4

Three Domains of Music?

PITCH, DURATION AND TIMBRE

That the foundations of music lie in the harmonic series is clear enough; and it is the mismatch between the curving spiral of precise 2:3 relationships – rather than tempered fifths – plotted against the fixed pitch grid of instruments like the piano, that mainly gives rise to the problems of tuning discussed in Chapter 2. However, although the notes from the equal-tempered scale don't exactly match the whole number frequency ratios of the harmonic series, it is useful to associate these ratios with their near note-letter equivalents – Figure 4.1. Also, the sharp sign # or hash, is generally used in written references, even where a flat sign appears on the staff, and in both cases they should be taken as referring to the same, acoustically true, whole numbered frequency ratio of the harmonic series. From the exclusively relational standpoint adopted in this book, tonal compositions are considered, in principle, as if they are self-organising systems, 'little worlds' of harmonic relationships, isolated from any external influence. All structure and coherence in these systems arises from their internal relationships, and all relationships are derived from, or computed by, the whole numbered ratios of the harmonic series. Thus a piece of tonal music, operating under these (theoretical) constraints, will necessarily involve a certain *flexing* of pitch relative to any fixed scale, as the composition computes its structural evolution through each harmonic (or metrical) progression – from the opening chord to the final cadence. Luckily, the tolerance of the ear and processes of aural cognition allow the true whole number relationships to be extracted from the inevitably approximate intervals embedded in the objective musical sound.

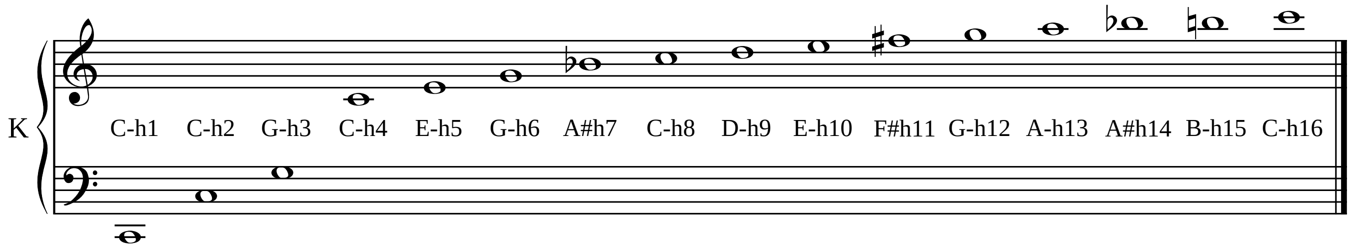


Figure 4.1 The harmonic series built on C, h1 through h16 with associated note letters.

Music's theory of scales or modes go back at least to the beginnings of Greek science and philosophy more than 2500 years ago, when Pythagoras discovered that: "there is a harmony in nature, a unity in her variety, and it has a language: numbers" to borrow Bronowski's[1] eloquent line. Much of our science rests on the (continuing) working-out of this deep insight; and it is remarkable that even before the tools of modern empirical experiment became available, the ancients were able to make such stunning progress. Not only in the Pythagorian association of number with the underlying structures of the natural world, but also in the later atomist theories of the school of Leucippus and Democritus, which anticipated somewhat, twentieth century particle physics. Pythagoras was probably the first to grasp the connection between music and mathematics, discovering the link between harmonious musical intervals and the ascending whole number ratios, 1:1, 1:2, 1:3, etc. that correspond to the natural or default modes of vibrations of a taut string. In effect, he discovered that the relationships of the harmonic series can be regarded as *nature's digits*; which is to say, physical phenomena encapsulating number relationships: oneness, twoness, threeness, etc. The story goes that Pythagoras realised this connection after hearing the ringing tones of different-sized hammers and anvils, on passing by a blacksmith's forge. However the realisation emerged, it remains arguably the greatest discovery of the human intellect: the connection between formal human math and objective physical reality. Even 'abstract' number systems like the decimal system, as discussed in Chapter 1, require physical tokens, some physical representation of the digits: 0, 1, 2, 3, through 9, to encapsulate number relationships. Nature's gift, however, is rather more generous in this regard, in that the digits of the harmonic series never run out – h1, h2, h3, through hn – providing a basis for all relationships. To quote Carl Friedrich Gauss[2] (1777–1855) one of the greatest mathematicians of the modern era, "Mathematics is thus in the most general sense the science [i.e. theory] of relationships". One might be tempted to add to this statement 'and nature, the practice [of relationships]'. Gauss also makes another highly relevant observation in *The Foundation of Mathematics*, pointing out that ancient Greek learning, and indeed classical learning in general, was probably skewed toward a geometrical view of mathematics by its lack of a positional number system. However, the Greeks' predominantly geometric approach could also be viewed as an attempt to extract a theory of relationships (i.e. mathematics) directly from reality[3], that is, the material world of (apparently) three-dimensional Euclidian space. Indeed, Plato laid down a knowledge of geometry as a necessary qualification for admission to the academy in Athens. And, to some extent – in a far humbler and incomplete manner – *Journey to the Heart of Music* is also attempting to recover some semblance of 'a mathematics' directly from physical reality; though not from the geometry of our spatial environment, but out of the auditory environment of *tonal music*.

To fathom a system by means of the human intellect, whether it be the great material world or the smaller arena of western tonal music, probably requires a coherent, all-encompassing, mathematical scheme for its ultimate realisation. However, a description framed in such terms is unlikely to tell us everything we might like to know about the system. The questions that remain unanswered, require other strategies. For example, the system of mutable numbers has little or nothing to say directly about the emotional content or effect of a musical composition. The analysis is fundamental yet limited in scope, indeed perhaps disappointingly barren at first sight. How can the exchange of a few digit sequences *explain* a great work of art? It can't and it doesn't. Mutable numbers explain only the foundations upon which a composition is built. Much the greater part of the magic of music develops later, emergent holistic qualities, found at higher organisational levels; where, within a coherent harmonic structure of commensurable mutable number exchanges, the interplay of melody, rhythm, counterpoint, timbre, meter, texture, etc., entertain and delight aural cognition. However, I suspect entropy at the fundamental level plays a vital role: in that the highly ordered exchanges of mutable number digit sequences embedded within tonal harmony, present to our ears and system(s) of aural cognition a supply of comprehensible, ultra-low-entropy aural structure upon which to feast. And, notwithstanding the limitations of applying mathematics to the field of music, from the ancients setting the stars to turn on harmonic spheres, to Niels Bohr's quantized atomic model of stepped integer electron energy levels in the early twentieth century, the harmonic series has been a key feature of many attempts to unravel Nature's puzzles.

THE DOMAIN OF PITCH

In music the integer sequence of vibrations of the harmonic series, oscillation or more generally, the principle of regular cyclic or periodic behaviour, is a simple yet most profound thing. The oscillation or vibration of a physical body, creates pitch and timbre, the very substance of musical sound; while simultaneously the pulses measuring out musical time – at the vastly lower frequencies of the rhythmic oscillations of duration – maps out the arena within which the substance of pitch and timbre finds its existence. This chapter investigates a far-ranging unity that runs through what might be termed the *three domains of music* – the oscillations of duration (rhythm), the oscillations of pitch (notes and chords) and the oscillations of timbre (tone color) – three levels of periodic behaviour, very different in scale and perceived character, yet basically partaking of the same intrinsic nature.

The connection between the domains of pitch (i.e. notes and chords) and timbre or tone color was firmly established by the scientific work of Hermann von Helmholtz[4] in the 1860s, although instrument makers and musicians, and in particular organ builders and organists, appreciated these connections in practice long before. Various different ranks of pipes in an organ as well as having descriptive names like Principal, Flute or Reed, are also given *foot length* designations referring to the resonating length of the largest pipe in the rank. (For some ranks the actual pipe length is a fraction or multiple of the designated foot length due to its design and voicing; nevertheless they are given the foot length of their sounding pitch.) As the compass of an organ keyboard by convention only extends down two octaves below middle C, it is the length of the pipe connected to this bottom C key, the largest pipe, which yields the foot designation. For a rank of pipes to play at normal pitch or thereabouts (circa A = 440Hz), the resonating length of the bottom C pipe is roughly eight feet – thus eight foot pitch equates with the normal playing pitch of a piano. Middle C played with an eight-foot organ stop selected would be (nominally) the same frequency as middle C played on a piano. By selecting a four-foot rank or stop on an organ, the middle C key would produce a note one octave higher than normal pitch, and with a sixteen-foot rank one octave lower. Indeed, on a large organ one could expect to find ranks of pipes ranging from thirty-two foot pitch to well beyond one foot, that is, ranks that play from two octaves below normal pitch to beyond three octaves above. And that is not all, in between these octave steps there will be ranks of pipes with fractional foot lengths, for example, the Gross Quinte at five and one-third foot, Nazard Flute at two and two-thirds foot and the Terz Flute at one and three-fifths foot. These three stops would play, respectively, tones a fifth, twelfth (octave + fifth) and seventeenth (two octaves + major-third) above normal pitch. The logic behind all these variously pitched ranks of organ pipes, from giants to tiny pipes smaller in diameter than a pencil, is revealed by selecting all the stops and playing middle C – Figure 4.2. (It is recorded that when assessing an organ's qualities, J S Bach would first draw all the stops.)

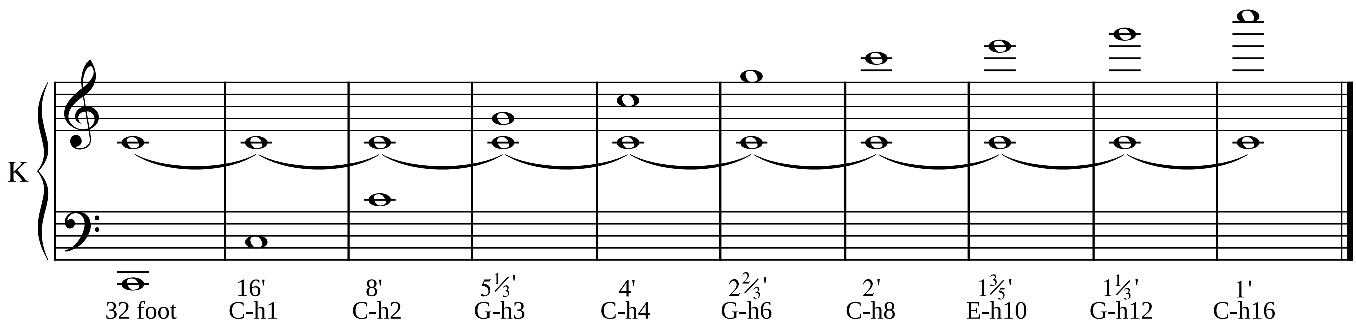


Figure 4.2 Foot length designation of organ stops with their actual sounding pitch for middle C reveals the outline of a harmonic series.

Effectively the various organ stops are a selection of ‘artificial’ partials of the harmonic series. Indeed, one of the great delights of playing the organ lies in the range of combinations and permutations of tone that can be achieved through the judicious selection of individual stops – the art of registration. All musical instruments produce a spectrum of partials – their characteristic tone palette or sound, sometimes termed the *formant* – however, the organ, chameleon-like, has developed the aspect of tone color into a whole dimension of expression. Basically, the organ has adapted notes into timbres, blurring the distinction between the two, and, in so doing, demonstrated the underlying continuity of the two domains of pitch (notes) and tone color (timbre). One particular example is the cornet registration where the flue pipes Flute 8', 4', 2' plus Nazard and Terz, mimic the timbre of a brass instrument. A cornet registration would normally be used for a single melodic line, illustrating the point that the five stops are acting as the components of single notes, not as five individual parallel melodies. From the tuning perspective this adaptation of notes into honorary partials or harmonics is clear, because the intervals between the integer and fractional foot ranks are tuned true, not *tempered*. When tuning a Quint rank the interval to normal pitch has the pure ratio 2:3 not the tempered ratio of 1:1.49.

The foot length designation continues upward in the form of both single rank or multi-ranked ‘mixture’ stops like the Scharf 1/2 foot or Zimbel 1/4 foot – pipes which could not have any sensible use other than as artificial partials – and in theory at least, these foot lengths could be extended downward beyond the 32 foot level. It is revealing to follow this line of thought by asking the question: How many octaves are there between middle C (256Hz) and a brisk Allegro of metronome 120? The answer can be found in Figure 4.3.

	Foot Length	Pitch Hz-cycles/sec.	Tempo MM-cycles/minute	
	1 foot pitch	C = 512 Hertz	30,720 Metronome	
Middle C-->	2 foot pitch	C = 256 Hertz	15,360 Metronome	
	4 foot pitch	C = 128 Hertz	7,680 Metronome	
	8 foot pitch	C = 64 Hertz	3,840 Metronome	
	16 foot pitch	C = 32 Hertz	1920 Metronome	
	32 foot pitch	C = 16 Hertz	960 Metronome	
	64 foot pitch	C = 8 Hertz	480 Metronome	
	128 foot pitch	C = 4 Hertz	240 Metronome	
	256 foot pitch	C = 2 Hertz	120 Metronome	<--Allegro
	512 foot pitch	C = 1 Hertz	60 Metronome	

Figure 4.3 Extending the domain of pitch (notes) downward until it reaches the realm of tempo. There are seven octaves, roughly the same distance as the compass between the highest and lowest notes on a piano keyboard, between the note middle C and an Allegro tempo.

Thus we have travelled, briefly, from the vibrations of note pitches upward into the domain of timbre and found no break in the continuity of oscillation between the two, a discovery pioneered and exploited by organ builders right at the beginning of the tonal era (e.g. the organ at St. Maarten’s, Groningen[5] built by Rudolf Agricola in 1479). Generally, pitch-frequencies lie at the lower end of the continuum, forming waves of lower frequency and often greater amplitude (volume), while the oscillations of timbre are found at higher frequencies and with lesser energies. The ear apprehends these differing ranges and from them generates perceived notes and timbres, but without any clear division. Adding amplitude to a harmonic of timbre can convert it into a perception of pitch, as illustrated when the guitarist lightly touches a plucked string at a fraction of its length (i.e. half, third, quarter) to produce a delicate ‘harmonic note’. On the other hand, notes in bulk – chords – can take on the character of tone, as for example, with an organ’s full chorus. However, in contrast to the upward continuum, the downward extension of the domain of notes much beyond 16 foot pitch, into the realm of tempo, is blocked by the limits of our ears. Sound waves become inaudible around frequencies of 16Hz (and above 10,000 to 16,000Hz depending on age) and so there is a discontinuity between the oscillations of rhythm and duration and the higher frequency realms of note-pitch and timbre. This disconnection allows a freedom and flexibility to exist between the oscillations of rhythm relative to those of pitch/timbre (i.e. the metrical pulse many vary while the elements of pitch and timbre remain fixed). The practical impact of this upon the application of MOS model to tonal music is to generally require the use of two systems: one to describe metrical duration and a second to encompass the broader vistas of pitch/frequency.

Waves

Though waves come in many shapes and sizes, including earthquakes and x-rays, fundamentally they all obey the same rules, with two basic types: transverse and longitudinal. In the former the direction of vibration is at right angles to the wave's direction of travel while in the latter the direction of oscillation is parallel to it. Sound is a longitudinal wave type, a mechanical waveform composed of compressions and rarefactions in a material medium. And as waves exhibit regular periodic behaviour it is convenient to express each cycle in terms of a circle, i.e. the 360 degrees marked off below the graph in Figure 4.4.

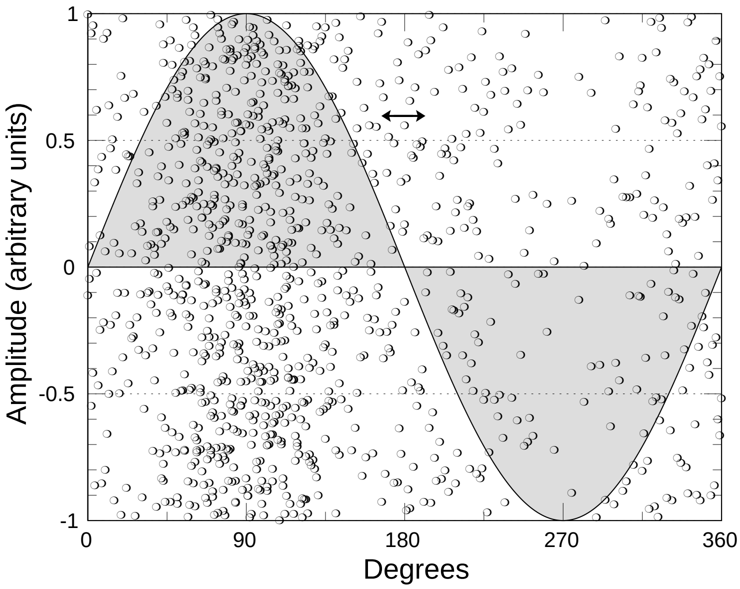


Figure 4.4 Compression (90 degrees) and rarefaction (270 degrees) charted onto an amplitude scale. One whole vibration or cycle: line graph and distribution plot. The 'particles' oscillate to and fro (parallel to the wave's direction of travel) as the wave moves through a material medium.

A standard (sine) wave in gray charts one simple cycle – Figure 4.4. The 'particles' schematically represent the squeezing and stretching of the medium through which the sound is travelling, while the shaded area translates this to and fro motion of the particles (indicated by arrow, center) onto a scale on the vertical y-axis. Though the sound wave is travelling through a medium like air at over 750 mph (331.4 m/s), the actual motion described by the individual particles of the medium through which it is travelling, is a forward and backward movement around a mean position. The wave front travels ever onward while the particles themselves, oscillating back and forth (along the axis of propagation), go nowhere. As each wave passes, the particles necessarily accelerate toward the center of their oscillatory motion and slow to a halt as they come to reverse their direction of travel. We shall return to this pendulum-like motion later in the discussion of duration. However, concentrating for the moment on the relationships between waves of the same type and with commensurate ranges – coherent waves – I would like to introduce (below) the term and concept of the *Metric*[6] of an interval or chord.

The Tuner's Calculator

Described in Chapter 2 was a 'black-box' process somewhat like calculus, guiding the tuner to a perfect match. As the tuning note approached a reference note, the beats (periodic fluctuations) formed by the combined tones became less frequent and tended eventually toward infinite length, that is, they disappeared. There were two notes involved in the intermingling process and their respective fundamental frequencies were brought into a one is to one, or other simple relationship. In Figure 4.5 there is an intermingling or interference pattern formed by two notes where the fundamental pitches are not brought together into a one-to-one relationship, but rather, standing apart at the frequency relationship of 4:5, an interval of a major-third, nominally C-E.

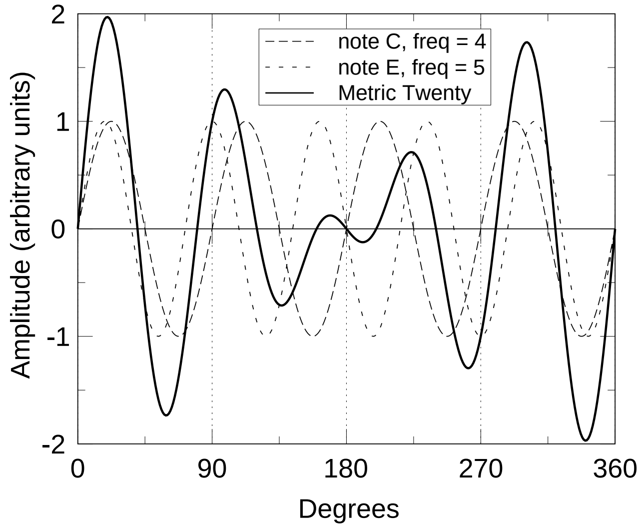


Figure 4.5 The Major-third interval of notes C and E, ratio 4:5, form an interference pattern of Metric Twenty. (Unless otherwise stated, all constituent waves are drawn with uniform phase and unitary amplitudes.)

In this interval what happens in terms of the black-box process of intermingling is that the two waves (the fundamental oscillations of notes C and E) amalgamate to produce a new more complex waveform, an interference pattern. The overall pattern of amalgamation is basically the same as seen earlier with tuning and reference notes, but in this example the frequencies of C and E are further apart, and so the much faster 'beating' undulations of amplitude – the continuous line in Figure 4.5 – are perceived as being integral to the sound. Audible beats do occur between the fifth harmonic of C (h5) and the fourth harmonic of E (h4) in the tempered major-third (4:5.04), but here we are concerned with the perceived true interval, a pure 4:5 relationship, and indeed, only with the note's fundamental oscillations, thus excluding all partials. Also, generally throughout the book, examples of interference patterns are illustrated *in phase;* in the real environment this is rarely the case; however, because the ear doesn't register phase differences in this regard, effectively musical sound is perceived as such: the remarkable filtering properties of the ear yield the relational essence of musical sound, shorn of its complex detail.

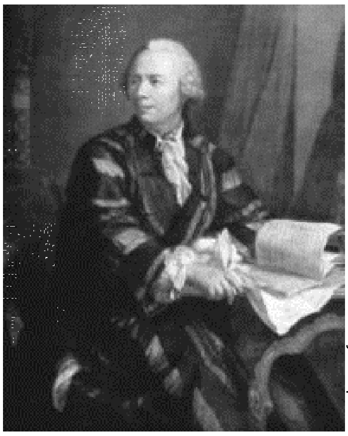
Looking at the graph, Figure 4.5, it could be said that a computation had occurred. Input, information in the form of the C tone's frequency and amplitude, interacts with the information carried by the second note E, according to a fixed set of rules (the physical characteristics of the medium, i.e. air, membrane, nerve, etc.) to produce an interference pattern – the output. Without a computer or calculator in sight, arithmetic is being done; input data is being transformed, manipulated by a *program or rule* and the result, output. At each instant, where the waves interact, a sum is computed:

wave[C] + wave[E] = wave[C+E], $x + y = z$.

Some of the time wave[C] will augment wave[E] and at others they will cancel each other out, in part or in full. The first is said to be constructive interference and the latter, destructive interference. The net result is an interference pattern, wave[C+E]. I label the resultant pattern with the *Metric* of the two waves.

Euler's Metric

It has been recognised from ancient times that our perception of concord corresponds with intervals of simple whole number ratios: unison 1:1, octave 1:2, perfect fifth 2:3, perfect fourth 3:4, major-third 4:5, minor-sixth 5:8, minor-third 5:6 and major-sixth 3:5. The eighteenth century Swiss mathematician Leonhard Euler developed this idea into a theory of consonance/dissonance based on the lowest common multiple of the nominal frequency ratios in a chord. Thus the major third C-E above, with the ratio 4:5, was given the value 20, a C-major triad C-E-G with the ratios 4:5:6 was assigned a value of 60, the rather more complex seventh chord C-E-G-A# with the ratios 4:5:6:7 would yield 420 and so forth. The lower the value assigned to a group of notes, the greater their concord.



Picture courtesy Wikipedia

Leonhard Euler (1707–1783), was born the son of a protestant pastor and a pastor's daughter: Paul Euler and Marguerite Bruckner, at Basel, Switzerland. Along with his two sisters he received an early education firmly founded upon his parents' Christian convictions, which remained a part of his character throughout his life. Although the family had moved away from Basel, Euler returned there, furthering his studies at the University of Basel and receiving private tuition from the renowned mathematician and family friend Johann Bernoulli. Though originally destined to follow his father into the church, Euler's self-evident mathematical talent and Bernoulli's influence swung him towards a mathematical vocation. In 1726 Euler obtained a position at the Imperial Russian Academy through his connection with Bernoulli. From 1727 to 1741 Euler lived in St. Petersburg, gradually advancing his position to become head of the mathematics section in 1733. In 1734 he married Katharina Gsell, the daughter of an art teacher, bought a house in St. Petersburg and began what was to become a large family. Many different areas of mathematics were enlarged and enriched by Euler; amongst his many contributions, he advanced the understanding of the relationship between primes and integers, and one piece of work, the Euler beta function, was used by Gabriele Veneziano in the twentieth century in the initial development of string theory. Another area of interest to Euler was the long-standing connection between mathematics and music, about which he wrote in *Tentamen Novae Theoriae Musicae* of 1739. In this work he set forth a mathematical approach to consonance and dissonance, and, in the final chapter, began to investigate the process of modulation. By this time, Euler's remarkable mathematical ability and productivity had been noticed abroad and in 1741 Frederick the Great of Prussia lured him to Berlin with the offer of a position. Euler was to spend twenty-five extremely productive years in Berlin, although, his relationship with the king was eventually to sour. Euler's unflinching religious convictions sat uneasily within one of the most sophisticated and cosmopolitan of European courts. Luckily, in 1766 Euler was able to return to St Petersburg under the patronage of Catherine the Great, where he remained for the rest of his life. His last years were overshadowed by the loss of his wife in 1773 and the general deterioration in his eyesight; however, he continued to work on mathematical problems right to the end of his life.

In Euler's scheme a unison at value 1 is the most harmonious interval, next the octave at 2, the fifth C-G 6, fourth C-F 12, the major-third C-E 20, and so on, with increasing values becoming progressively more inharmonious. (This sequence: 2, 6, 12, 20, 30, 42, etc. will reappear in the guise of the primary conjunctions of the algorithm of symmetrical exchange in Chapter 9, Figure 9.2.) At the further end of the spectrum, noise, near random, complex sound, when reduced to its many, awkwardly proportioned, simple ratios (using Fourier analysis, see below) would produce very large, inharmonious values indeed – a sign of a high raw information content. Beautiful sounds of themselves, as defined by naive hearing – that is, low whole number frequency relationships – don't contain great amounts of syntactic information. Interestingly, Euler's consonance/dissonance system based on the lowest common multiple of intervallic relationships, mimics the nerve pulse encoding algorithm used by the ear – though this is something Euler could not have known.

Implicit in the older identification of small whole number ratios with consonance, lies the more modern concept of the separation of resonances on the ear's detector membrane: the *critical band*. Over most of the ear's range, frequencies of ratio 5:6 and wider (minor-third to unison) do not destructively interfere with each other significantly on the ear's basilar membrane. (In the bass, the critical band widens somewhat causing thirds to sound less attractively harmonious in lower registers.) The amount of mutual overlapping increases gradually: for the intervals from octaves (1:2) to minor-thirds (5:6), the ear's detector membrane manages to accommodate the multiple resonances. However, as the increasingly narrow ratios of the harmonic series draw the wave peaks and troughs ever closer together, from the major-second (8:9) onward, the interference between the wave resonances, becomes destructive; resulting in increasingly random nerve signals being generated. Random signals are the hallmark of noise – discord. (Sitting right on the edge of the critical band is the interval of ratio 6:7 which finds no place in the equal-tempered scale, a flattish minor-third: a loss perhaps for classical tonal music but a subtle harmonic nuance exploited in jazz and by barber's shop groups.) Overall, the older view of consonance and dissonance, based on the abstract mathematics of small whole numbers and string division, has not been overthrown, but rather enmeshed more thoroughly in the material world through the empirical approach taken by modern physiology and acoustics.

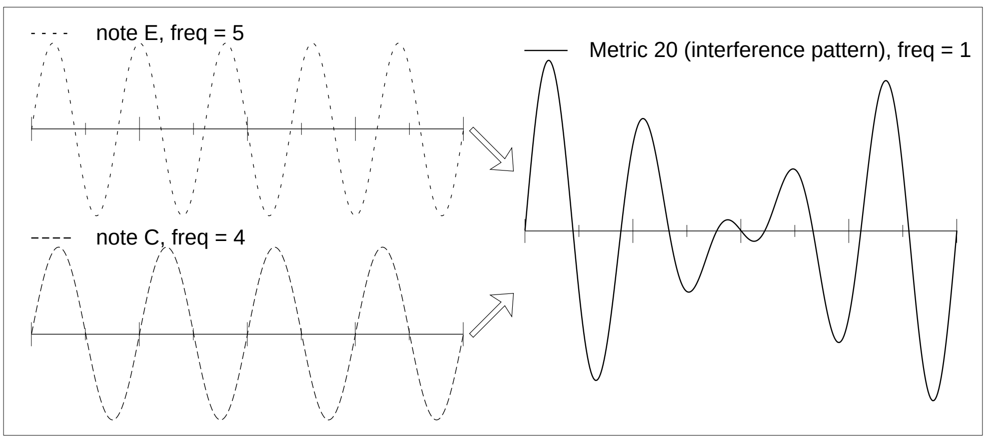
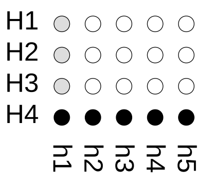


Figure 4.6 The two constituent waves of the major-third interval C and E (left) and the Metric 20 waveform which results from their combination (right).

Joseph Fourier a man of humble birth but great natural ability, demonstrated in a theorem that bears his name, that any complex wave can be reduced to a group of constituent simple waves. Even the most complex sound wave can, in principle, be rendered down to a set of basic simple tones. Figure 4.6 provides an example of this process, where the complex contour of the interference pattern (which was shown in Figure 4.5 by the continuous line) is separated out from the two simple (sine) waves of notes C and E.

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Metric: 20        H20/h5 - - H20/h4
Harmonic          H16/h4 - -
Harmonic                 - - H15/h3
Harmonic          H12/h3 - -
Harmonic                 - - H10/h2
Harmonic           H8/h2 - -
Note: E                  - - H5/h1
Note: C            H4/h1
               H3
           H2
Period: H1
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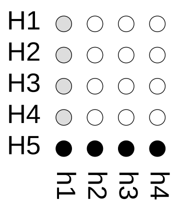

Figure 4.7 The two notes C and E in the wider context of the period of their interference pattern (H1) and common harmonic (H20). The ‘handle’ of Metric Twenty or M20 for short, encapsulates these relationships. (Notice that the fundamental harmonic series in this example is identified by the upper-case and the nested series by lower-case.) To the left and right are the associated number patterns of MBN $5_4\,0_1$ and $4_5\,0_1$ – two digit sequences out of mutable twenty’s total of eight individually distinguishable arrangements of digits.

In essence the concept of Euler’s Metric is capturing a part of Fourier analysis (in addition to Euler’s harmonic theory) and provides a most useful ‘handle’ for an oscillatory system, both at the level of constituent parts and as a whole interference pattern. I call this handle the *Metric*.[6] It is the smallest single value, the lowest frequency wave if you like, which can coexist or cohabit with all the constituent frequencies in the system[7]. And for shorthand the abbreviation ‘Mn’ for Metric ‘n’ can be used, i.e. M20 for Metric Twenty.

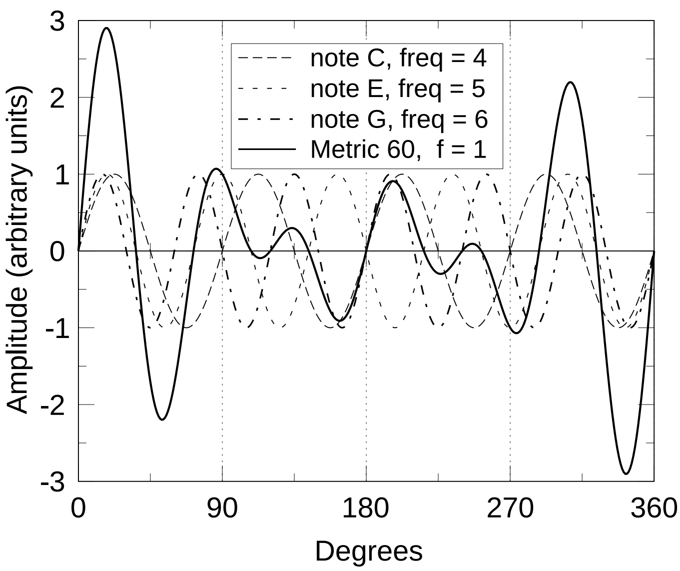

Figure 4.8 Major triad: note ratios 4:5:6, Metric Sixty or M60.

Chords can of course consist of varied numbers of notes, however the basic combination is the three-note triad. Figure 4.8 illustrates an interference pattern or Metric formed by three notes with frequencies of ratio 4:5:6 (i.e. C-E-G) – the common major triad. The perception of a chord's consonance or dissonance ultimately arises from the effects of the interaction of multiple resonances upon the ear's detector membrane, and, in the simple smooth curve generated by the ratios of the major triad some impression of harmony can be gleaned. Yet even for the major triad the basilar membrane has to accommodate considerable overlapping of resonances. A circumstance which favors simple whole number ratios and one that also produces a mixing together of the sense impressions produced by the separate objective frequencies. With the minor triad, the inversion of the interior intervals of major and minor-third again allows a sufficient distance between resonances for the nerve impulses generated from the membrane's oscillation to signal a harmonious commensurability, though one of a more complex internal makeup. However, once another interval of a third is added to either the major or minor triad to form a seventh chord, the mechanism of membrane and nerve cells is pressed to the very limits of its ability to generate commensurable streams of pulses from multiple resonances. Ultimately whether seventh chords are defined as consonant or dissonant is a matter for debate and historical perspective. Euler's theory of harmony basically encapsulates the situation of increasing degrees of dissonance (overlapping resonances) as chords pass from simple triads to more extended forms, leading relentlessly to ever more destructive interference.

Chords with interior intervals closer than a third and chords consisting of more than three successive thirds will produce a natural perception of dissonance. This occurs even when the notes are widely spaced, due to the lower notes' individual overtones falling close to the higher notes' fundamentals and thus still causing destructive interference. The irregular streams of nerve impulses generated by extended chords inevitably give rise to a perception of dissonance in normal musical performance. (However, in the exceptional circumstances of electronically engineered sound production this situation may be averted through artificial adjustments to the position and range of overtones.)

Picture courtesy Wikipedia

Jean Baptiste Joseph Fourier (1768–1830), was the son of a tailor, born at Auxerre, France. Orphaned at the age of eight, he was brought up at St. Mark's convent in Auxerre, where he also received his early education. With few prospects for advancement due to his lowly birth, Fourier entered the army as a teacher of mathematics. He was an ardent supporter of the Revolution, and his participation in the overthrow of the ancien regime opened up prospects of promotion. Fourier advanced to take a position at the Acole Polytechnique and soon came to the attention of Napoleon who supported his academic work and made use of his organisational talents by appointing him Governor of Lower Egypt during his abortive eastern expedition. In the years following the French expulsion from Egypt, Napoleon continued to make use of Fourier in administrative posts in France. Alongside his political duties Fourier carried out his scientific and mathematical investigations into the nature of waves and the propagation of heat. After his patron's final fall from power in 1815, Fourier moved to England where he continued his researches; returning eventually to Paris, he died in 1830. The theorem which bears his name and the Fourier analysis which flows from it are probably the most used and useful mathematical procedure ever discovered or devised.

THE DOMAIN OF DURATION

You might perhaps remember the opening (and rapid closing) of the Millennium Footbridge across the Thames in London some years ago; where the engineers had not taken account of the formation of a marching meter out of the chaos of each individual pedestrian's arbitrary gait. The bridge's elasticity provided a means of communication between the throng of pedestrians, leading to an unconscious synchronization of their pace. As more and more were drawn into step, their self-reinforcing pace, echoing a natural resonance of the bridge's span, resulted in a swaying of the structure.

We live entangled in countless metrical schemes – patterns of recurring fluctuations. From the ticking clock and ringing phone (an interesting 5/4 meter: ring, ring, 3, 4, 5) to nature's changing seasons, the cycle of youth and age and, most intimate of all, the complex rhythm of a beating heart. Mostly we take them for granted, their presence made invisible by familiarity and reliability but every now and then, as in the case of the Millennium Footbridge, the cyclic pattern turns out to have the character of a boomerang! And interestingly there was no external agency or direction bringing the crowd into step on the Millennium Footbridge; unlike the incident where a sergeant-major famously marched his company to disaster on the bridge at Angers – which taught soldiers ever after to break step over bridges. The Millennium marching meter arose naturally out of a propitious set of conditions – it organised itself – with the pedestrians unwitting pawns in a larger metrical scheme.

Time Signatures

Continuing the theme of oscillation, we now leave the frequency domain of notes and go on to look at how the longer wavelengths (lower frequencies) of rhythm and duration are woven into music's fabric of notes. Like the simple ratios of the consonant intervals, meter in its normal usage is also a legacy of ancient Greek civilization, indicating a system of durational patterns in poetry – pentameters, etc. and in the ordinary musical usage, patterns of notes which form the basic durational (i.e. metrical) units of music.

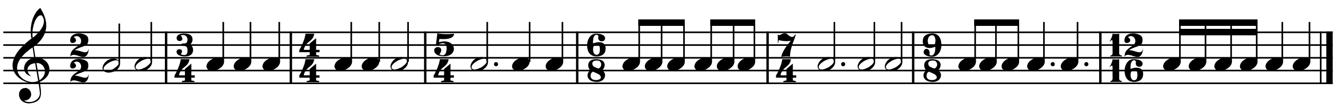

Figure 4.9 Time Signatures: the upper 'numerator' gives the number of notes or pulses (metrical units) per metrical period (bar or measure) and the lower 'denominator' gives the note fraction which expresses this metrical unit. For example, 3/4 yields 3 quarternote beats per measure.

The metrical units in a score, the measures of music, are governed by time signatures. And this is what gives the game away immediately: the same simple integer ratios turning up again – just as found in the harmonious intervals of pitch. For example, with time signature 2/4 the octave ratio 1:2, or 3/4 time the fourth 3:4 (C-F); but here the low whole number ratios specify the *horizontal* (temporal) relationship between notes rather than the *vertical* (harmonic) relationship. Also the scale is of a different order: harmonic relationships are expressed in cycles per second (hertz, Hz) whereas temporal relationships are specified in notes and beats per minute (Malzel Metronome, MM). The ticking pendulum of the metronome, counting out one hundred and twenty 'cycles' of durational oscillation per minute of an

Allegro, is in essence no different from a tuning fork counting out 440 ‘pulses’ of middle A in each second. The underlying periodic process at work is no different, though operating at very different scales.

In normal usage, small ‘m’ metrical patterns are ordinarily associated with music’s larger scale structures – note durations, rhythmic groupings, melodic phrases and sections of contrasting meters – rather than the fine texture of the audible domain of note frequency, harmony and timbre where the shorter waves form interference patterns. But essentially, metrical patterns and interference patterns are two differing perspectives on the single principle of oscillation.

In music, meter is a recurring scheme of durational values which forms a basic framework somewhat above the level of individual notes, imparting an underlying character to a piece of music and providing an extendible unit for the construction of larger structures, e.g. melodies. If one imagines that notes are the bricks from which a composition is built, then the meter might be thought of as the style of brickwork – English bond, Flemish bond, Herring-bone pattern, etc. The character that a meter imparts is largely down to the arrangement of stressed and un-stressed durations.

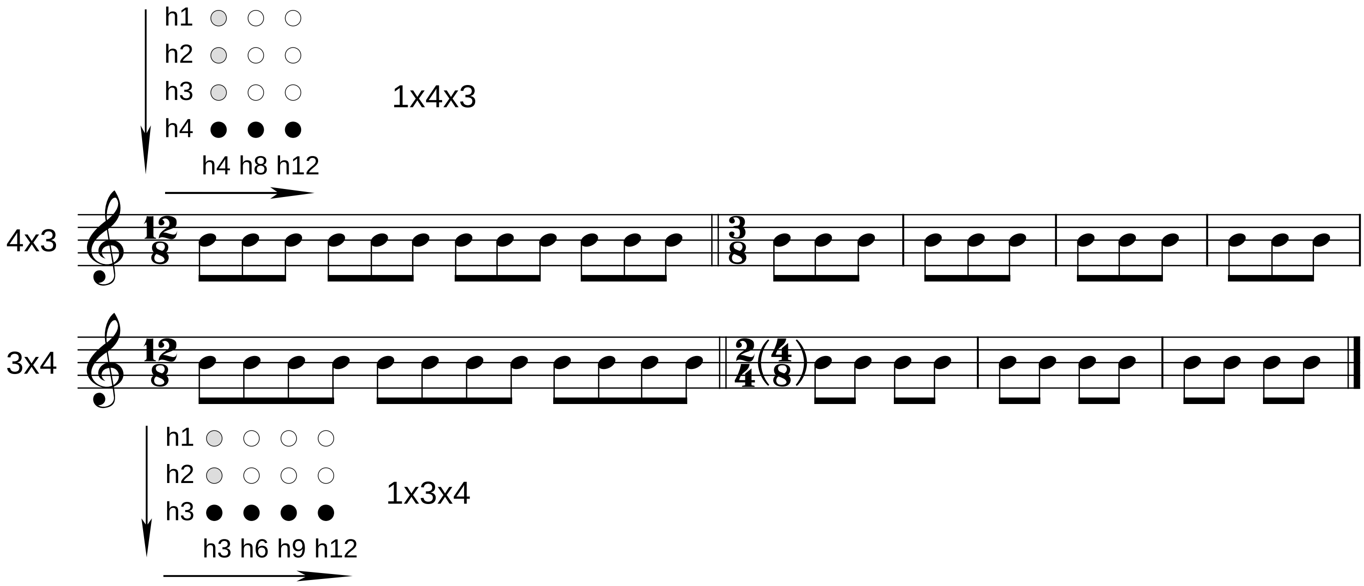

Figure 4.10 Twelve-eight (12/8) meter nesting sub-groups of three and four, with associated number patterns.

Time signatures can be divided into two groups: those which form *simple* meters and those which form *compound* meters. The simple meters are traditionally classified as: duple – 2/2, 2/4, 2/8, triple – 3/2, 3/4, 3/8 (Figure 4.11) and quadruple – 4/2, 4/4, 4/8: With compound meters nesting one simple meter inside another; for example, compound duple – 6/2, 6/4, 6/8, compound triple – 9/2, 9/4, 9/8 and finally compound quadruple – 12/2, 12/4, 12/8. (Though simple quadruple meter with one stress per bar usually translates in practice to a nesting of two duples.) Thus, for example, 12/8 usually forms a compound quadruple meter out of four nested 3/8 simple triple meters but could express a compound triple meter with three nested 4/8 quadruple sub-units (Figure 4.10), as well as some more eccentric combinations. Other, more unusual meters are used occasionally, of which quintuple meter 5/4 is the most common and septuple 7/4 quite rare (though often found in Romanian folk music). In performance these meters almost always break down into groupings of sub-units, for example, 3/4 + 2/4 or 4/4 + 3/4. Even stranger time signatures can be found in some contemporary compositions, for example 3/5 or 5/8. Compound meters, as illustrated below in Figures 4.12-14, are in essence nested harmonic series, that is, nested whole numbered frequency relationships, where the measure plays the role of fundamental period.

A Broader View of Meter

The specific use of meter outlined above is only one application of what is a broader principle. It is a common observation that the nature of music is closely bound to that of proportion and number but what is less commonly observed is that these are abstractions useful in dealing with what are actually patterns of periodic oscillation occurring in the material world. In a broader interpretation, meter (and *Metric* – borrowed from Leonhard Euler and impertinently rebranded) provides a useful method of understanding the relationships in complex patterns of oscillation (interference patterns) as it follows the 'grain' of the system it seeks to model and explain.

The metrical principle is both a way of reconciling differences and of organising detail into convenient units, for oscillatory systems. Rather like a mixture of pebbles in a vibrating container sorting themselves by size, likewise, ordered patterns emerge from the 'grading' process of nesting oscillatory patterns within each other. (See the Table of Harmonic Series, Figure 9.2.) Where oscillatory units make square pins and round holes, meter is the salve, the solution that will allow patterns of three and four to live amicably together, two and five and seven to rub along. Meter is the compromise that all parties can live with – the lowest common multiple within which their differences can be reconciled. Moreover, the necessary 'dance' that a meter performs to accomplish this task, creates flexible, extensible (and characterful) units of structure in music and perhaps more generally, in other oscillatory systems.

Rhythm is a most physical thing: it lives, it is expressed through (bodily) movement and dance. One need only think of the pendulum-like oscillations of keeping time with foot tapping or the conductor's swaying body and *waving* arms. To beat time is to tune to the frequency of a reference tempo – no different from tuning to a reference pitch – and to accent counter to the pulse is an equivalent process to that of suspensions and passing notes acting counter to the harmony. If you watch musicians, as they play rhythmic syncopations they often jerk or throw some part of their body counter to its normal flow of movement – deflecting or counterbalancing the pendulum-like regularity of the meter. In Figure 4.4 the motion of a particle of the medium through which a sound wave travels is illustrated describing a back and forth oscillation. The particle is behaving like a tiny pendulum swinging to and fro, swapping kinetic and potential energy around within each oscillation. Similarly, musicians embody the characteristics of a pendulum as they measure out note durations and keep track generally of the metrical and rhythmic units of the music they are performing. Essentially, they mimic the motion of the particle in Figure 4.4 with parts of their body, or imagine doing so in the mind – in order to *resonate* in sympathy with the low frequencies of rhythm and meter. Just as material tokens are required for number processing, so is a physical pendulum, in some form, required for metrical processing – keeping a pulse. Of course, the bodily movements of musicians don't actually generate objective physical waves at these frequencies (circa 1Hz), *but rather mime the frequencies*, which are synthesised through the changing patterns of note durations and rhythms, as the music flows along. A forensic Fourier analysis of musical sound, in performance, would be able to objectively identify the metrical parameters from the dynamic accents applied to the notes; so these long waves of meter do gain objective existence, but by the back door as it were. In the domain of note pitch (and timbre) represented in Figure 4.4, the particles are dancing with the energy imparted by the wave itself but in the 'cooler' low frequency realm of note duration the energy flow is reversed, and the musicians must *dance* to give *substance and sustenance* to the *long waves* of meter. In testament to this difference, in the realm of pitch, the harmonic spectra generated by musical sound is unquantified, theoretically unlimited. That is, until an exchange between two chords takes place – so defining a particular conjunction frequency between them – no specific mutable number value can be ascribed to each (nested) harmonic series. By way of contrast, a meter is 'got-up' by musicians immediately, in performance, and without any exchange occurring between different meters, a precise mutable number value is specified from the start, encompassing the particular meter, pulse and figuration.

The 'waves' of temporal duration and meter in music have frequencies spread around the period of the human heart beat, at rest, about seventy per minute, MM = 70 or 0.86 hertz. Remarkably, this range of durations turns out to be precisely the order of magnitude needed by the MOS model for the fundamental frequencies of nested harmonic series capable of describing the chord progressions of normal (i.e. non-trivial) tonal compositions. Which is to say, the unit value of mutable number digit sequences which happen to well describe harmonic progression in tonal music turn out to be broadly equal to the music's pulse. While there is unlikely to be an exact concordance between the frequency values of the notes and the pulse in any normal piece of music – and indeed if there were to be, it would severely limit the harmonic range of a composition – there certainly is an approximate resonance between the fundamental tones of the mutable base numbers delineating the chords of tonal music and the metrical units measuring out their temporal duration. To jump a little way ahead, Figure 5.18 in the next chapter shows a schematic representation of an ideal system of nested harmonic series, describing a simplistic piece of music which does have an exact concordance in all its frequency relationships, and in Figure 13.8 this union of harmony and meter is applied to the first Prelude from the Well-tempered Clavier. Although in a sense an approximate concordance is no concordance, and this feature of tonal compositions could be argued away as merely coincidence or chance, equally it could be seen as indicative of the underlying unity one would expect to find in a system based on a single category or phenomenon – oscillation.

Figure 4.19 below gives a schematic picture of the *loosely coupled* relationship between the domains of pitch/timbre and duration. However, notwithstanding that these domains are united from the MOS perspective by their shared whole number structures and mechanisms, they are also divided from one another by the nature of their fundamental unit – one is constant, the other variable. In the domain(s) of pitch/timbre the basic units of frequency are held steady, and motion (harmonic progression) is obtained through the agency of the algorithm of symmetrical change traversing these discrete elements; in contrast, in the domain of duration, overwhelmingly, change is obtained by *varying the absolute size* of the pulse/ meter, the absolute duration of the unit – ritardando, accelerando, rallentando, meno mosso – either gradually or all at once. Only occasionally does the modulation algorithm come into play: at points of metrical change within continuous passages of music (e.g. hemiola). Many pieces, or movements within pieces, contain no change of meter at all, but almost all will exhibit some degree of tempo variation and in most pieces this expressive device is all pervading. A catalog of the range of standard meters in given in Chapter 10 and an illustrated discussion of how the MOS model accommodates melodic and figurative elements, plus the aspect of tempo variation, is discussed in Chapter 10 and illustrated with Example K.

Interestingly in music, temporal durations can exist without pitch or timbre but not the other way around; for example, the 'silent' composition for piano by the American contemporary composer John Cage, has a set duration but no notes! If pitch exists (i.e. notes and chords are sounded) then there must be temporal duration, but not the reverse. This hints at a flow of structure from the period of the whole piece, the fundamental H1 that contains all the *partials of subdivision* – timbres, notes, bars, phrases, sections, etc. Music's elemental unit, as with the harmonic series itself, might perhaps be its longest wave, the most fundamental 'H1', rather than its shortest. As a piece of music evolves from the first instant of sound, this longest wave of the whole composition would touch or, passing through all the constituent ratios as it grows ever outward, to reach its full extent only at the last moment, the final cadence. Indeed in the domain of duration alone, using the period of a whole composition as a fundamental frequency, one could construct mutable numbers in an analogous fashion to mutable numbers of harmony, whose conjunction frequencies encapsulate the pulse(s) in the piece. And in a rather theoretical way, it would be possible to encompass the whole of a tonal composition in *one very large mutable number* founded upon this ultimate fundamental. That is, by relying on the exponential growth of digit sequences in fecund mutable numbers, a single large value can be found with a sufficient variety of mutable digit sequences to accommodate all the chords in the composition.

Meter, Metrics and Interference Patterns

Implicitly, an interference pattern created by finite non-random constituents forms a meter, a recurring pattern of fluctuations – a metrical structure. Though often complex and irregular in the real world, where an interference pattern's constituent frequencies are full or complete – in the sense that all the integer frequencies are present and they have equal or near equal amplitudes – they possess a simple regularity. Indeed it is an interesting feature of simple waves of uniform amplitude, that if they are combined in ascending whole number order, h1 through hn, an interference pattern of exactly 'n' equal divisions will be formed. Although a special case, it is useful because it highlights relationships between Metrics formed from frequencies of consecutive whole number steps – h1 to hn (and/or multiples thereof, e.g. h4, h8, h12, h16 – Metric Twelve built on h4). And many real systems, though less uniform, will probably possess some level of regularity and proportionality as described by such idea systems; for example, the interference pattern produced by the ratios h1, h2 and h3 – a Metric Six (M6), Figure 4.11.

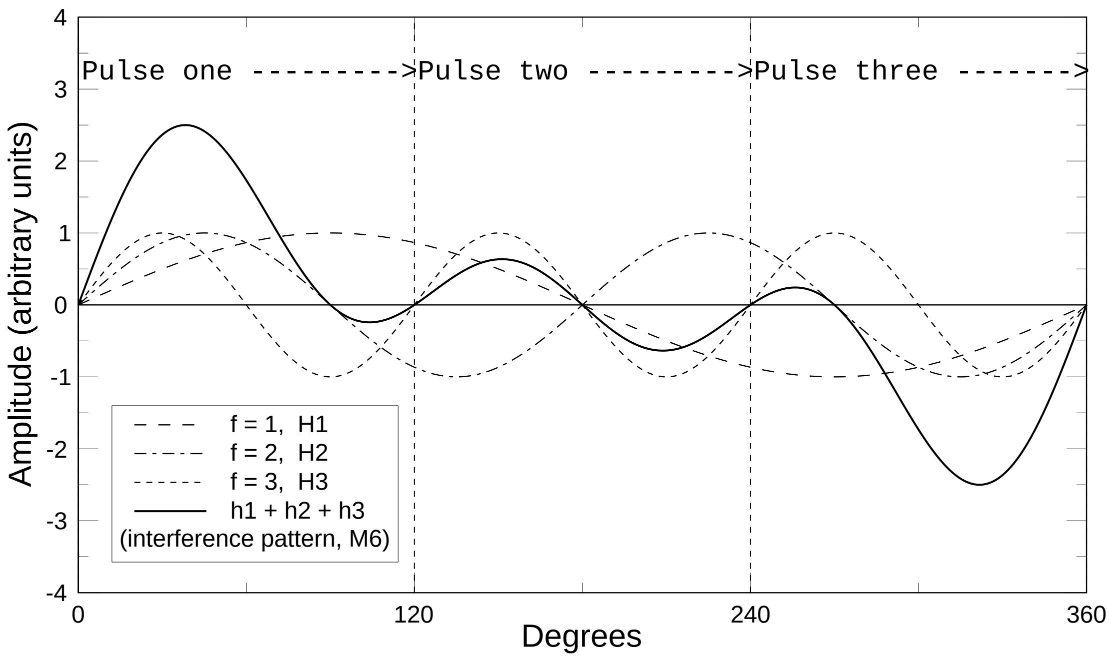

Figure 4.11 Metric of six (3/8 time), the interference pattern of frequency ratios 1:2:3 produces one strong pulse followed by two weaker pulses. This is an example of a additive, single column mutable number: 1x3 or MBN 3_1.

Simple meters of any description can be derived by the process of addition, but for the more complex compound meters the nesting of simple meters, one inside another, is required. These two metrical groups, simple and compound meters, equate with single column mutable numbers (i.e. prime state) and multi-column mutable numbers (intermediate/ground states) respectively. In the case of compound meters the absolute regularity of the subdivision of the period is lost when equal amplitudes are applied across the board; however, as noted before the accommodating tolerance of the ear and processes of aural cognition may be relied on to rectify such anomalous detail.

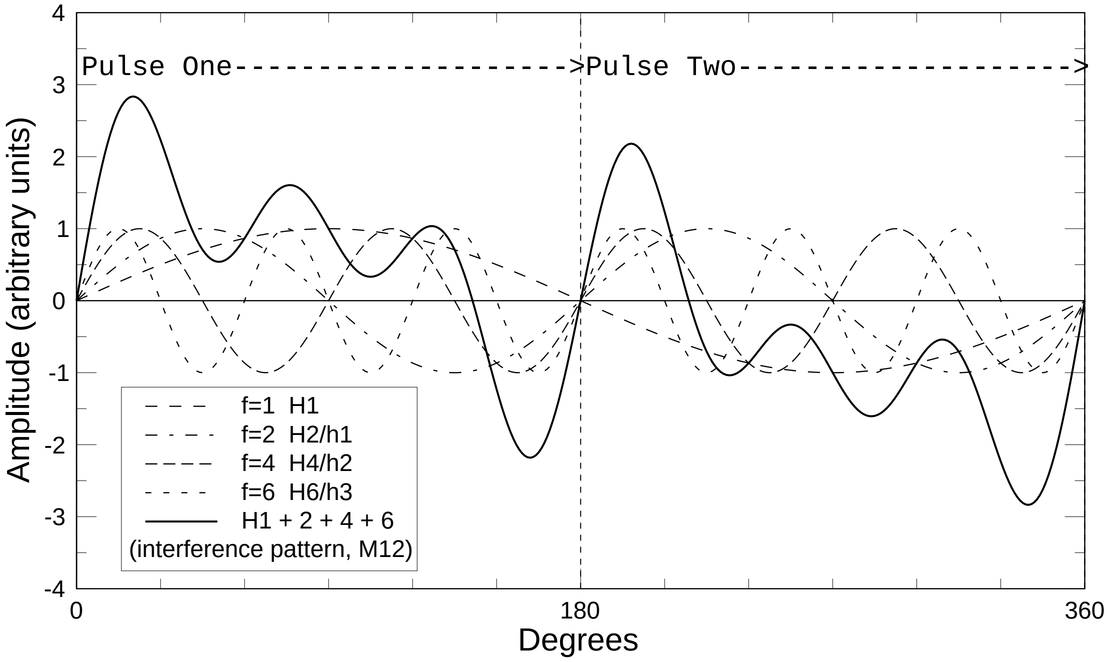

Figure 4.12 An interference pattern with a compound duple 6/8 meter, constructed by nesting three in two. It is also an example of a multi-column mutable number: 1x2x3 or MBN $3_2\ 0_1$.

A Simple Algorithm

Having reviewed something of the metrical realm of music, now it is time to consider meter in motion, in particular the act of changing from one meter into another. There are two reasons for doing this: firstly meter is the fundamental organisational principle in the domain of duration, which is perhaps reason enough, but secondly, in the ear, audible sound (i.e. pitch/timbre) is converted to nerve pulses, and these pulse streams have a metrical character. That is to say, once the objective physical sound waves articulate the ear’s detector membrane and stimulate the attached hair cells to fire nerve impulses, effectively the wave motion is translated into a metrical representation of the objective musical sound – a succession of pulses. And, if there is truly only one underlying domain in music, that of oscillation, rather than the three apparent domains of pitch, timbre and duration, then, also, there should ultimately be one organisational scheme, one algorithm which applies equally to duration, pitch and timbre.

Right at the beginning of this book, in the introduction, a small portion of a courante from the *English Suites* by J.S.Bach is given as an example of metrical change, and it was noted that this composition veered between measures of three metrical units and measures of two metrical units: measures with time signatures 3/2 and 6/4. A snatch of the music is shown in Figure 4.13, a measure of triple meter in 3/2 time and a measure of duple meter in 6/4 time. Above the staff in Figure 4.13 each measure is counted out in six quarternotes. Quarternotes are the lowest common durational multiple shared between 3/2 and 6/4 time and as such both measures are the same in this regard. Both measures count up to six; the lowest common multiple of 3 and 2 is 6. Equally, below the time signature, counts of three pulses followed by two pulses, the period of the whole measure is counted, one pulse per measure. Here again, as at the level of the measure, there is stability. When the melody is played and listened to, the change between meters is the main impression gained. However, as the courante continues on, swapping back and forth between 3/2 and 6/4, and somewhat more in the background, there is also a feeling of overall rightness and commensurability. Picking up on this shadowy background level of LCM (lowest common multiple) at the top and measure-lengths at the bottom – both of which are implicit in the foreground play between the meters 3/2 and 6/4, encourages the emergence and recognition of an overarching unity. The broader context of LCMs and measure-lengths, conjoin the disparate and interesting foreground metrical patterns, thereby rendering the piece whole and intelligible overall.

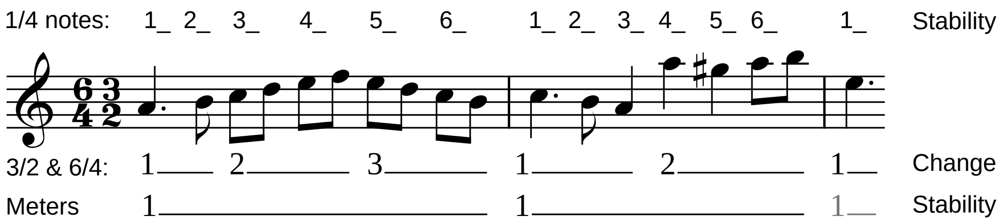

Figure 4.13 The opening two measures of the Courante from the second English Suite, with metrical analysis.

Out of these two measures a simple model may be constructed encapsulating the metrical essence of the whole piece – Figure 4.14 – where the shifting foreground groups of three and two pulses are shown in bold black type and the more shadowy background relationships in gray type. This simple abstraction embodies an overarching unity in the courante's metrical relationships; and, from this kernel I hope to demonstrate it is possible to develop a powerful new approach to, and understanding of, tonal music.

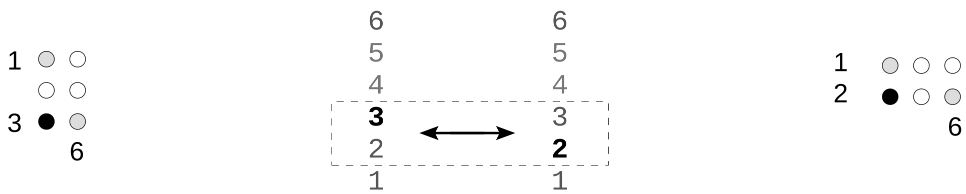

Figure 4.14 A simple model of the metrical scheme of the Courante, with number patterns to left and right.

Yet such a basic construction could have applications in other situations, beside that of meter. For example, the explicit groups of three and two pulses might be nerve impulses signalling an objective frequency detected by the ear, equal to their lowest common multiple. Or the explicit groups of three and two pulses might be the frequencies of two harmonics (i.e. strong overtones of a weak or absent fundamental tone) from which the internal processes of aural cognition construct a fundamental pitch sensation (illustrated for bassoon tone below). Or the explicit groups of three and two pulses might be the root pitches of chords forming a dominant to tonic (V-I) full cadence, in a harmonic analysis. The potential uses of this simple model or algorithm will be further explored and developed in the coming chapters; it is the basic mechanism upon which the modulating oscillatory systems model of western tonal music is built.

THE DOMAIN OF TIMBRE

I'm sure that you're ahead of me now. Yes, indeed, look at the distribution of fundamental and overtones in the sound produced by most musical instruments and it reveals the same patterns of close whole number ratios. Figure 4.15 gives a few examples and Figures 4.16-17 show the overtone profile of a note played on the violoncello.

Instrument	Formant	Metric
Flute	1:2	M2
Oboe	1:4:5:6:(10)	M60
Clarinet	1:3:8:9:10:12	M360
French Horn	1:2:3:4	M12
Violin	1:2:3:4:5:...n	M60 (variable)

Figure 4.15 Instrumental tone color formant expressed as a Metric by calculating the lowest common multiple of the most prominent harmonics.

The clarinet with its upper register obtained by over-blowing to the third and fifth harmonic (the twelfth and seventeenth) has a particularly interesting and ambivalent tonal palette with the strongest harmonics h8, h9 and h10 equally prominent – strong, sour and sweet tone all in one. However, there are also exceptions to this integer regularity in overtone structure – so painstakingly developed by instrument makers over hundreds of years – as in tuned percussion instruments, where generally weak non-whole numbered frequencies add a piquancy to an overwhelmingly strong fundamental. For example, the typical overtone profile of tuned metal bars is: h1, h2.75, h5.40, h8.93, etc. and for the traditional European bell, the tuned harmonics, which include a minor-third, are: h1, h2, h2.4, h3, h4. However, instruments lacking integer overtone structures, such as these do, form an exceptional group, and generally their use in western music has been limited and peripheral. The main thrust of developments has been to enhance the perceived tone, improve the accuracy of tuning and extend the range of musical instruments. All of these goals largely rest upon the production of clean whole-numbered overtone series. And here the critical nexus between a scale system based on simple (low whole number) interval ratios, harmonious music played in parts (chords) and instruments which generate pure, integer-related overtone series, becomes clear. Each of these three elements co-evolved, each reinforcing the selection of the other two characteristics: with perhaps the fifth (ratio 2:3) generated scale allowing, if not encouraging, the creation of polyphonic harmony, and both the scale and harmonious polyphony, informing the design paradigm of western musical instruments.

Joseph Fourier's work has provided a means of dismantling a complex waveform into a set of simple constituents. In Figure 4.16 a single sustained note, G below middle C, has been decomposed into individual frequency components, by Fourier analysis, revealing peaks of energy in the waveform corresponding to the note G (approximately 200Hz) and its overtones h2 (400Hz), h3 (600Hz), h4 (800Hz) and h5 (1000Hz). The pattern is continued with rather less detail in Figure 4.17, where the spectrum of harmonics for the perceived note G is traced over the most significant range of hearing for musical sound. The sequence of harmonics or overtones can be seen to extend up to, and beyond, the region of maximum acuteness in human hearing – roughly between 1000 and 4000 hertz. These graphs clearly illustrate two points:

Firstly, what is perceived as an individual note (with timbre) is in fact generated by composite frequencies, arrayed in the integer relationships of a harmonic series. That is to say, the phenomena of note-pitch, probably the most basic constituent of tonal music, is largely the creation of aural cognition.

Secondly, this objective array of integer harmonics (i.e. note plus overtones) will project upward from the whole-number relationships present in chords, thus providing the raw material from which the upper reaches of mutable numbers may be built. Essentially, the acoustic structure of musical sound, as charted in these two graphs, Figures 4.16-17, is the objective foundation upon which the application of mutable base numbers, to music analysis, rests.

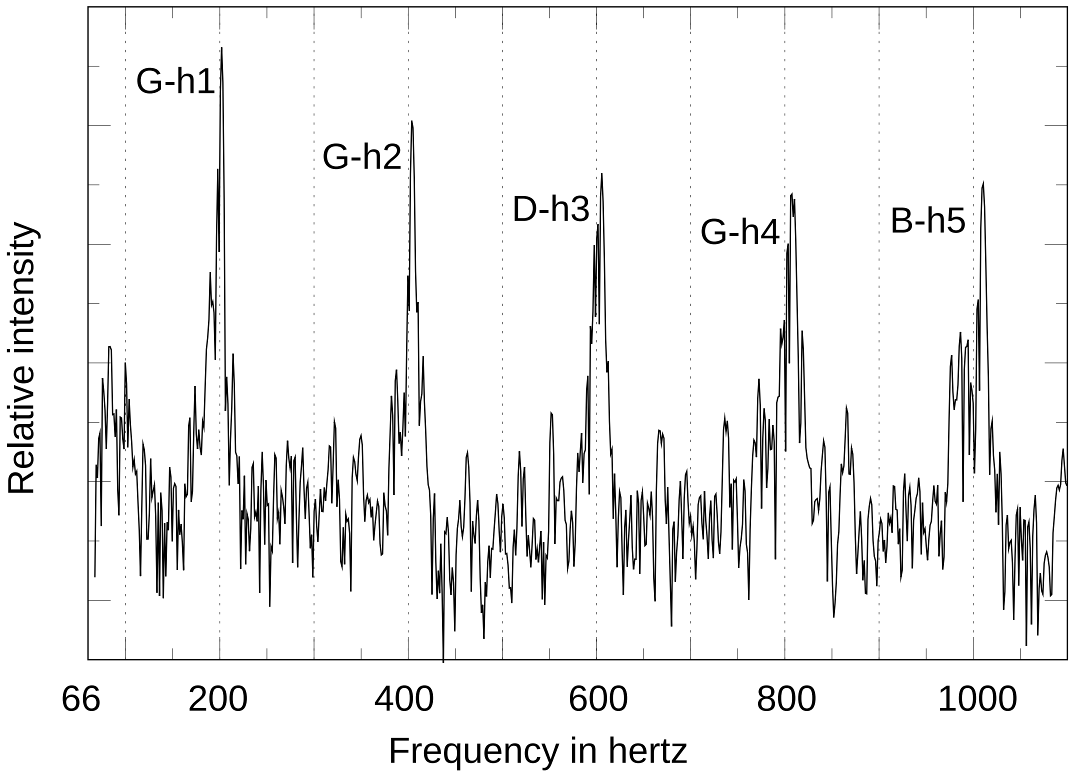

Figure 4.16 The lower frequency spectrum of the note G below middle C, played on a violoncello. (Based on the Fourier analysis of approximately three-quarters of a second of continuous sound – 32,000 samples.)

There is remarkably little to distinguish between the various peaks, apart from a gradual overall decline in power. In combination the energy contained in the overtones far outweighs the single peak representing the fundamental ‘note’, and yet we hear one single pitch with an integral attribute of tone color. Indeed, as the human ear is most sensitive in the region beyond the nominal range of most written music (i.e. the two octaves beyond a soprano’s top C) one might expect to hear overtones rather than fundamentals, yet we don’t – for normal musical sound. In some more exotic sounds, like those devised by John Pierce and for some tuned percussion sound, we may hear the overtones as separate entities, but, for normal musical sound, physiology, physics and processes of aural cognition function in such a way as to amalgamate the harmonics (fundamental and overtones) into a unified perception of pitch with timbre.

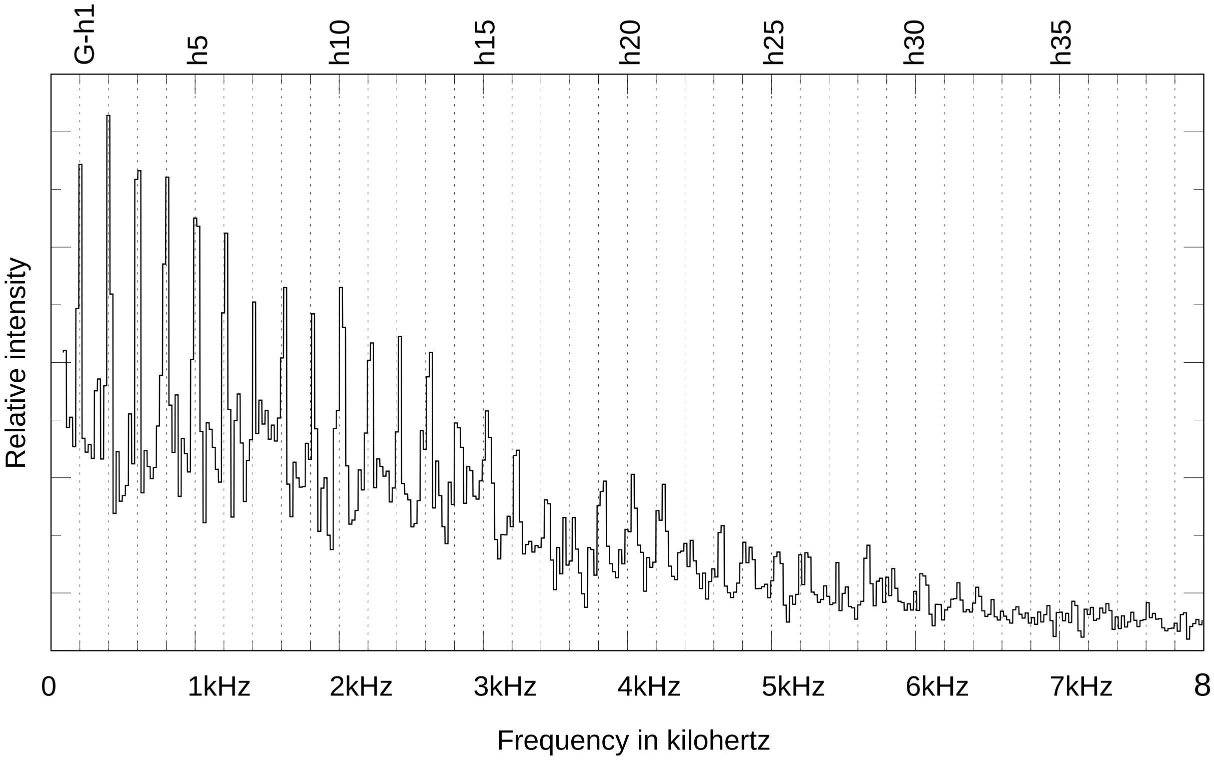

Figure 4.17 The frequency spectrum of the same note G below middle C played on a violoncello, extended across the principal range of hearing. (The intensities are compacted by averaging over groups of sixteen elements.)

Natural sound – in other words noises – do not often have the regular discrete structure of musical sound, but are generally formed of irregularly varying mixtures of frequencies and intensities. At one end of the continuum of natural sound a completely random mixture of frequencies and amplitudes yields an amorphous ‘white noise’ – typical of ‘road noise’ from tyres, the crackle of a hot frying pan or hiss of a radio off station. If the white noise is contained within a broad frequency band it might convey a sense of being a high noise (hiss), or very high noise (whistle), or low noise (buzz), or very low noise (thump). Somewhat better defined are noises where a narrowish band, or bands, of frequencies contain the majority of energy allowing something of a pitch characteristic can be apprehended, with the remaining weak, randomish, wide spectrum frequencies (perhaps with peaks linked to a core high energy band) possibly adding some sense of tone or character to the sound. Finally, moving to the opposite end of the continuum, musical sounds will present highly focused sets of frequencies to the ear – relatively clear harmonic series of steady integer-related frequencies – which the ear, and processes of aural cognition, can separate into the sensation of pitch (the fundamental period) with timbre (essentially the integral division of the pitch period). It is interesting that in experiments conducted by John Pierce, using non-harmonic, i.e. non-integer, stretched intervals of pitch and timbre (built on an equal-tempered semitone ratio of 1:1.0757, compared to the normal 1:1.0594), all sense of harmonic progression was lost. He concludes:

“This demonstration appears to show that the coincidence or near-coincidence of partials we find for normal (harmonic partials) musical sounds and for consonant intervals (with frequency ratios in the ratio of small integers) is a necessary condition for Western harmonic effects [...]”[8]

Acoustical phenomena in the ‘wild’ are rarely quite so clear cut as the preceding passages might imply. For a start the harmonics of timbre are subject to the same constraints of destructive interference on the ear’s detector membrane as are note frequencies. The limits to consonance imposed by the width of the critical band, discussed above with reference to Euler’s harmonic theory, apply equally to harmonics of timbre. As each note within a chord generates its own set of harmonic partials, all these partials can be the cause of either constructive or destructive interference depending on whether they coincide or not. Most harmonics of timbre generated by chords will not coincide, and so as chords grow in constituent notes, so also will the amount of destructive interference/noise gradually creep downward from higher frequency levels – and even a single harmonic series causes progressive, self-inflicted, destructive interference, i.e. increasingly from harmonics of 8:9 and beyond.

Notwithstanding such degradation, interference/noise is not entirely unwelcome: part of the tone palette of many instruments includes a leavening of noise, the humming or buzzing quality of lower registers in many instruments – piano, ‘cello, bassoon, etc. – is quite attractive, and useful in the ‘bite’ it adds to what might otherwise be rather bland or dull sounds. Equally, in higher registers, interference between closely or irregularly spaced harmonics can add a dissonant piquancy to musical sounds, for example in the harpsichord, glockenspiel and bells. Likewise the violence of the hammer blow upon three high tension steel strings in the piano, generates harmonics of timbre which waver and interfere with each other in a chaotic way that can produce a warm, rounded, holistic tone. In part, the particular attraction of this fine wavering or transience in the harmonic spectrum of piano tone, probably lies in the wealth of material it provides for the direction-finding components of aural cognition to work upon. (It is believed that direction finding is the last evolved and predominant feature in the hearing mechanism.) Human tastes, of course, vary enormously but most individuals find the depth and complexity of traditionally produced ‘acoustic’ musical sound more satisfying than the often thin and sterile offerings of electronic generation. Indeed, most modern electronic instruments now mimic traditional musical sound through sampling, which rather underlines the point.

For the most part, timbre is set by the instrumentation of the piece: the organ apart, individual instruments like leopards cannot change their spots. The tone color of whatever instrumentation is used in a composition, while allowing subtle differences to be made by the performer, overall firmly stamp the notes of individual parts with their instrument’s particular character. However, in pieces written for the organ, some hint of a music involving a free transformation of timbre can be gleaned from the organist’s choice of stops and changes of registration occurring within the composition. And indeed, in concerted instrumental pieces these transformations of timbre can be rendered with great fluidity and subtlety of shading through imaginative orchestration and skilled performance. Although a mutable number analysis of changes in timbre in an orchestral composition might be possible to some degree, it is perhaps unlikely to prove as compelling and comprehensive as that which can be achieved for harmony.

However, one can discern parallels between the realm of timbre and the domain of pitch. For example, in the care which must be taken in handling the orchestration of instruments with complex formants, such as the clarinet or oboe – echoing the difficult character of augmented/diminished harmonies – in that clarinet/oboe tone, like exotic chords, possess rather awkward, extended, hard-to-nest overtone series. A two part passage that succeeds using clear-toned flutes, may not please so well on clarinets where their complex arrays of harmonics can interact awkwardly. The interesting case of bassoon tone is another example: here the ear largely syntheses the perceived note out of higher harmonic frequencies[9]. The tone of the bassoon is full and rich, with the partials far more energetic than the fundamental. The processes of aural cognition, by means of generating something akin to difference tones from these strong, closely spaced partials, provides the sensation of a firm fundamental tone which measurement indicates to be absent from the objective sound. This acoustic phenomenon lies somewhat parallel to the basic approach adopted in the MOS model, where a deep fundamental frequency is postulated to link together a composition's harmonies, while not being objectively present in the musical sound.

Scores are very misleading guides to the actual aural make-up of compositions: *in analysis we see a handful of notes, but in performance we hear vastly more*. Though our conscious attention is drawn toward the written notes – the melodies and chords – unconsciously the ear is hard at work behind the scenes, sweeping up an extensive and almost imperceptible halo of faint, and not so faint, frequencies which accompany the principal tones. Each written note becomes a harmonic series of greater or lesser degree in performance: a fundamental *and overtones*. Indeed, not just one harmonic series, but due to the resonant qualities of the formant of instruments and buildings, a varied host of many different frequencies are also generated. In addition to this, our ears construct further *combination* tones: the difference and summation tones created in the ear's asymmetrical structure. Though we read a few notes on the page, in performance the ear is enveloped by a veritable cloud of frequency relationships stretching from well below the bass notes to the very edge of audible sound, and beyond. The ear and aural cognition respond to sound over the full range of perception. Though many of these frequencies are faint and are not usually consciously detected, they are apprehended in some considerable degree – as demonstrated, for example, in the determination of tone color, subtle differences of reverberation or the determination of the direction from which sound comes. Just a few notes on the page of a score translate into many components, spread over a wide frequency range in performance. It is therefore reasonable to suggest, that *our understanding of sounding music is based on the full range of objective and synthetic frequencies received by the ear and perceived by the brain*.

Conclusion

Instinctively, one feels that music, which exists in one continuous physical realm, has an underlying unity and logic. However, we perceive a divergence between rhythm, notes and timbre which appear to inhabit the very different domains of audible sound and temporal duration. This disjuncture, which is more apparent than real, at least for pitch/timbre, arises as much from the nature (and limitations) of our perception as from within the music itself. Our ears can distinguish audible frequencies – notes and tone color – from approximately 16,000Hz down to 16Hz, around which point the lower frequency cycles in the music's structure precipitate out into the temporal domain, to be perceived in the form of rhythm and duration. But this is just our perception of what, in musical terms, is a consistent structure of nested oscillations ranging from the highest harmonic of timbre, down to the period of the complete score. Illustrated in Figure 4.18.

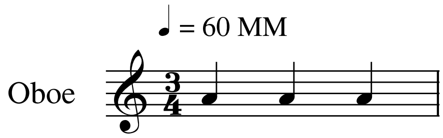

Figure 4.18 Music's single dimension of oscillation.

Measure/Bar Frequency	0.33Hz (1 cycle per 3 seconds)
Note Duration Frequency	1Hz (1 cycle per second)
Note Pitch Frequency	440Hz (440 cycles per second)
Timbre/Formant Frequencies	1760, 2200, 2640Hz

Perhaps a good analogy of the unity of musical phenomena over its full range of oscillation is to be found in the different states of water. When the temperature is below freezing, water molecules vibrate slowly allowing firm bonds to form solid ice – hard, chunky units of duration. Increase the temperature (rate of vibration) and the electrostatic bonds become less firm, and liquid water results – fluid melody and malleable harmonies emerge. Boil the water and the high frequency of vibration (kinetic energy) breaks the bonds completely, making steam – cloud-like hues of tone color. Water ice, liquid water and water vapour are not different things, they are one thing, H_2O, under different conditions. Similarly rhythm/duration, notes/harmony and timbre/tone color are one thing, *oscillation*. They merely appear to be different when viewed in the differing contexts of cycles per second or cycles per minute – the human perception of audible sound and temporal duration. We apprehend music as notes, chords, tone colors, rhythm, meter, etc.; this is the mind's interpretation, our way of sifting and sorting the complex patterns that reach our ears... but *the music itself*, the ideal system embedded within the empirical sound, is not solely these *human perceptions*, it is the sum of all the objective and implied *relationships*. Ultimately, I would like to suggest that tonal music has only one domain, the one realm of integer relationships of nature's gift, the oscillations of the harmonic series.

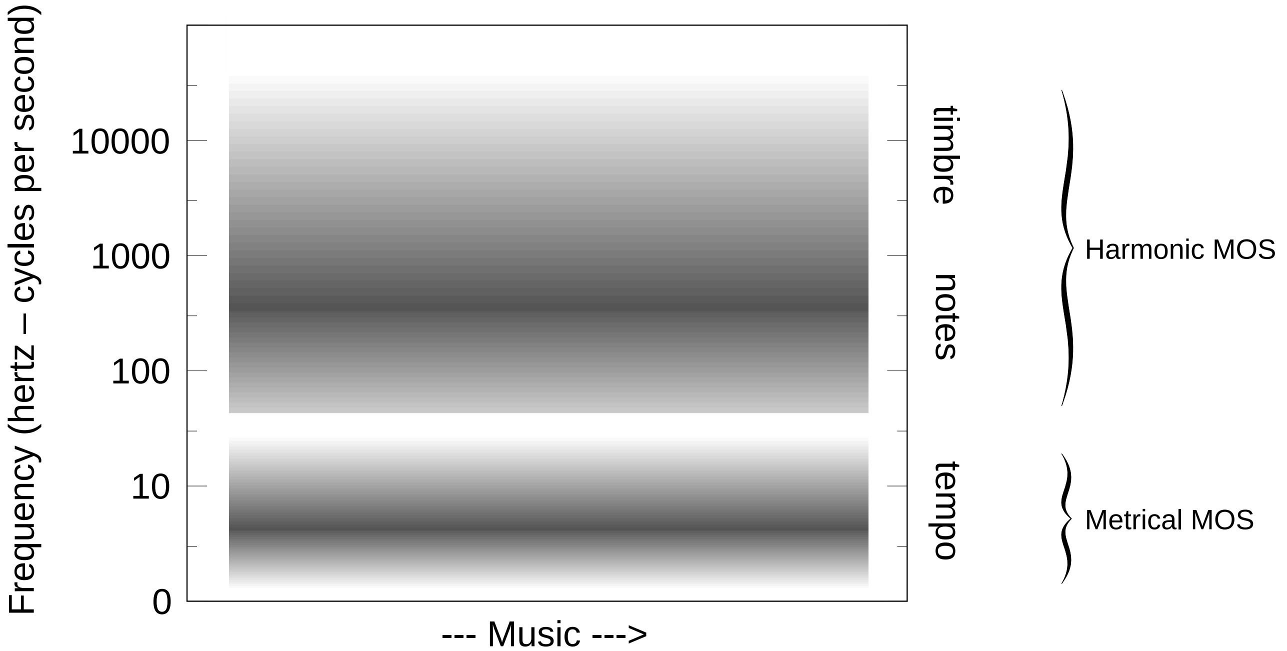

Figure 4.19 Schematic representation of a piece of music as a sequence of one-dimensional relationships (the vertical y-axis) with darker shades for the frequency ranges of most intense activity. The separation of the domains of pitch and timbre from that of tempo/duration is echoed in the use of two separate modulating oscillatory systems for the analysis of the harmonic and metrical aspects of a composition.

Notes:

1. Bronowski, J., *The Ascent of Man,* (Book Club Associates, London, 1977).

2. Gauss, C.F., *The Foundations of Mathematics,* tr. G.W. Dunnington, *The Treasury of Mathematics: 2,* Ed: H. Midonick, (Pelican Books, London, 1968).

3. Penrose, R., *The Road to Reality,* (Jonathan Cape, London, 2004). Chapter 3: Kinds of number in the physical world.

4. Helmholtz, H. von, *On the Sensations of Tone,* tr. A. J. Ellis, (Dover, New York, 1954).

5. Andersen, P-G., *Organ Building and Design*, tr. J. Curnutt, (Allen & Unwin, London, 1976) page 132.

6. In earlier versions of this document the term 'Meter' was employed, rather than 'Metric' – hopefully the term Euler's Metric or simply 'the Metric' will be less ambiguous while also acknowledging its origin. The concept of Euler's metric (formerly termed Meter) was principally used in the early years of developing the MOS model in terms of a pseudo-physical system, that is, before I realised that a formal interpretation as a positional number system was also possible.

7. Beament, J., *How We Hear Music,* (Boydell Press, Woodbridge, UK, 2005). The concept of a Metric as given here, is broadly synonymous with Beament's description of the repetition patterns of chords in Chapter 7, section 7.1, page 76. Both the graphs above and Beament's bar charts (e.g. Fig. 7, page 77) present the essential argument shorn of the added complications of phase differences.

8. Pierce, J.R., *The Science of Musical Sound,* (Freeman & Co., New York, 1992) pages 89-92.

9. Taylor, C., *The Science of Musical Sound*, in *Music and Mathematics*, Eds: J. Fauvel, R. Flood, and R. Wilson, (OUP, 2003). page 58, residue effect.