

Journey to the Heart of Music

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Chapter 2 – The Piano Tuner’s Tale

SCALES AND KEYS

Any journey, however long or short, must begin from somewhere; and the point of departure for this journey came from my work as a piano tuner; so it is amongst the somewhat arcane details of notes and scale and keys, but above all *ratios*, that we set out.

One of the luxuries of this line of work is that it requires, even allows me to demand, a quiet and tranquil environment, free from interruption. Quite a luxury; but one not without dangers, time to think, time for the mind to wander. A thought that strayed into my mind recently was that, as a piano tuner, my work involved endless hours of performing a process not unlike calculus: And indeed, ultimately, what is a piano tuner but a humble mathematician of ratios, the journeyman-geometer of sound? The process of tuning involves listening to the interference or interaction between the tuning note and another reference note and adjusting the tuning note in accordance with changes heard in the combined sound. A visual analogue of these combined sounds could be imagined as somewhat like throwing two pebbles into a placid unruffled pond and watching the ripples from the impacts intermingle. The two radiating patterns on the pond’s surface being something like the mixing of sound waves that a tuner relies on to gauge fine differences between notes, and when notes are in perfectly tune.

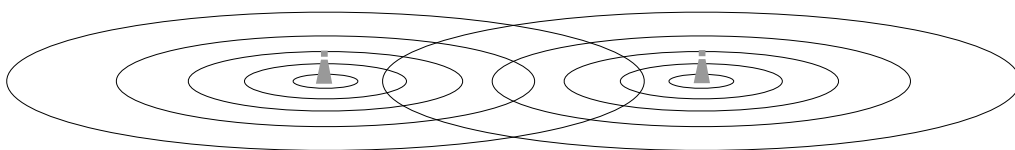


Figure 2.1 The intermingling of waves on the surface of a pond.

GRAPHS

If a cross-section were cut through the pond surface it would produce the same effect as looking through the side of an aquarium.

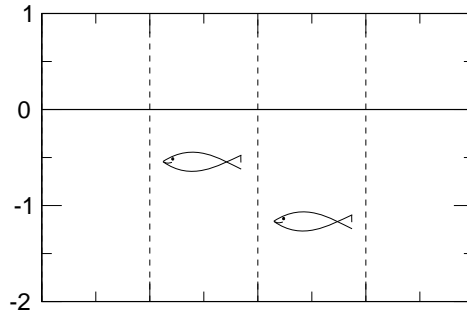


Figure 2.2 A fish-tank or aquarium with a placid unruffled surface at level zero.

If a pebble were dropped into the left-hand end of the aquarium, the ripples would spread out across the formerly smooth surface.

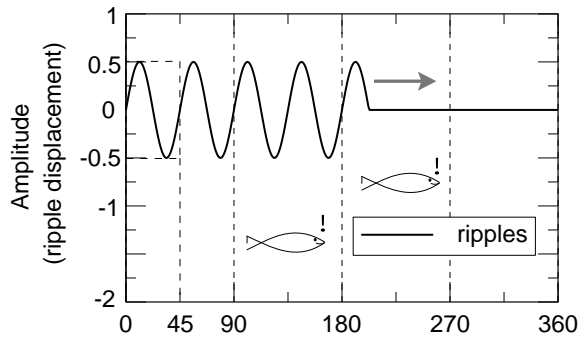


Figure 2.3 The ripples from a single pebble dropped into the aquarium travel across the unruffled surface. Four and one-half ripples shown, with one complete ripple enclosed in a dash-line box on the left-hand side between plus 0.5 and minus 0.5 displacement.

The units on the left-hand side allow us to measure the distance or displacement between the top/peak and bottom/trough of each ripple. The overall amplitude of this ripple is one unit, as it rises to plus one-half at the peak and falls to minus one-half at the trough; and, over the time period from the pebble hitting the water to the front ripple reaching the far side, there will have been eight ripples created in all. Therefore forty-five of the units shown below the aquarium measure the time or frequency of each individual ripple. If the front ripple took one second to span the length of the aquarium (eight ripple lengths) the time or frequency of each ripple would be one-eighth of a second or eight ripples-per-second. The time measurement of ripples, or fluctuation, or cycles per second, are by convention named hertz (abbreviation Hz) after the nineteenth century scientist Heinrich Hertz.

Now if the ripple or wave pattern of the first pebble were marked on the side of the aquarium as a dotted line so that the ripples of a second pebble could be viewed against it, this might reveal a picture somewhat like Figure 2.4: with two ripples of slightly different frequency which begin and finish together but diverge gradually toward the center.

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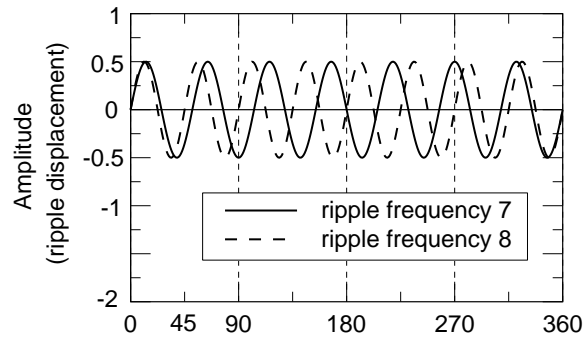


Figure 2.4 Two ripples or waves viewed one against the other.

Next the question is what would happen if the two pebbles were dropped in the tank simultaneously? What effect would the ripples have on each other? One way to approaching this question is to imagine one ripple as the surface over which the other ripples are appended. (For convenience I shall call them wave frequency seven and eight and abbreviate to wave[f=7] and wave[f=8].) So instead of the formally flat surface of the aquarium, now the surface is imagined to be the dotted line trace made by wave[f=8] and on to this undulating surface the additional undulations of wave[f=7] are imposed.

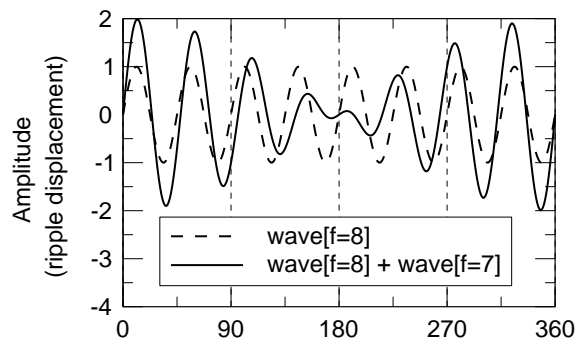


Figure 2.5 The superimposition (sometimes termed superposition) of wave[f=7] onto the 'surface' of wave[f=8].

Comparing Figures 2.4 and 2.5, it can be seen that at the left and right edges the two superimposed ripples or waves augment each other, with peaks and troughs roughly coinciding, so in combination doubling the amplitude or displacement. However, in the middle section the opposite is true, wave[f=8]'s trough coincides with wave[f=7]'s peak and vice versa, resulting in the two waves cancelling each other out. In the middle the amplitude of the combined waves dies away to a fraction of the amplitude found at the left and right hand sides. Thus by causing two slightly different waves to intermingle or *interfere* the result is to produce large (double) waves at the edges which die away to small waves around the middle. The effect doesn't necessarily happen in such a straightforward way with all waves, the right conditions for obtaining this result are that the waves begin and end together over a commensurate period and their frequencies and amplitudes are not too different. Waves that begin and end together, that keep in step with one another over some portion of time are said to be *in phase*, and the complete interval over which a system repeats is often referred to as the *period*. For the most part, out in the real world, sound waves are all jumbled up and out

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of phase: as they stream away from their source(s), bouncing off solid objects at odd angles, perhaps repeatably, before being absorbed by softer surfaces or lost to the open air. However, as the human ear doesn't register phase differences to a significant degree, much of this chaotic intermingling is filtered out, leaving the mechanisms of aural cognition with a relatively clear signal to process.

By convention, matters relating to the interference of waves (and many other phenomena connected with waves and periodic behaviour) are linked to a circle and the period divided up into 360 units or degrees – the angle through which the radius of a circle travels (anti-clockwise) before returning to the starting position. The units running along the base or x-axis of Figures 2.1 to 2.5 are degrees, 360 per period; while the units of amplitude of the wave (the volume or loudness of a sound wave) are written on the y-axis: on the left hand side. There are many more fish-tanks in this book, *graphs* that is; there is no reason to be daunted by them. Essentially they are snapshots of a (usually repeating) pattern of relationship between quantities. Here we have been looking at the repeating pattern of amplitude (displacement from the equilibrium of a smooth surface: zero on the y-axis) of two intermingling waves of slightly different frequencies – wave[f=7] and wave[f=8]. Graphs are a compact, easy to read, way of describing a relationship. Just imagine how difficult it would be to express in words, or write in figures, the precise ever changing contour of the interference pattern of Figure 2.5 – that the eye grasps in an instant.

Graphs are worth making your friends; they have a lot to offer. Another way graphs are used in this book, in addition to mapping out the effects waves have on each other, is for the charting of harmonic series. This is rather more straightforward. The vertical axis (y-axis) counts off the subdivisions from the fundamental tone – h1; while the horizontal axis (x-axis) registers the increasing energy and complexity of the 'system', in unspecified amounts. Again the great advantage of the format is that the eye immediately picks up the relationships: in this case those of one harmonic series contained (nested) within another. This is illustrated in Figure 2.6. where the fundamental series based on C-h1 contains a second harmonic series built on G-h3.

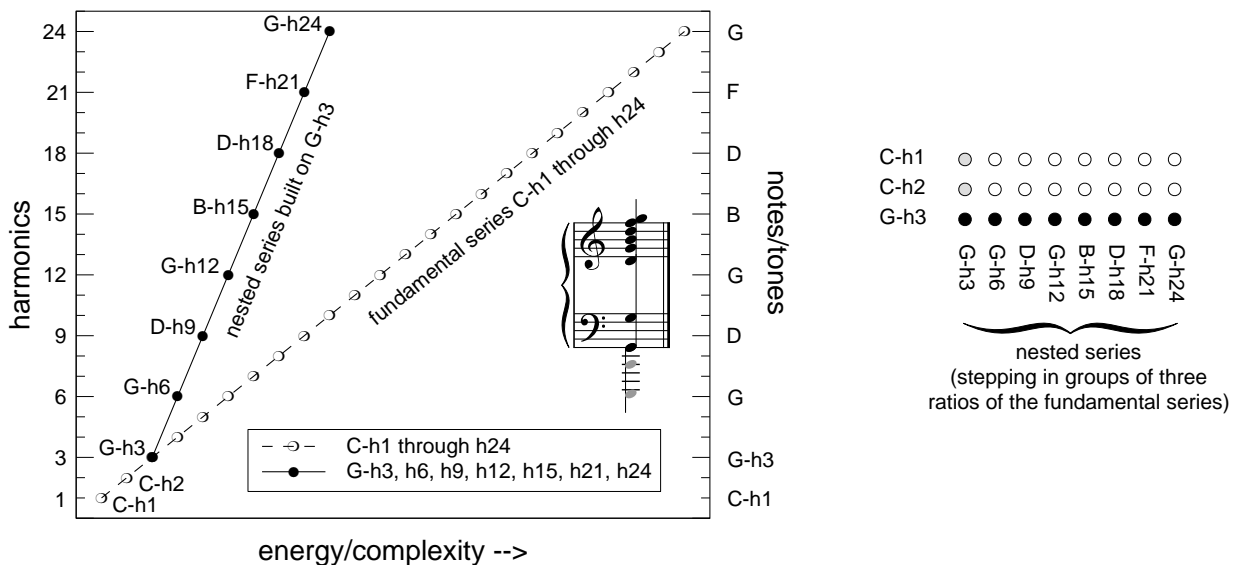


Figure 2.6 On the left, in graph format; two harmonic series, one based on G-h3 (continuous line) contained or nested within the ratios of a broader, more fundamental series built on C-h1 (dotted line); and to the right the same arrangement illustrated in music notation and as a rectilinear number pattern.

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Both series illustrated in Figure 2.6 reach up to the twenty-fourth harmonic, the fundamental series by brute addition of every harmonic. In contrast, the nested series built on G-h3 steps more lightly in groups of three: h6, h9, h12, h15, h18, h21 and h24. Viewed from the standpoint of mutable numbers, the fundamental series represents the prime state digit sequence of mutable number twenty-four; whilst the nested series, plus H1 and H2 (from the fundamental series), represents an intermediate digit sequence of the same number. Written in the factor and subscript formats, they are:

Mutable Base Numbers: Prime state: 1×24 (MBN 24_1) and Intermediate state: 3×8 (MBN $8_3 0_1$)

Returning to the topic of tuning a piano (we never really left it), and the thought that piano tuning could be viewed as a process not unlike that branch of abstract mathematics called calculus. It can be seen in Figure 2.7 how two waves start out together, but because the wavelength of one note is slightly longer, by the time they approach the right-hand side they are totally out of step. The waves begin together, in phase, but end completely out of phase, with peak against trough. If the diagram were extended further to the right they would come back together again just as in Figure 2.5. If these two waves are summed up into a single interference pattern, clearly they will compound each other on the left-hand side and cancel each other out around 180 degrees, the middle of their period, on the right.

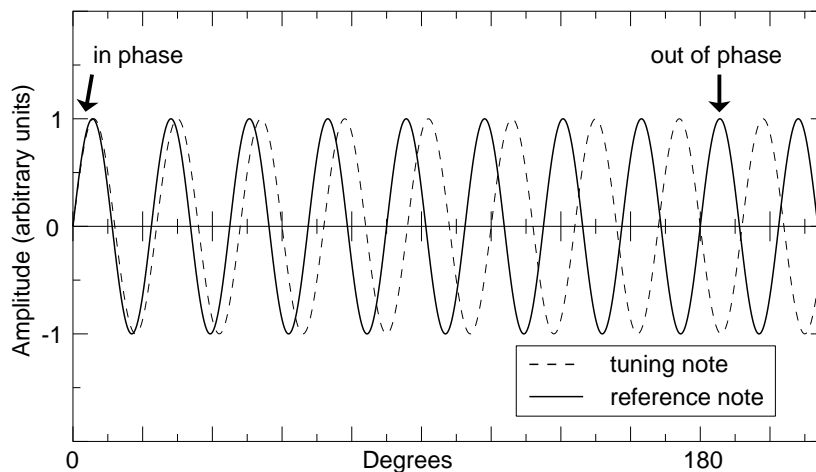


Figure 2.7 Two waves of close but not identical wavelength, a tuning note and reference note.

In the graph Figure 2.8 there are again two waves of nearly the same frequency, a tuning note and reference note. In this graph the waveforms of both notes are shown as dotted lines, and because many cycles of the tuning and reference notes are packed into the overall period of the graph, they appear as a rather fuzzy band between plus one and minus one on the vertical y-axis. When these two notes are summed together, that is the two waveforms intermingle, the resulting interference pattern – the sum of the two waves – is shown by the continuous line. The effect produced by the interference between tuning and reference notes is termed *beats* and the shape of the continuous line vividly illustrates the swelling and dying away of beats when heard. Three times in Figure 2.8 this process of addition and subtraction creates the swelling sound of beats. The rate of beating, the regular swelling and dying away, depends on how quickly one wave catches up with the other – how quickly they come into, and out of, phase. Subtracting the frequency in hertz

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(vibrations per second) of one note from the other, will yield the absolute rate of beats per second heard. For example, reference note A440Hz minus tuning note 437Hz equals three beats per second.

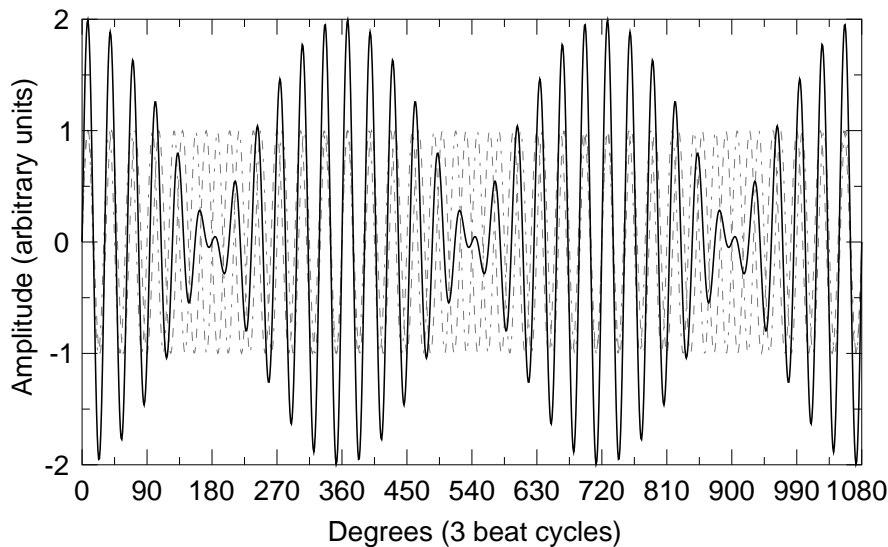


Figure 2.8 Two notes of nearly the same frequency/wavelength (dotted lines) begin in phase, compounding their amplitudes, but then gradually work their way out of phase, cancelling out each other's amplitudes. Three cycles or periods of this process or 'interference pattern' are illustrated.

If the beats are relatively slow, say below four or five per second, they are not unpleasant, adding an undulating character to the sound. This feature is used in some compound organ stops and can be heard in the tone of the string section of an orchestra. Once the rate of beating climbs much above six per second it soon becomes most unpleasant to the ear as the resonant vibration of the two frequencies diverge sufficiently to destructively interfere with each other on the ear's detector membrane. However, going in the opposite direction – slowing the beats down further – so the wavelength (and therefore also the frequency of each complete vibration) of the tuning note approaches that of the reference note, causes the interference pattern, the time between beats, to lengthen. This slowing to the point of vanishing guides the tuner to a perfect match between the two notes. Once the beats disappear, the tuning note has reached the same wavelength and frequency as the reference note. (The trick also works for tuning notes which achieve a close multiple of the reference note, most usefully the octave multiple of two and intervals of a fifth and fourth). Thus in a calculus-like manner, as the tuning note tends to unity with the reference frequency, so the period between beats lengthens, tending towards infinity.

The process of tuning, of listening to beats and adjusting the string tension accordingly, is like pulling a lever which is attached to a 'blackbox' process. The tuner doesn't have to know how or what goes on inside the blackbox, they only have to move the lever according to the signal that emerges from the process, until the beats, the fluctuations in volume, disappear. The necessary and quite complex calculations, concerning string tension, mass, elasticity, which could be done in abstract mathematics, thankfully, have been devolved into the *physical process*. Essentially piano tuners have an *oscillatory computer* to help them in their work... and it comes free with a pair of ears! Similarly, via the MOS model and mutable numbers, the performance of tonal music (harmonic progression) can be understood as the execution of mathematics in sound.

SCALES

An interval of a fifth, such as the notes A below E played together, produce a sonorous and harmonious combination. The Figure 2.9 shows the wave pattern which results when these two notes are sounded together with an equal volume (amplitude) and uniform phase. It can be seen from the graph that this pattern repeats at a frequency equivalent to the note which is exactly one octave below the A – ‘frequency 1’ in the figure. It is the introduction of this lower period which *unites* the objective sounding notes A and E – creating an harmonious union. The lower period is the *one* the unit within which the two elements of the ratio 2:3 can coexist. In essence, the interval has created, or at least implied, a little harmonic series A-h1, A-h2, E-h3. Any two (or more) notes viewed as whole numbered ratios, whatever their frequency, have a shared h1, a common fundamental frequency which acts as a reference point to the relationship they have with each other. For example, the more dissonant and awkward relationship of A#h17 and B-h18 (17:18) equally find a common bond in A-h1; and should a ratio consist of entirely irregular numbers (what mathematicians call irrational numbers) the ear would construe them as a close approximation to something more regular.

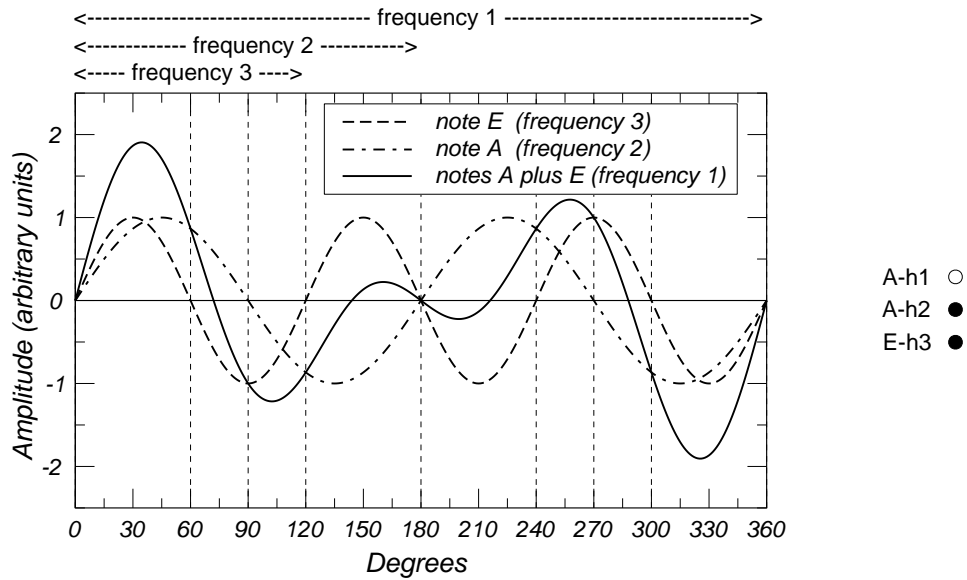


Figure 2.9 The interference pattern formed by notes A and E (frequency ratio 2:3) has a period twice that of A and three times that of E. On the right is a number pattern which encapsulates the harmonic relationships.

The pitch or frequency ratio between the notes of a *truly* perfect fifth is 2:3, or 1:1.5. However, though the interval A to E on the piano is often referred to as a perfect fifth, it is normally a perfect *tempered* fifth, with the ratio 2:2.9966 or 1:1.4983. Pianos are not tuned to perfect harmony! Beats, which are the hallmark of out-of-tuneness, are surprisingly present in all intervals of a correctly tuned piano; though difficult to detect except in the most simple intervals (octaves, fifths and fourths) where the interfering harmonics are close to the fundamental tones. Even the octaves, which are in theory tuned true, are deliberately stretched by tuners to brighten the perceived tone and keep pace with the sharp harmonics generated by the thicker strings of lower notes. The reason why a piano, or many other fixed pitch instruments, are tuned with ever so slightly out of tune intervals (i.e. the tempered fifth A to E, 1:1.4983) is to accommodate playing in all keys,

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harmoniously. Each key of the piano keyboard must produce a range of notes, not just one, and contribute to many varied relationships within different chords. The F-sharp key must also do service as G-flat, not to mention E-double-sharp. The compromise that has come to be generally accepted to allow each keyboard key to stand in for a range of different notes, is *equal temperament*: the equal spacing of all semitones within the octave. The adjustment of note pitches to something other than the true whole number relationships of the harmonic series, that is, the deliberate introduction of out-of-tuneness, is called *tempering*. Historically many different systems of temperament have been used, though today equal temperament prevails.

The perfect octave, should one ever be found, has the ratio 1:2, and to make twelve equal divisions between one and two, we need to find the twelfth root of two – the number which multiplied by itself twelve times will yield two. The number is 1.05946 approximately and this number delivers the sequence of frequency relationships shown in Figure 2.10. Thus, while a perfect octave is 1:2, the ratio of the tempered major-second A to B is 1:1.1225 and that of a tempered major-third A to C#, 1:1.2599 – considerably wider (sharper) than the natural or perfect interval of the harmonic series 1:1.25 (4:5). For the most part, we accept these small imperfections, the accommodating tolerance of the ear smoothing over the irregularities. But this is not how piano tuners set a scale, not by using abstract mathematics to multiply 1×1.05946 .

A	=	2.0000	}	1:2
G#	=	1.8887		
G	=	1.7818		
F#	=	1.6818		
F	=	1.5874		
E	=	1.4983		
D#	=	1.4142		
D	=	1.3348		
C#	=	1.2599		
C	=	1.1892		
B	=	1.1225		
A#	=	1.0595		
A	=	1.0000		

Figure 2.10 The relationships of the equal tempered scale. The true fifth A to E, 1:1.5, is adjusted downward to 1:1.4983 and the true fourth, 1:1.3333..., up to 1:1.3348 (A to D).

To set an equally tempered scale, twelve imperfect fifths, must be squeezed into seven perfect octaves. The slightly squashed fifths are only about one tenth of one percent flatter or smaller than the true fifth (1:1.5). The tuner listens to the beats generated by the second harmonic of the upper note and third harmonic of the lower note in the interval, as a guide. The process is essentially the reverse of tuning two notes to be identical frequencies described above, as the aim here is to start with a truly perfect fifth and introduce a tiny (and controlled) amount of imperfection! Tuners ‘un-tune’ the true fifth as it were, into a tempered fifth.

Starting from the lowest note on a standard piano keyboard the ascending sequence of fifths, A, E, B, F#, C#, G#, D#, A#, F, C, G, D and A – thirteen notes forming twelve fifths set within seven octaves, illustrated by gray keys in Figure 2.11 – yields in open array, the twelve semitone notes of the scale. For reasons of practicality, tuners abbreviate this to a pattern of up a fifth, down a fourth, in the middle range of the keyboard: where the beating partials two to three octaves higher, lie near the ear’s most sensitive range.

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twelfths from the beginning. Interestingly, from this expanded scheme of octaves and twelfths it is apparent that no matter how far the measuring rods are taken (that is the positive non-zero powers 2^n and 3^m) they will never perfectly coincide, as no power of one prime number can equal the power of another. There will always be a comma.

To use Rameau's language, out of the most basic relationships furnished by nature's gift – the whole numbers one, two and three – comes the structure of the twelve note scale. In terms of mutable numbers, the two measuring rods are formed from the ground states of the two large and similar magnitudes (decimal 524288 and 531441), both numbers capable of producing highly efficient mutable digit sequences –i.e. many nested layers of low prime oscillatory patterns.

MBN factor format: $1 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2$ and $1 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3$

We will return to the spiral of fifths/twelfths with the more dynamical approach of the modulating oscillatory system model in Chapter 9, but for the present, see Figure 2.18 and perhaps peek a look at Figure 9.20.

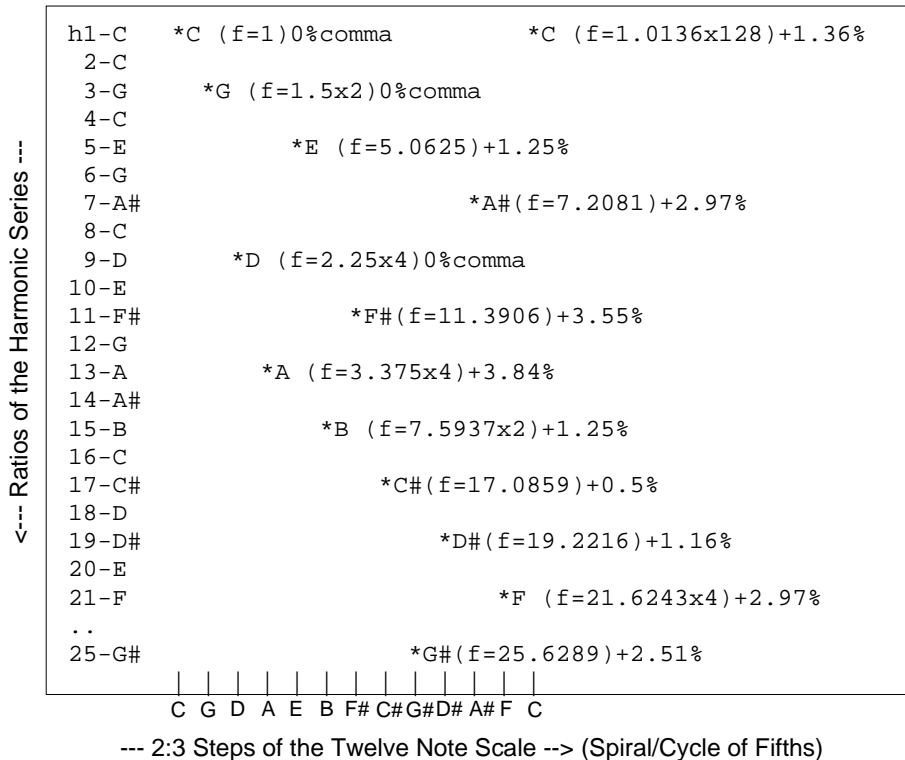


Figure 2.13 Graph of the twelve note natural scale, i.e. derived from exact 2:3 rather than tempered relationships (frequencies reduced or augmented where required), plotted against the fundamental harmonic series of origin.

As well as being derived from the most fundamental relationships capable of extension within the harmonic series, the twelve note scale possesses two other attributes. Firstly, it has a balance between economy and variety: neither too few, nor too many notes to sustain a variety of interesting relationships whilst also maintaining an underlying unity. And secondly, the twelve note scale presents a particularly rich set of low whole number ratio combinations relative to the total number of intervals (as noted by James Jeans¹) which the smoothing effect of equal temperament generalizes across all notes. In its natural state, and

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allowing small commas/margins-of-error, the twelve notes of the scale stray no farther from their fundamental tone (h1) than the twenty-fifth ratio of the harmonic series – illustrated in Figure 2.13.

What the configuration of twelve tonal centers of traditional music offers is a particularly full set of (exact or close) inter-relationships between the fundamental harmonic series of origin and the frequency positions of the cycle/spiral of 2:3 steps. The economical scale of twelve notes contains a wealth of intervals with low integer ratios: fifth 2:3, fourth 3:4, major-third 4:5, minor-third 5:6, etc. And perhaps significantly, the first three steps of C major, G major and D major, the tonal centers of the II-V-I harmonic relationship, are perfectly in tune with the fundamental harmonic series of origin, setting the scale on sure foundations, combined with a firm sense of direction.

Overtone Series of True Harmonic Partial	^	A-h27	--	26.908Hz	-A	Twelve Equal-tempered Notes
	A-h26					
	G#h25	--	25.398Hz	-G#		
	G-h24					
	-h23					
	F#h22					
	F-h21	--	21.357Hz	-F		
	E-h20					
	D#h19	--	19.027Hz	-D#		
	D-h18					
	C#h17	--	16.951Hz	-C#		
	C-h16					
B-h15	--	15.102Hz	-B			
A#h14						
A-h13	--	13.454Hz	-A			
G-h12						
F#h11	--	11.314Hz	-F#			
E-h10						
D- h9	--	8.980Hz	-D			
C- h8						
A# h7	--	7.127Hz	-A#			
G- h6						
E- h5	--	5.040Hz	-E			
C- h4						
G- h3	--	2.997Hz	-G			
C- h2						
C- h1	--	1.000Hz	-C			

Figure 2.14 The ‘true’ whole number relationships of the harmonic series compared with the adjusted frequencies of the equally tempered scale. Both columns can be read as ‘h’ numbers or hertz (cycles per second). The frequencies of the twelve notes of the equal tempered scale have been raised to the octave where their equivalent partial first appears in the harmonic series.

As Figures 2.13 and 2.14 make clear, there is some disagreement between the notes of the tempered scale and the true whole number relationships of the harmonic series from which they are ultimately derived. Notwithstanding this fact, it is convenient and useful to label the partials of the harmonic series with the note letters of the scale, and generally, to treat the adjusted relationships of keyboard music making, in principle, as if they were the whole relationships of the harmonic series. It is the graceful tolerance of the ear and processes of aural cognition which allow this extraction of a precise relational ‘meaning’ from the approximations which are inevitably the objective experience. Noticeably, toward the top of Figure 2.14, the divergence between the relationships of the *uniform* tempered scale and the *curve* of the harmonic series begin to break down. There is no note which corresponds to h23. The semitones shrink, from a wide B-h15

to C-h16, down through C#h17 to D-h18 which comes closest to the equal tempered semitone (1.0588 against 1.0594 for the twelfth root of two), to the narrow G-h24 to G#h25 (1.0416), while the double entry for A-h26/h27 represents the beginning of a separation between the keys (tonal centers) of C major and G major. Such variations in intervallic distance are generally of no greater significance than the irregularities of tempered relationships, given the accommodating flexibility of aural cognition.

Other Scales

Apart from the various forms of pentatonic scales² common to many cultures and of great antiquity, which probably first arose through experimentation but were latterly also generated by sequences of fifths or fourths in the Chinese tradition, there are other 'measuring rod' type sequences possible. For example, two and five (1:2 and 1:5) producing a three-note scale in major-thirds: A, C#, F, (A) – with a negative comma of minus 2.4%; or two and seven yielding a four-note scale A, C, D, E, (A) – with a comma of +2.5%. Rameau, the greatest of the eighteenth century theorists, knew of these relationships and incorporated them in his harmonic schemes under the heading double, triple and quintuple geometric progressions. However, such odd scales were never more than theorising and lack sufficient variety.

Through the nineteenth century the equally tempered form of the twelve-note scale came into almost universal application for keyboard and other fixed-pitch instruments. However, many other systems have been suggested or experimented with at different times. A system extending the measuring rods of octave and fifth to fifty-three fifths spread over thirty-one octaves (2^{31} and 1.5^{53} ! with a comma of just 3.57% of a semitone) was proposed by Nicolas Mercator, the mathematician and map-maker, but as we have only ten fingers at our disposal, with fifty-three notes in an octave, it never gained support. Just intonations – or more generally all non-equally tempered tuning schemes where some keys are more in tune than others – were in wide use up to the end of the eighteenth century but, in the long run, the demands of composers for ever broader tonal reference condemned them to history. However, in the context of early music performance, they are probably aesthetically superior: encouraging the ear to rediscover a subtle purity – which later generations traded for the courser utility of equal temperament – and perhaps one day, music technology will provide keyboard instruments with a smoothly varying intonation in all keys. (For comparison the file SCALES.PDF in the EXTRAS directory/folder charts all the equally tempered octave scales from twelve to twenty-four steps. Of these the division of the octave into nineteen equal-tempered steps might support an interesting tonal music with some subtle micro-intervallic effects. The rest make for rather difficult listening.)

Distinctly different from the fine-grained equal-tempered fifty-three note scale or the rather exotic scale of nineteen notes, both of which would support traditional tonal harmony, are the non-twelve-note scales proposed and investigated by John Pierce³ and William Setheares⁴ amongst others. With these scales the positions of the overtones are adjusted to non-whole numbered positions and/or deleted so as to match or 'harmonise' with the particular scale steps chosen (e.g. ten-tone equal temperament). To gain a perspective on these various scales – without going into the detail – one could draw an analogy between the four-square 'Euclidian' space of traditional tonal music and the many possible 'curved' non-Euclidian geometries representing these other possibilities (see also Chapter 13, page 2). The parallel rule of Euclid might similarly be conceptualized as the 'straight lines' of whole numbered harmonic relationships basic to tonal

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music. When this rule of allowing only whole relationships of harmony and tone is relaxed, completely new and strangely different musical possibilities emerge. It is perhaps too soon to know what can or will be made of these unusual scales, but certainly the linking of scale structure to the overtone spectra of the constituent scale notes is a most powerful idea and a concept which may be equally well applied to traditional tonal music with 'straight' whole numbered harmonic spectra. I suspect here lies music's real frontier.

Significantly, the twelve-note scale is generated from intervals which have their origin in the harmonic series and so the scale is not the fundamental object of music. Scales are devised by mankind, the tempered fifth merely a utilitarian measure accepted by our ears as near enough true. Since Rameau's time the ultimate origin of music has been placed in the physical world, lying within the natural modes of vibration of a material object. Prior to this realisation the divisions of a string were taken to be the point of origin; this gave a more abstract numerical slant to music's foundations. Latterly, in the twentieth century, the rise of atonal music has revived this more mathematical approach to some extent, with the equal-tempered twelve-note scale being viewed in terms of the integers (modulo 12) in atonal set theory. However, the essentially mathematical procedures of atonal theory and practice are not intuitively understood by the ear – in the way that tonal procedures are – and therefore must appeal to other than traditional musical sensibilities.

Though equal-tempered scales can look rather like the set of integers modulo 'n'; and in particular the twelve-tone equal-tempered scale like the numbers – C-0 C#1 D-2 D#3 E-4 F-5 F#6 G-7 G#8 A-9 A#10 B-11 – under recursive division by twelve. There is, I believe a difficulty, an issue at least for me: The ear appears determined to mishear such abstract arithmetic in favor of the simple relationships and ratios of the harmonic series. To be wholly relevant musically, the identification of the equally spaced integers with the equally spaced tempered scale would require the human ear and processes of aural cognition to be able to fully appreciate such equality – to be able to hear the beautiful mathematics of atonal set theory. I do not believe that this is the case; indeed contrariwise, the ear does its damndest to hear inequality, turning the equal irrational relationships of the tempered scale into understandable unequal whole ratios when possible, thereby fostering an intuitive apprehension of tonal music's own embedded mathematics – mutable numbers. I fear, perhaps obtusely, the wonderful and extensive mathematics of atonal theory is just that, wonderful and extensive (abstract) mathematics, conducted in musical sound.

TONAL CENTERS (KEYS)

Although the twelve-note scale provides the raw material for music-making, generally, most pieces of music emphasise a limited number of tones, by singling out just seven notes for particular prominence, from the twelve available notes (though the other five notes can and do appear). These seven note groupings are the *diatonic scales*, which are mainly constructed in steps of whole tones (i.e. two semitones), rather than the entirely regular semitone steps of the twelve-note scale. The core difference between the two types of scale lies in their contrasting structure. The totally uniform regularity of the equally tempered twelve-note scale doesn't favour or give prominence to any one note above the others. The twelve-note scale is amorphous, undivided; it is *atonal* in itself. In contrast, the diatonic scales distinctly isolate and emphasise one particular note, the tonic or home tone of the key or tonal center, by means of the seven note scale's unequal structure. The diatonic scales create a form of *relational gravity*, a sense of being drawn toward a tonal center, the tonic note of a key.

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The pattern of spacing between the notes for a major diatonic scale is: (whole)tone, tone, semitone, tone, tone, tone, semitone – C, D, E, F, G, A, B and C. There are also a variety of separate minor diatonic scales recognised in theory; however, in practice music often slips back and forth between the major and minor modes as if the minor is no more than an inflection of a single tonal framework, based on the one tonal ‘center of gravity’; and the exact details of their construction need not trouble us. For the major diatonic scale – the pre-eminent scale of a tonal center or key – it is principally the attraction between the notes separated by the narrow interval of a semitone, that single out the tonic or ‘home’ note above all others, as the central tonal reference point.

The diagram shows the C-major scale on a treble clef staff. The notes are C, D, E, F, G, A, B, C. Brackets above the staff indicate semitone intervals between E-F and B-C. Below the staff, the notes are labeled with Roman numerals I through VII (I). A legend below the staff defines the functions of each degree:

- I - Tonic
- II - Supertonic
- III - Mediant
- IV - Subdominant
- V - Dominant
- VI - Submediant
- VII - Leading Note

Figure 2.15 The seven degrees of the C-major scale.

The leading note B, tends generally to rise to the tonic C and the subdominant F to fall to the mediant E. This force, often referred to as *voice leading*, is most strongly felt in a full cadence (V⁷- I).

G7th resolves to C major

h8-G	→	G-h6
h7-F	↘	E-h5
h6-D	←	C-h4
h5-B	↗	G-h3
h4-G	→	C-h2
h3-D	↘	C-h1
h2-G	↘	
h1-G		

Figure 2.16 The voice leading of notes B and F in a full cadence are marked by black arrows (G-major7th to C-major).

On the keyboard of a piano, playing any seven consecutive white notes (plus upper octave) will produce a diatonic scale or mode, for example, FGABCDEF. This is the Lydian mode, the names deriving from ancient Greece, although it is essentially a product of the medieval epoch. It is one of a wealth of

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different church modes or scales used in that period. However, only two of these medieval scales, the Ionian CDEFGABC and the Aeolian ABCDEFGA survived into the modern era to become our major and generic minor scales. On the keyboard these are the seven successive white notes beginning on C and A. It is an interesting question as to why only these two scales escaped the great ‘extinction of modes’ that occurred at the dawn of the tonal era – circa 1500 to 1600 AD.

Both scales, the Ionian/major and the Aeolian/minor, traverse an octave but the spacing of the notes is sequentially dissimilar, the positioning of the semitone intervals are different in the major and minor scales (and also somewhat variable in the minor); and this gives each a distinctive relational ‘flavor’ – the bright major scale and a sad or intense minor. By using a combination of black and white keys, both these scales can be reproduced on any note; for example, DEF#GABC#D is the major scale (the same spacing of notes as CDEFGABC) but built on D instead of C. A melody using these notes starting with D would be said to be ‘in the key of D major’. Equally, a piece using a minor scale beginning on D would be ‘in the key of D-minor’. However, as noted above, most non-trivial compositions mix elements of both scales and it is usually preferable to think in terms of the *tonal center* of D, or the tonal center of whichever other tone of the twelve note scale lies at the center of a web of (in principle) whole number relationships. Indeed, in sophisticated and extensive compositions there is likely to be a hierarchy of temporary tonal centers (tonicisations) circling around an overarching key-center. For example, temporary movements from an overarching key of D major, to A major (D major’s dominant’s key-center), to B minor (D major’s relative minor, i.e. same key signature), to D minor (D major’s parallel minor) and so forth. [In general, the term *key* is used in these documents as meaning the twelve *tonal centers* of the cycle of fifths rather than twenty-four major and minor modes; however, where specific reference to the major or minor mode is required, then, for example, ‘the key of D major’ is used.]



Figure 2.17 The harmonic relationships of an example melody exhibit the ratios:
 D:D 1:1, D:G 3:4, D:A 2:3, D:F# 4:5, D:D 1:1 to the key note D;
 and sequentially D->G 3:2, G->A 8:9, A->F# 6:5, F#->D 5:4.

The building of a diatonic scale on a particular note, such as C or D or F# or any other, is crucial for our ears and aural cognition to be able to understand and interpret music. The sound (pitch/frequency) of the tonal center, in this discussion D, is held in our memory and used as a reference point, whereby the ratio to the frequencies of the other notes in the melody can be assessed. In the key of D major, a simple melody such as that given in Figure 2.17 consists of a few simple whole number relationships – disregarding the approximations introduced by equal temperament. Most of the time not even a musicians would think in terms of these ratios; rather, unconsciously, that fabulous processor attached to our ears crunches the numbers and delivers the intervallic information – that the music makes sense, that the ratios are commensurable (or not). This happens without conscious effort, just as with spoken language, vast amounts of low-level processing go unnoticed.

The remarkable achievement of European music was to develop this natural scheme of melodic and harmonic reference (the ratios of the scale) into an extended system of twelve interlocking tonal centers. As

we have seen in Figure 2.12, in actual fact the natural set of 2:3 relationships (fifth) doesn't quite close the circle, but overshoots the seven octaves – by 1.36%. Thus while the adjusted or equal tempered fifth (1:1.4983) forms a *cycle of tonal centers or keys*, in its natural state the true fifth produces, a *spiral of 2:3 relationships*. While composers kept to just a few closely related keys, or even better one key, the mismatch between the notes of keys generated by a spiral of exact 2:3 relationships remained tolerable. However, as composers explored more keys and more key relationships within individual compositions, the difficulties for fixed pitch instruments became more acute. After attempts to adapt the keyboard with divided keys failed, e.g. different keyboard keys for G# and A-flat, equal temperament or something approaching it, was the only answer.

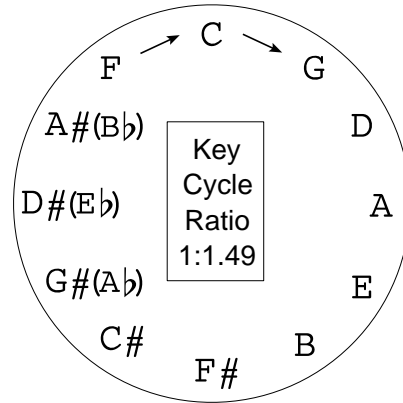


Figure 2.18 The closed cycle of fifths: formed from the tempered ratio 1:1.4983 between twelve adjacent tonal centers or keys.

Within this system of twelve tonal centres, usually referred to as the cycle of fifths, a piece of music could move from one reference *home key* to another, and another, and so on... and, usually, back to the initial home key to finish the piece – a journey through the relationships of the scale(s) set within the broader relationships of tonal centers.



Figure 2.19 The example melody of Figure 2.17 steps from the reference tonal center (key) of D to the next adjacent, downward, tonal center of G. (The relationship between the D and G tonal centers is 3:2 – descending the cycle of fifths – but here 3:4, as the melodic line rises upward to G.)

Nesting Scales in Keys

The music in Figures 2.20a/b illustrates a real example of the nesting of scales (i.e. melodies made from notes of a particular scale) within keys (tonal centers). The example is a short prelude from a trio sonata by Corelli (Op.2/07). There are two pages of music with keys and melodic proportions (of the upper part) marked. All proportions are shown as idealised pure intervals, which is what aural cognition divines from

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musical sound, rather than tempered ones.

The prelude's 'home' key is F major, a scale built on F – bars 1–4. The music then moves to C major – bars 5–8, a scale built on C with a 2:3 relationship to the home key F. At bar 8 the music just 'touches' the key of G major. In bar 9, the piece falls back into C major, where it continues until bar 14. At bar 14 the music moves dramatically forward into the key of D minor, a scale built on D. This is the denouement, marked by the rapid tonal motion from C to D, a relationship of 4:9 with the added intensity of also moving to an *alternative* D (minor) scale. Bars 15–17, in D minor, are succeeded by an equally rapid fall back toward the home key of F, dropping through C at the beginning of bar 18. The home tonal center is re-established in bars 19–22 (see Figure 2.21).

Below the topmost staff of Figures 2.20a/b are the proportions of the upper melody line relative to the current key or tonal center. Thus 'F: 2:3, 3:5, 2:3, 3:4, etc.' indicates that the scale is built on F and the relationship between this reference note F, and the note in the melody is 2:3, etc. In these ratios the first figure represents the reference tonal center and the second figure the melody note.

The tonic key or home tonal center is a scale built on F; this is the fundamental reference point. And the sequence of tonal centers or keys is:

F major, C major, G major, C major, D minor, C major and back to F major.

Thus the sequence of key relationships to the tonic key (home scale) could be written:

F: 1:1, 2:3, 4:9, 2:3, 8:27, 2:3, 1:1.

This might be called the '*key melody*' – the route mapped out by the succession of the prelude's large-scale units of structure. In this example by Corelli, a rather restricted definition of the extent of a key is being used to help illustrate the crucial role that ratios play in music; under different circumstances a more liberal view of key might well be appropriate.

Within each of these large units of structure are nested small units of structure – each melodic phrase – which exhibit similar proportional relationships, for example, in bars 10 and 11, where we find a pattern not fundamentally different from the 'key melody' above.

C: 1:1, 16:15, 3:2, 2:3, 3:4, 4:5, 8:9, 1:1

This pattern of nesting will usually extend out to the relationship between the home keys in each of the movements of a larger piece of music, a typical pattern being: C major, G major, A minor and back to C major.

C: 1:1, 2:3, 3:5, 1:1

Thus, with a large musical work, it is possible to perceive levels of nesting from the whole score down through movements, sections, sub-sections, composite phrases, single phrases to an individual fragmentary motive.

Trio Sonata Opus 2 No. 7

Preludio

Arcangelo Corelli

Adagio $\text{♩} = 66$

I
II
K

F: 2:3 3:5 2:3 3:4 4:5 3:5 8:9 (C: 2:3) 8:9 (C: 2:3)

Key: F

5 6 7 8

C: 2:3 3:5 2:3 3:4 4:5 3:5 8:9 (G: 2:3) 8:9 (G: 2:3)

2:3 Key: C 2:3 Key: G

9 10 11

C: 4:5 8:9, 1:1, 16:15, 3:2, 2:3 3:4 4:5 8:9 1:1

3:2 Key: C

Figure 2.20a Prelude Opus 2 No.7 by Arcangelo Corelli

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12 13 14

3:5,15:8,1:2... 1:1 3:4 4:5 3:5 2:3 (D: 3:4)

Key: C

15 16 17 18

Dmin: 5:6 8:9 1:1 9:8 17:15 4:3 2:3,3:4,5:6 F: 2:3,3:4,3:10
(C: 2:3,3:4,4:5,3:2,1:2)

4:9 Key: D 3:2 Key: C, 3:2 Key: F

19 20 21 22

F: 8:9 1:1,6:5,16:15, 1:1... 16:15,1:1 3:2 6:5 16:15,1:1 1:1,16:15,1:1

Key: F

Figure 2.20b Prelude Opus 2 No.7 by Arcangelo Corelli

The graph Figure 2.21 charts the course taken from one tonal center to another, and eventually back to the first, described above. It is typical of the patterns of tonal (center) structure found in music. Significantly once the home key has been established most of the action and development happens in tonal centres at some distance away. This maintains a tonal tension throughout the arching span of the piece, holding the whole structure taut and together, while providing for a satisfying sense of return, of arriving unmistakably at a meaningful destination.

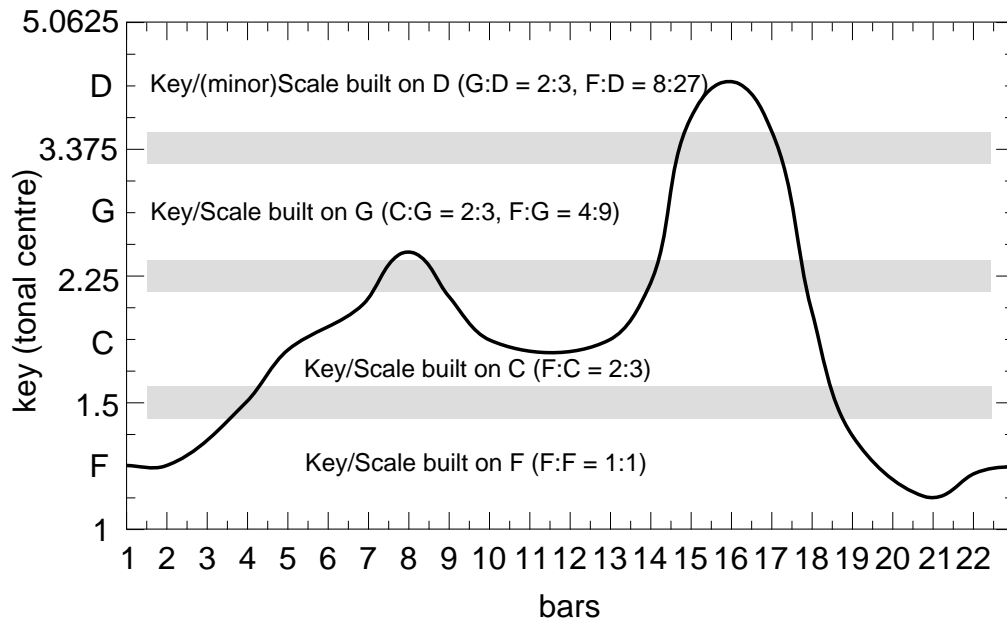


Figure 2.21 Tonal center (key) trace of the prelude by Corelli (Op2/7) with key relationships marked as ratios (e.g. F:C = 2:3) and on the y-axis as a cycle of fifths – 1 : 1.5 : 2.25 : 3.37 : 5.06.

Thus the twelve-note division of the octave provides twelve possible reference frequencies for our ears and aural cognition to keep track of where we are, in regard to both the current scale being used by the melody at any point in the music, and how this current scale (key) relates to a fundamental home scale (the key where the melody began). You will no doubt have noticed that this is an example of a nested structure – twelve notes nested within twelve keys.

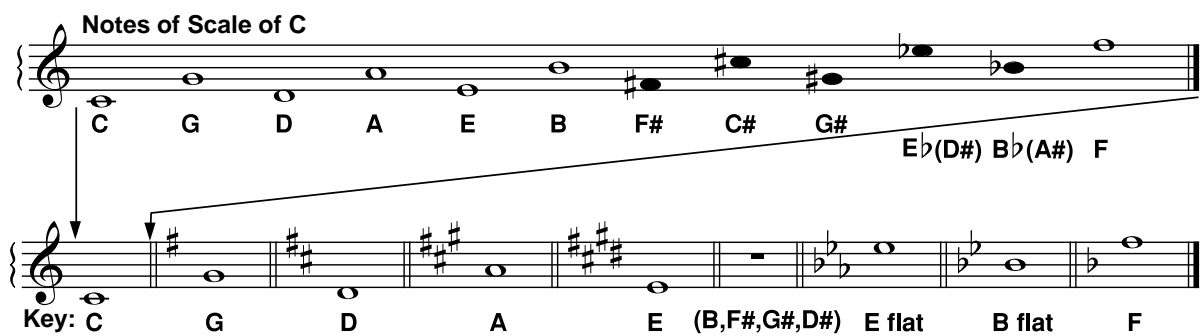


Figure 2.22 Twelve notes nested within each of the twelve tonal centres (keys)

THE LANGUAGE OF MUSIC

Music has an almost universal appeal; there are few people indeed to whom music communicates little or nothing. And if there is communication there must be a code, a language commonly understood. It is reported by Josef Haydn's friend and biographer A.C. Dies⁵, that at their last meeting in Vienna, on the eve of Haydn's departure for London, Mozart remonstrated with his esteemed friend, that England was a faraway and dangerous place, and besides:

"Papa you can't speak English."

To which Haydn made his famous riposte:

"My language is understood throughout the world."

Indeed, though they speak to us with different accents, we all understand Haydn's language, Bach's language and Beethoven's language, Josquin's and Janacek's: it is also the language of Louis Armstrong, Cole Porter and the Beatles. We all understand the *language* of tonality – the common tongue of traditional classical and popular music – sometimes called the music of common practice. The language of music, at its most fundamental structural (harmonic) level, in performance, is, more or less, a scheme of relative magnitudes – a physical number system. It is not a representation of relative magnitudes by material tokens, that is *abstract* mathematics, a symbolic formal system; but rather tonally organised music appears to mimic the *real thing* – physical mathematics in the raw state. In principle, a tonal composition could be viewed as a dynamic system in the physical world. Thus the language of music is everyone's native tongue because it is the language of everyone's native land: we are all inhabitants of the great, wide, material world.

Almost certainly, there are both elements of nature and nurture involved in the acquisition of this common musical tongue, just as the young human mind has attributes which favor the learning of spoken languages, which only require a suitable external environment for language skills to quickly develop. Similarly, and at an earlier stage, musical cognition is apparent. Indeed, many babies will have learnt to recognise the theme tunes of popular television before birth; and later, they will be taught their first words by sing-song repetition. In all musical traditions, these learning faculties no doubt unfold under the influence of the particular culture, but probably along broadly similar lines. The basic elements of rhythm and pitch are pretty universal. However, the various musical cultures of the world have followed different paths, more or less, in regard to the development of particular characteristics. One might almost call them 'specialisms', in the skills of their members: the rhythmic complexities of many African traditions, micro-tonal inflections in Indian and Arabic music or the refining minimalism of Buddhist influence in Japanese music, for example.

Under the influence of western tonal music, it is the computational aspect of harmonic progression that has been intensively developed and taken to atypical lengths in European musical culture. In so doing, the western tradition came upon an awesome discovery: *a number system*. This might well have occurred in any musical culture; it was just that the western culture happened to explore this particular aspect through the development of polyphony. (One wonders, has it occurred before in another culture and then fizzled out, in a similar way, for example, to the disappearance of the central American Mayan positional number system?) Although the musicians of the European tradition didn't understand exactly what they were uncovering, they could feel its visceral power, and once the flame of harmonic computation had ignited in their minds, there was to be no turning back. Indeed, I suspect, once the system of harmonic computation becomes embedded in a culture, it becomes almost impossible to turn back – that is without breaking the flow of the tradition

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through the generations. Sadly I suspect, the members of a music culture which have learnt the art of tonal computation will have difficulty again hearing the music of their past without placing it within the framework of this powerful system. There is a parallel here with the growth and spread of the decimal number system, though many other cultures developed their own counting schemes, e.g. Roman numerals; and some their own positional systems, e.g. Chinese stick notation, all have fallen before the power of the now near universal Indo-arabic (decimal) number system. Interestingly, a similar 'takeover' in reverse appears to be occurring in India, where, care of Bollywood and general exposure to western influence, indigenous melodies, tonescapes and forms are being fitted onto an underlying structure of western harmony. And in a way reminiscent of the spread of the subcontinent's great invention – Hindu positional number notation – the West's *tonal positional number system* is infiltrating India's native musical culture. A similar intrusion upon indigenous music cultures is also evident in East Asia. In Japan, South Korea and now China the influence of western music both classical and popular is finding a growing acceptance: indeed the piano and its tonal literature has acquired a respected position in the education of the young in these countries.

The fully fledged tonal system of twelve scales nested in twelve keys (tonal centers) was not invented overnight, but grew up rather slowly in Europe as each generation of composers explored the boundaries... until, moving forward through ever more keys, eventually brought the system back to where it had started from, approximately. It was to take about six hundred years of the 'random walk' of artistic creativity, from the beginnings of music in parts – *polyphony* – in the cathedrals and monasteries of the late medieval period, to the writing of *The Well-tempered Clavier* in 1722 – Bach's definitive essay on the key cycle. This remarkable artistic device, the cycle of tonal center relationships, built on the foundations of *nature's gift* – the harmonic series – was a collective, evolutionary achievement, the work of many, over an extended period of time. It was, and still is, the keystone topping out the edifice of tonal computation – the system of mutable numbers.

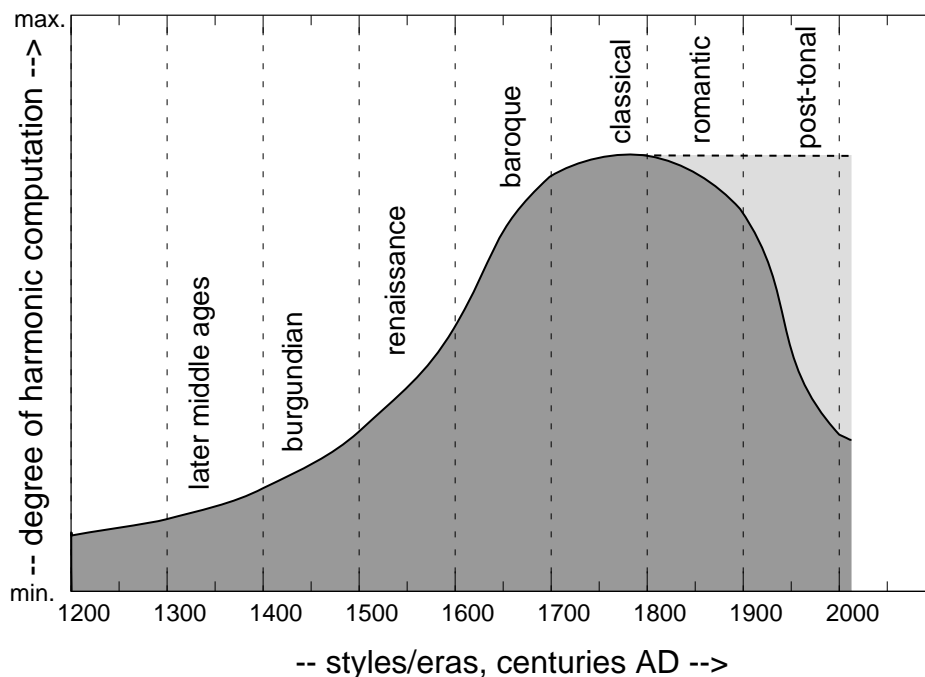


Figure 2.23 A schematic representation of the growth of harmonic computation: the discovery and development of an underlying number system in western tonal music.

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Tonality appears to be more than a transitory style, though many composers of the twentieth century attempted to 'move on', they struggled to find a convincing alternative language or system. The lighter gray area, in Figure 2.23, stretching out from the zenith of harmonic computation during the later baroque and classical periods, indicates the continuing hold the underlying number system has maintained over the aural perceptions and predilections of the vast majority in the western musical tradition. Though most twentieth century composers generally diluted or blurred the sharply focused tonal computation of earlier times, through the addition of extraneous elements and the avoidance of previous formulas, few dared to dispense with the language of mutable numbers entirely, and of those that bravely did, none appear to have found a wholly convincing substitute. [07/05/09]

Notes

1. Jeans, J, *Science and Music*, (Cambridge University Press 1937/Dover: New York, 1968) p189.
2. Pentatonic scales are spaced like the five black piano keys, beginning on each of these five black keys results in five different arrangements of intervals. See James Beament, *How We Hear Music*, (The Boydell Press, Woodbridge, Suffolk, UK, 2005) from page 18, for an interesting discussion of their nature and origin.
3. Pierce, JR, *The Science of Musical Sound*, (Freeman W.H., New York, 1992).
4. Sethares, WA, *Tuning, Timbre, Spectrum, Scale*, (Springer, London, 2005).
5. Dies, AC, *Biographical Accounts of Joseph Haydn*, (Vienna, 1810). Eng. trans: Gotwals, V, *Haydn: Two Contemporary Portraits*, (Uni. of Wisconsin Press, Milwaukee, 1963).