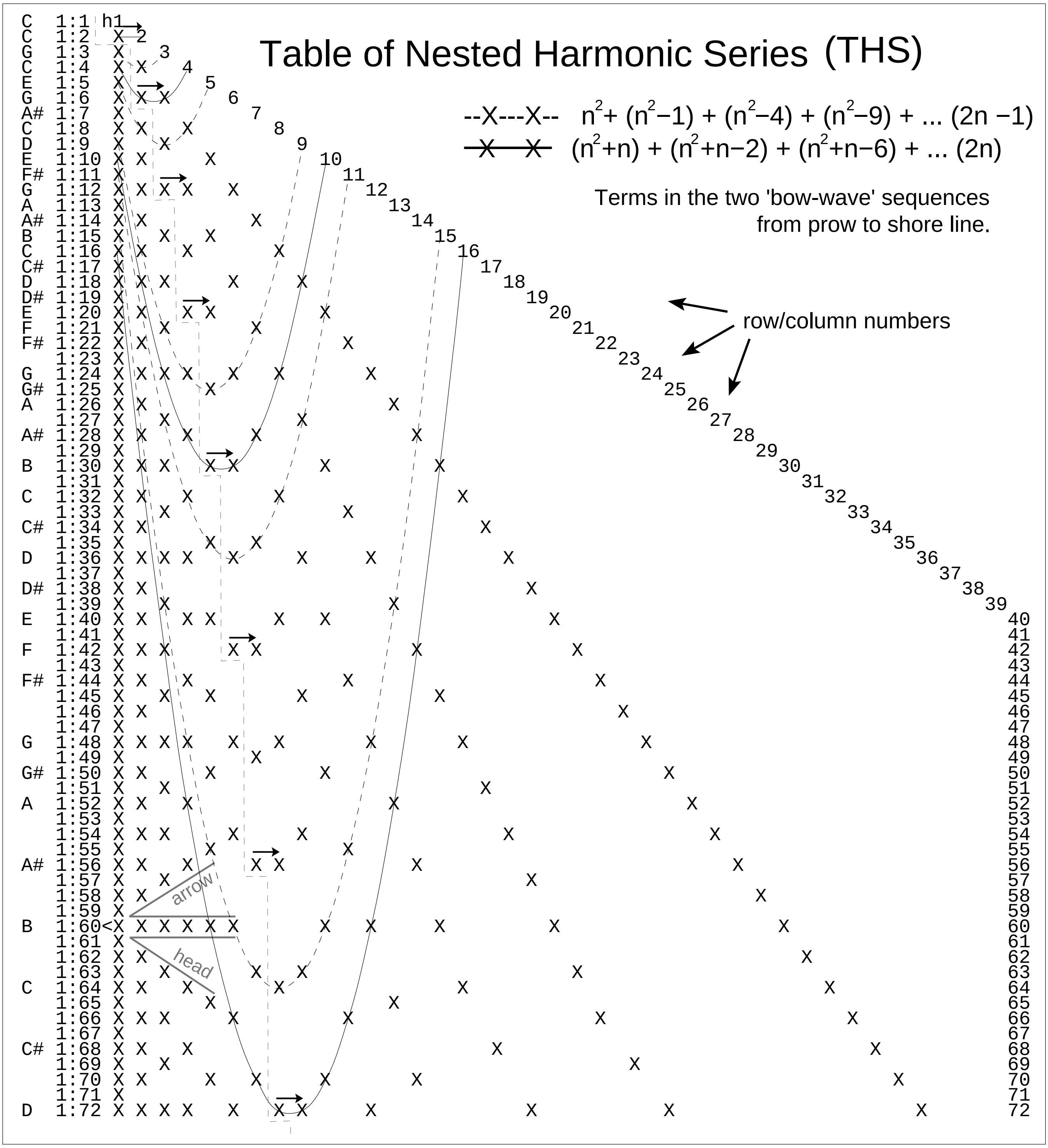

Figure 7.17 Superimposed upon the sequence of primary modulation exchanges (gray stepped dashed line) are a few example ‘bow waves’ which connect the modulation conjunctions to the ‘shore line’ of the THS through a continuous black line. The intermediate steps between conjunctions are connected by the black dashed lines. The conjunction bow waves connect with even numbered rows only, while the dashed back lines from intermediate steps (square numbers) touch both odd and even rows but always reach the shore line at an odd number.

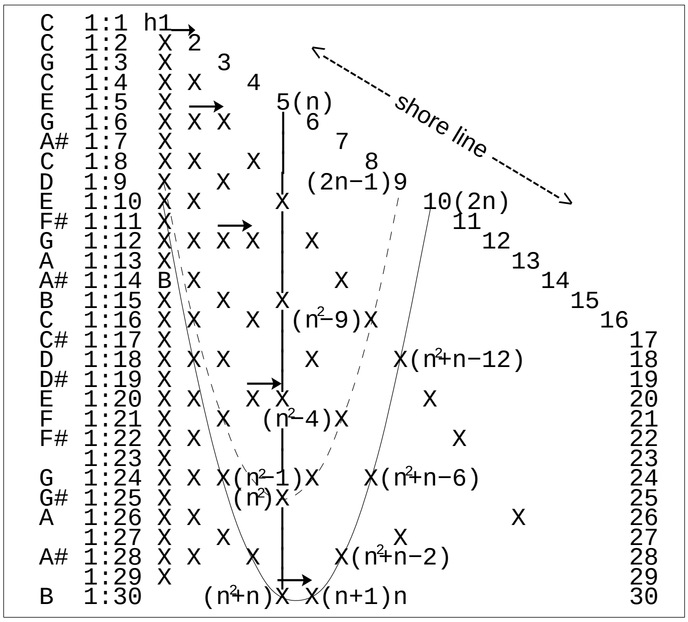

Figure 7.18 The patterns of bow wave connections exhibit regular symmetric relationships which link a central column 'n', to points on the shore at 2n or 2n−1.

If one imagines an ideal system of modulation exchanges, enveloped within a cloud of energetic 'particles of frequency', so that it was impossible for the system to release energy and relax internally, but only to absorb more 'particles' and grow; such a system's bow wave might sail forward through the THS indefinitely, becoming ever more complex and spanning an ever greater number of columns and rows.

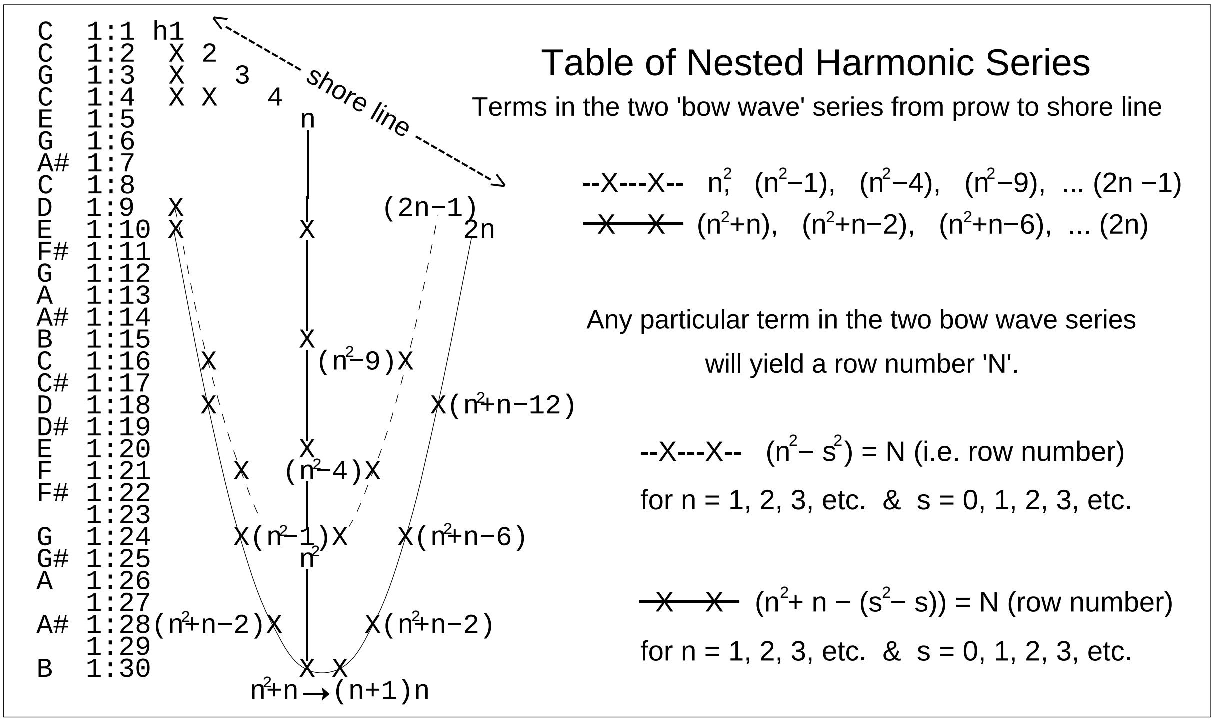

Figure 7.19 A less cluttered view of the bow wave relationships shown in Figure 7.18.

In this hypothetical system, as the modulation computations of 'column n' gradually reach outward through the ratios of the THS, expanding through the intermediate step of n^2 to the conjunction at $(n^2 + n)$, a bow wave of connections ripples out, linking ever more columns and rows. For example, the prime number 11 (2n–1) is connected to the none prime 6 (n) and the none prime 85 (2n–1) is connected with the prime 43 (n).

The two sets of terms for points on the bow waves can be reduced to the expressions:

$$n^2 - s^2 = N \quad \text{and} \quad n^2 + n - (s^2 - s) = N$$

for the intermediate and conjunction lines respectively, where N is a row number, n a column number and s a whole number. In the illustrations above, the intermediate bow waves are shown linking ‘column n’ with the shore line at point/row $(2n - 1)$. This is the limit of the bow wave as it breaks on the shore line, where the expression $(n^2 - s^2) = N$ takes the form:

$$n^2 - (n-1)^2 = N, \ \text{– i.e. } 25 - 16 = 9$$

but in some cases it is possible that other whole numbers will also yield the same row number N, for example:

$$n^2 - s^2 = N, \ \text{– i.e. } 3^2 - 0^2 = 9$$

and these multiple solutions appear to interconnect some bow waves. The same approach holds true for conjunction bow waves as well.

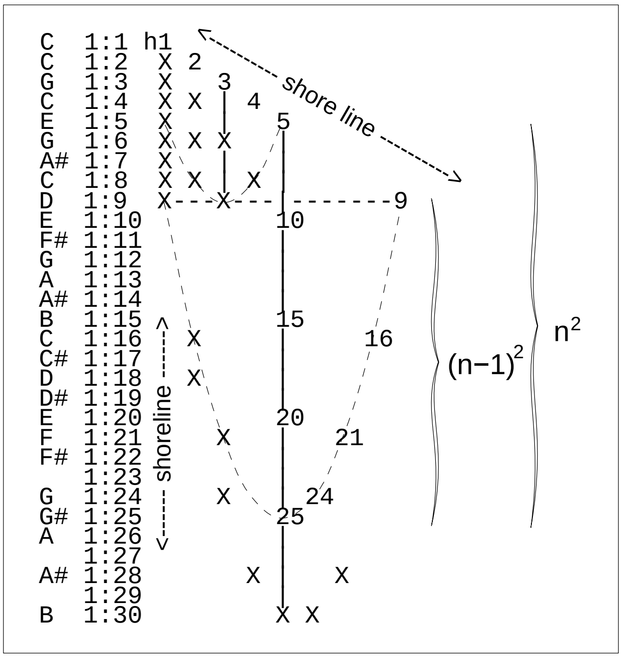

Figure 7.20 The intermediate bow wave for row 9 and column 5 with the connected bow wave of row 5 column 3.

Of the many relationships that arise from the features and characteristics of bow waves a representative one focused on here is that of two interconnecting intermediate bow waves. In Figure 7.20 the lesser bow wave of column three just reaches row nine ($n^2 = 9$); however, as shown in Figure 7.21, it is equally possible that a lesser bow wave which fits the expression $(n^2 - s^2) = N$ could cut through the row number of the greater bow wave at two points. Indeed, in Figure 7.20 it would be better to think of the single X at column three row nine as two Xs, one on top of the other, as each of these points of intersection marks two factors/divisors of the row within which they fall. Thus 3×3 yields ‘row 9’ in Figure 7.20 and 5×7 yields ‘row 35’ in Figure 7.21.

Overall, the relationship between lesser and greater bow waves is that of a number and its factors, the lesser bow waves which satisfy the expression $n^2 - s^2 = N$ or $n^2 + n - (s^2 - s) = N$, for n and s in whole numbers, will contain terms which are factors or multiples of factors of the number/ratio of row N. Conversely, prime numbered rows will have no whole number solution to the expression other than that of the limiting case: $n^2 - (n^2 - 1) = N$. Figure 7.22 shows in a little more detail how these relationships fit together.

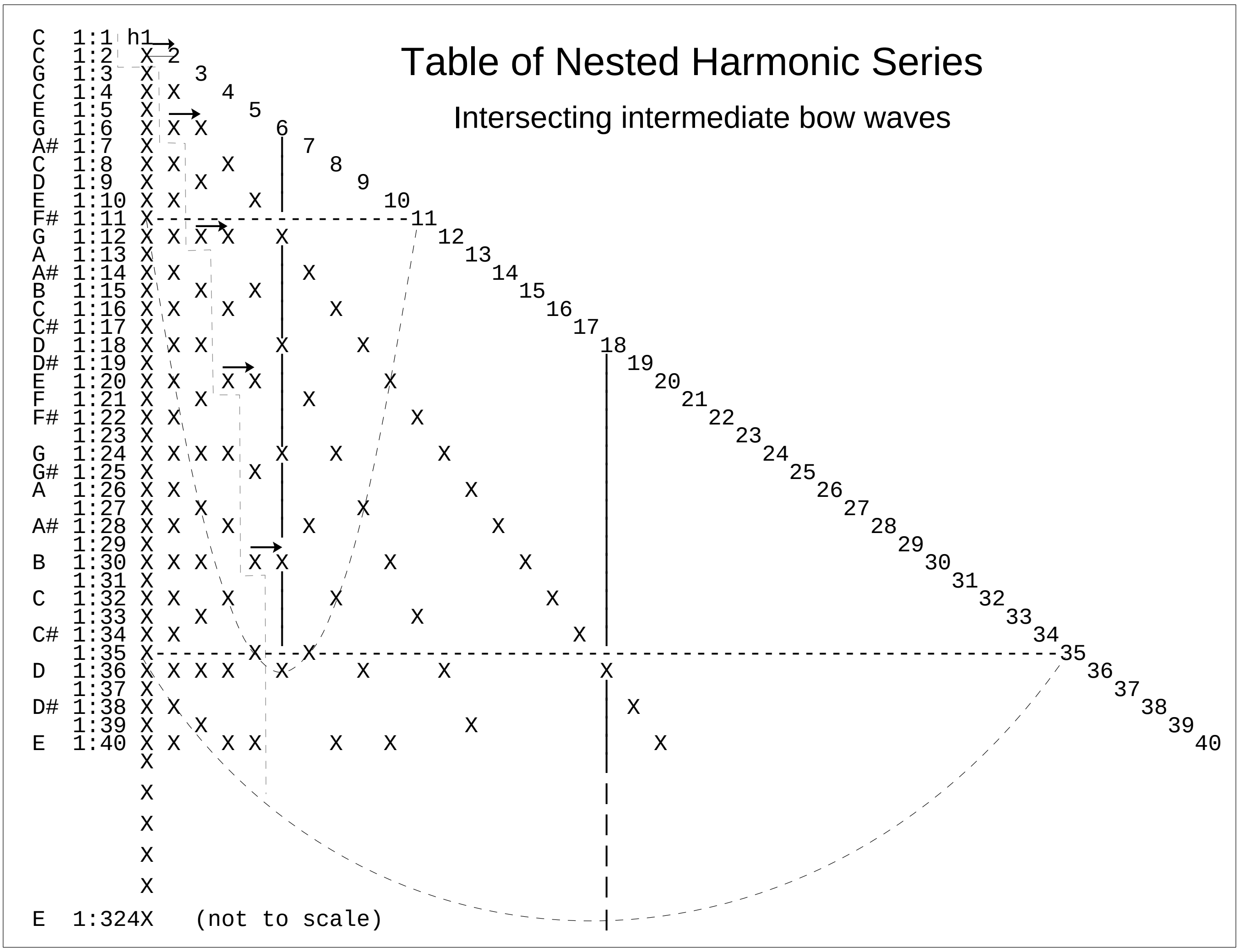

Figure 7.21 The intersecting lesser bow wave of column six cuts through row thirty-five at columns five and seven.

Later, in Chapter 13, this topic of bow waves will be picked up again and used there as the basis for a computer program to find the divisors of a given number. The program may be run on a digital electronic computer using binary numbers, or alternatively, be performed on musical instruments, as a tonal composition written in the mutable base position-value number system. Though I should warn the reader not to expect that such a piece of music will excessively delight the ear; however, it does underline the point that a tonal composition, viewed from the perspective of the MOS model, is a sequence of numbers processed in accordance with the algorithm of symmetrical exchange and that this mechanism – which equates to commensurable chord progression – can be turned to overtly mathematical use as well as its more common purpose of entertaining the ear and mind.

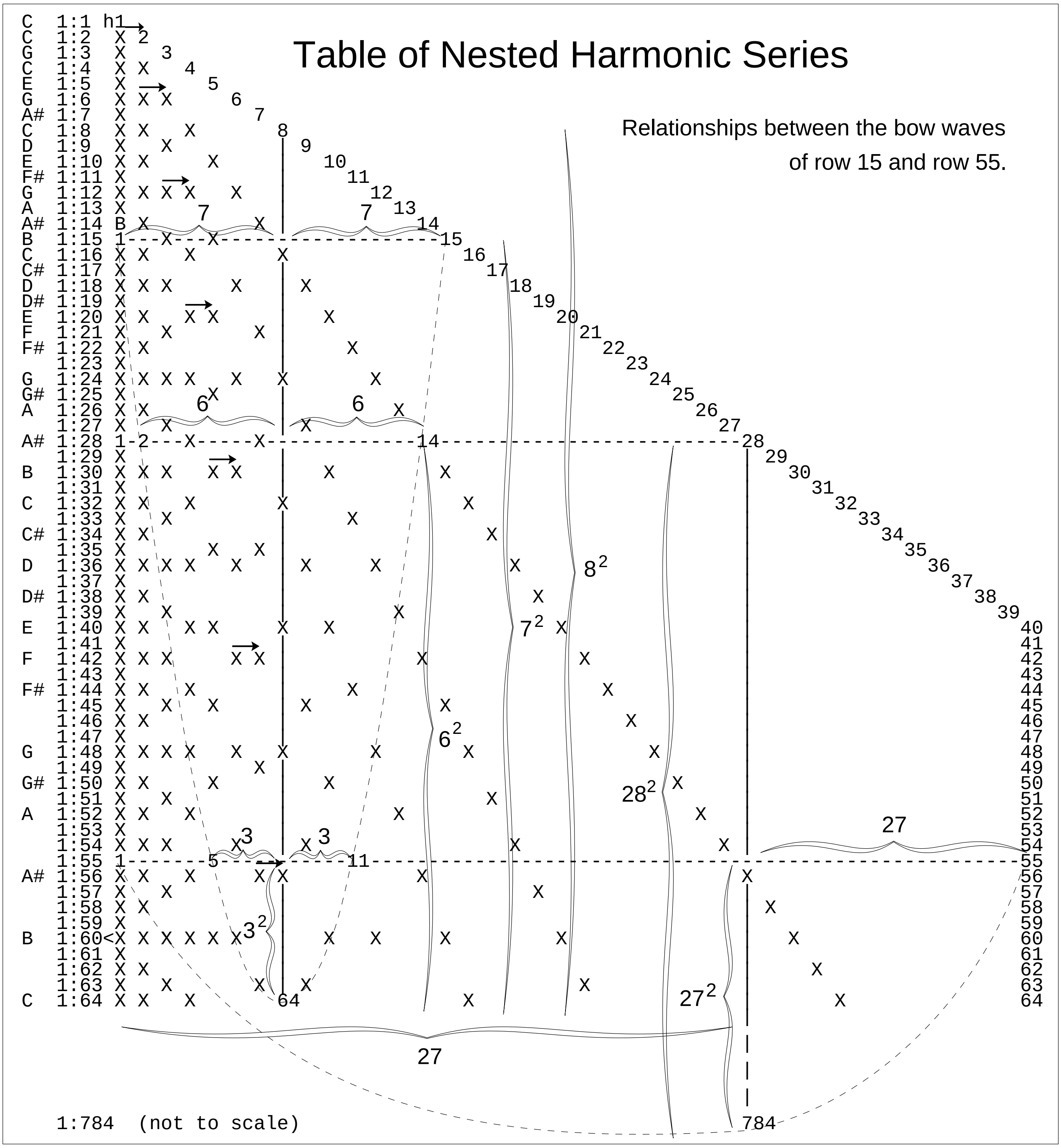

Figure 7.22 The 'resonances' between row 55 and row 15 are revealed by the squares of whole numbers.

Notes

1. Strictly the arithmetic mean between the upper and lower boundaries.