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The Heart of the Matter

"In the snow, flat-topped hillocks and shoulders, outlined with wavy edges, ridge below ridge, very like the grain of wood in line and in projection like relief maps. These the wind makes I think and of course drifts, which are in fact snow waves. The sharp nape of a drift is sometimes broken by slant flutes or channels. I think this must be when the wind after shaping the drift first, has changed and cast waves in the body of the wave itself. All the world is full of inscape, and chance left free to act falls into an order as well as purpose".

Gerard Manley Hopkins – lines describing a snowy landscape, from his Journal: 24th Feb. 1873.

HARMONY AND MUTABLE NUMBERS

I suspect that the majority of mathematicians would hold to the view, more or less strongly, that 'mathematical truth' has some form of existence which transcends the material world: an appealing idea of ancient lineage, reaching back at least to the Greek philosopher Plato. Another view, perhaps less ancient and less well supported, is that mathematics finds its existence wholly within the material world; indeed, that it is a product of the way the universe is. In this book the latter more structuralist and somewhat Aristotelian approach is pursued, through viewing and modelling some aspects of music *as mathematics*, expressed in the form of a pseudo-physical oscillatory system. Or put another way, a likeness is drawn between tonal music in performance and the evolution of self-organising periodic physical systems, both of which, are interpreted as mathematical computation. To this end a somewhat arbitrary distinction is drawn between 'abstract' symbolic mathematics, that is the normal traditional math and, 'physical' mathematics – the behaviour of physical (or pseudo-physical) oscillatory structures[1]. Whether any such self-organising physical systems actually exist in the material world, truly following tonal organisational principles, I do not know and leave that question to others better able to judge[2]. However, placing that question aside, western tonal music does appear to fit the proposed model when viewed as a system standing on the threshold of self-organisation – i.e. a pseudo-physical oscillatory system. Musicians and composers give material life to their symbolic scores – they *play* them – and through the discipline of 'performance validation', composers, musicians and audiences choose between acoustic structures and relationships which are aurally meaningful and those which don't succeed as well. Over time the cumulative effect of these choices appear to have guided western music toward the discovery of a powerful underlying structural device – *a system of number processing by harmonic progression and metrical change*.

The train of thought being pursued here, is that underlying the organisational imperatives of tonal music there lies a physical form of computation, not oscillation described by 'abstract' mathematics, but physical processes which actually *are mathematics*. Such physical systems are here termed *Modulating Oscillatory Systems* (abbr. MOS). Though in tonal music these processes are only partially realised and incomplete, ultimately, the harmonic and metrical core of tonal compositions (when viewed as pseudo-physical oscillatory systems) can be reduced down to the outlines of a position-value counting scheme: A number system which is here termed *Mutable Base Numbers* (abbr. MBN).

Overall, this mathematical view of tonally organised western music rests on the more general, and it must be admitted extremely speculative, conjecture that self-organising periodic systems in the material world, operating under the influence of the Second Law of Thermodynamics, form a broad set of phenomena that create and develop structures which are in essence physical examples of the *fundamental theorem of arithmetic* – i.e. that every integer is constructed from a unique set of prime numbers or *prime factors*. For example, the number thirty can be expressed as the product of three relatively small numbers: 2×3×5. Thirty, or whatever number, of positive units (+1+1+1...) added together to form a given magnitude is inefficient compared to multiplying out a few well chosen factors. Which is to say structural relationships that take on a multiplicative (rather than additive) form are inherently more efficient. And, as we shall see, the effect of nesting one harmonic series within another reproduces this superior multiplicative organisational principle in oscillatory form; in particular, in music, producing the acoustic structures we recognise as harmony and meter: An analogue of which, caught Hopkins eye on that cold February day in 1873. Thus by combining a process of *physical factorisation* with the principles of *positional notation*, it is conjectured, a useful model of systems in the material world might possibly be derived, and subsequently, applied to tonally organised music. Essentially, this is simply a restatement in part – with some degree of elaboration – of the connection drawn between numbers, music and physical reality, made by the Pythagoreans more than 2,500 years ago.

In order to easily grasp the concept of mutable numbers and their encapsulation of harmonic motion and change of meter in western music, we must first remind ourselves about the structure and mechanics of counting, how units or ‘digits’[3] are put together to make multi-column positional numbers. Next, review the nature of the harmonic series, for it is to the ratios of the harmonic series that we shall need to look, for the actual digits with which mutable numbers are made. And finally, bringing these two fields together, construct these mysterious, beautiful, mutable base numerals, and apply them to music.

POSITIONAL NOTATION

The aim of this chapter, and indeed the whole book, is to establish the proposition: that, in principle at least, a tonal composition is effectively a number system computation – a representation and manipulation of number relationships in the material world. So what precisely comprises a number system?

Fundamentally, a number system is a scheme employing physical tokens, either real or symbolic, to describe numerical relationships. From the earliest times, the most convenient physical tokens to ‘come to hand’ for counting – keeping track of magnitude relationships – were fingers (and toes) and occasionally the spaces between these digits; also pebbles, beads, seeds, sticks, shells, etc. have been employed[4]. These early, instinctive, counting systems were probably all additive in nature. In such systems ever more fingers, toes, pebbles or beads were required to represent larger numbers, with the more sophisticated systems using symbolic tokens and symbols to represent groups of tokens. For example V, X, C, D and M in Roman Numerals – the ‘V’ probably began as a picture of an open hand of five digits, later stylised, and ten as two hands ‘X’. It was also common for letters of the alphabet to stand double use as numerals as in the Roman, Greek and Hebrew traditions. Interestingly, the early intuitive counting schemes generally began counting natural numbers from one rather than zero – a feature (we shall see below) they share with mutable base counting. However, a number of societies independently discovered a new and powerful way of representing number relationships: position-value number systems (often termed place notation); and, through the mechanics of these systems, some found a pressing need for a zero token to fill empty positions. Of these various forms of positional counting, one has come to dominate mathematics.

We are all familiar with this leading position-value number system, so familiar in fact that we rarely give it a second thought: It is of course our familiar decimal number system, written with the ten units/digits: 1, 2, 3, 4, 5, 6, 7, 8, 9 and 0. Though often referred to as Arabic, the system originally developed in India, and later spread to Islamic cultures through trading contacts with the Subcontinent, before being passed on to the West. Indeed, the derivation of the word *algorithm* testifies to the interrelated roots of today's system – originating from the name of a great Persian mathematician al-Kuwarizmi, working in Baghdad in the ninth century. Al-Kuwarizmi brought together the advances of Arab mathematics and astronomy with the existing body of Indian and Greek learning; the translation of his work on arithmetic into Latin: *Algoritmi de numero Indorum*, in the 12th century, helped to make this knowledge available in Europe[5] – as also did the contacts between European and Islamic culture in Spain.

The salient advantage of positional notation for the representation of numbers lies in its inherent flexibility and economy – in essence it represents a loss-less compression algorithm. The digits in a position-value number system have two meanings, firstly their *intrinsic meaning* of oneness, twoness, threeness, etc. plus a second meaning derived from their *column position*. Thus the threeness of the unit column's three is multiplied by ten, to become 30, when the digit three appears in the tens' column; and by 'ten by ten' in the hundreds' column to become three hundred, and so on. Each column shift produces a multiplication by ten – from 3 to 30 to 300 to 3000. It is the physical positions of the digits, relative to one another, which is the source of a digit's second, extra, meaning. Should the physical relationships between material tokens be lost, usually so is the number. Thus the zero digit became a vital place-holder, clearly identifying empty columns which otherwise might be a source of confusion. In general, additive number systems are not so sensitive to the spatial relationships of their tokens. For example in the ancient Greek additive decimal system, three sequences of nine capital letters, plus three extra symbols, represented 1 through 9, the tens (10 through 90) and the hundreds (100 through 900) which, written in any order, would represent one and the same value. The great advance made by schemes involving positional notation is to combine the additive principle with that of multiplication, producing a system that can still express every integer through the application of the 'step by step' additive principle while also remaining light enough to fly efficiently to relatively large numbers by means of multiplication.

The decimal system, as its name suggests, is based on ten, or in the jargon, is a *base ten number system*. The example numbers above (plus some others) could be written explicitly showing their base tens, thus:

Number	=	1000-column	100-column	10-column	units
3	=				(3 × 1)
30	=			(3 × 10) +	(0 × 1)
300	=		(3 × 10×10) +	(0 × 10) +	(0 × 1)
3000	=	(3 × 10×10×10) +	(0 × 10×10) +	(0 × 10) +	(0 × 1)
3456	=	(3 × 10×10×10) +	(4 × 10×10) +	(5 × 10) +	(6 × 1)
7890	=	(7 × 10×10×10) +	(8 × 10×10) +	(9 × 10) +	(0 × 1)
9999	=	(9 × 10×10×10) +	(9 × 10×10) +	(9 × 10) +	(9 × 1)

Other bases are perfectly feasible, indeed any base of two or more units can be used to construct a positional number system. Of the non-decimal positional systems in use today the binary, octal and hexadecimal systems, with bases two, eight and sixteen respectively, are perhaps the most familiar – from the domain of computing; and not forgetting the remnants of a base sixty number system evident in our measurement of time. (A *base one* positional system is theoretically possible, but it lacks flexibility and is effectively an additive system of the type a convict or castaway might use to record the passage of days.

SYSTEM:	DIGITS
Binary:	0,1
Octal:	0,1,2,3,4,5,6,7
Decimal:	0,1,2,3,4,5,6,7,8,9
Hexadecimal:	0,1,2,3,4,5,6,7,8,9,A,B,C,D,E,F
Mutable:	h1,2,3,4,etc. hn (unlimited)

.... DIGIT SEQUENCES ....

Decimal	Octal	Hex	Binary	Mutable Base	(Factor Format)
0	0	0	0000 0000	0	(no physical zero digit)
1	1	1	0000 0001	1	(h1)
2	2	2	0000 0010	2	(h1,h2)
3	3	3	0000 0011	3	(h1,h2,h3)
4	4	4	0000 0100	2×2	(h1,h2,h4)
5	5	5	0000 0101	5	(h1,h2,h3,h4,h5)
6	6	6	0000 0110	3×2	(h1,h2,h3,h6)
7	7	7	0000 0111	7	(h1 through h7)
8	10	8	0000 1000	2×2×2	(h1,h2,h4,h8)
9	11	9	0000 1001	3×3	(h1,h2,h3,h6,h9)
10	12	A	0000 1010	5×2	(h1,h2,h3,h4,h5,h10)
11	13	B	0000 1011	11	(h1 through h11)
12	14	C	0000 1100	3×2×2	(h1,h2,h3,h6,h12)
13	15	D	0000 1101	13	(h1 through h13)
14	16	E	0000 1110	7×2	(h1,h2,h3,h4,h5,h6,h7,h14)
15	17	F	0000 1111	5×3	(h1,h2,h3,h4,h5,h10,h15)
16	20	10	0001 0000	2×2×2×2	(h1,h2,h4,h8,h16)
17	21	11	0001 0001	17	(h1 through h17)
18	22	12	0001 0010	3×3×2	(h1,h2,h3,h6,h9,h18)
19	23	13	0001 0011	19	(h1 through h19)
20	24	14	0001 0100	5×2×2	(h1,h2,h3,h4,h5,h10,h20)
21	25	15	0001 0101	7×3	(h1,h2,h3,h4,h5,h6,h7,h14,h21)
22	26	16	0001 0110	11×2	(h1 through h11,h22)
23	27	17	0001 0111	23	(h1 through h23)
24	30	18	0001 1000	3×2×2×2	(h1,h2,h3,h6,h12,h24)
25	31	19	0001 1001	5×5	(h1 through h5,h10,h15,h20,h25)
26	32	1A	0001 1010	13×2	(h1 through h13,h26)
27	33	1B	0001 1011	3×3×3	(h1,h2,h3,h6,h9,h18,h27)
28	34	1C	0001 1100	7×2×2	(h1 through h7,h14,h28)
29	35	1D	0001 1101	29	(h1 through h29)
30	36	1E	0001 1110	5×3×2	(h1,h2,h3,h4,h5,h10,h15,h30)
31	37	1F	0001 1111	31	(h1 through h31)
32	40	20	0010 0000	2×2×2×2×2	(h1,h2,h4,h8,h16,h32)
etc.	--	--	---- ----		
255	377	FF	1111 1111	17×5×3	(h1-h17,h34,h51,h68,h85,h170,h255)

Figure 1.1 Five position-value number systems. By convention binary is usually written with leading zeros and the letters A through F are used for the additional digits required for a base sixteen system. The columns of mutable number digit sequences in ‘factor format’ contain decimal numbers separated by an ‘x’ (or on occasion ‘~’ or ‘n~’).

Getting to grips with non-decimal systems can be difficult at first, as all our ingrained instincts lead us to treat digit sequences as individual (decimal) numbers. For example, the digit sequence ‘12’ *is* the number twelve ordinarily, however, this is only true for the decimal digit sequence ‘12’. In the base eight, octal positional number system, the digit sequence ‘12’ represents the value ten, and in the hexadecimal system, the value eighteen. *Digit sequences and number are not the same thing*. Digit sequences are structural (material) representations of magnitudes, and vary as to what actual magnitude they represent, depending on the base(s) of the system being employed. Moving between number systems with different bases is endlessly confusing, and for the most part, it is easier to stick to decimal numbers and use a chart of equivalent digit sequences for other bases. A short comparative chart is provided in Figure 1.1. With some relief, it can be seen that the digit sequences of mutable base numbers are much less confusing than thinking of ‘34’ or ‘1C’ as the number twenty-eight!

Amongst the positional number systems invented by various civilisations, the one which developed in India during the sixth century AD had a particular advantage, in that it used separate compact glyphs for each symbolic digit. The Babylonian, Mayan and Chinese systems all economised in this area, using the additive principle to generate some digits from others. For example in the Chinese ‘rod’ notation, one was written |, two was || and four ||||. The advantage of separate compact symbols for a number system lies in the facility it lends to the execution of written calculations.

A position-value number system does not have to have a single fixed base, though most do – for quite self-evident reasons of clarity and ease of use. Historically there have been examples of systems with more than one base. For example the Mayan of central America mixed base eighteen in a predominantly base twenty positional number system, presumably to obtain the approximate year-length figure of 360, from two ‘columns’. While the Sumerian/Babylonian system contains traces of parallel bases: ten, twenty and sixty. The old British imperial measures present many other examples like pence, shillings and pounds (bases 12 and 20) or gallons, pecks, bushels and quarters (bases 2, 4 and 8). Indeed going further, a different but fixed base could be used for each and every column, or even more esoteric, a shifting pattern of bases could be applied to the columns! That is, a positional number system where columns can dynamically change their base from number to number, *or even within one number*. If you are reeling from getting to grips with octal and hex, do not despair. This last, *mutable* base form of positional notation – the notation that is needed to model tonally organized compositions as computations in a positional number system – though a generalization over all systems, rests on a straightforward natural logic derived from the harmonic series which helps to make it intuitively easy to grasp. Figure 1.2 gives a glimpse of how the unit and column digits of a *physical* mutable base number fit together, as nested series, inside one fundamental harmonic series. (Mutable Base Number may be abbreviated MBN.)

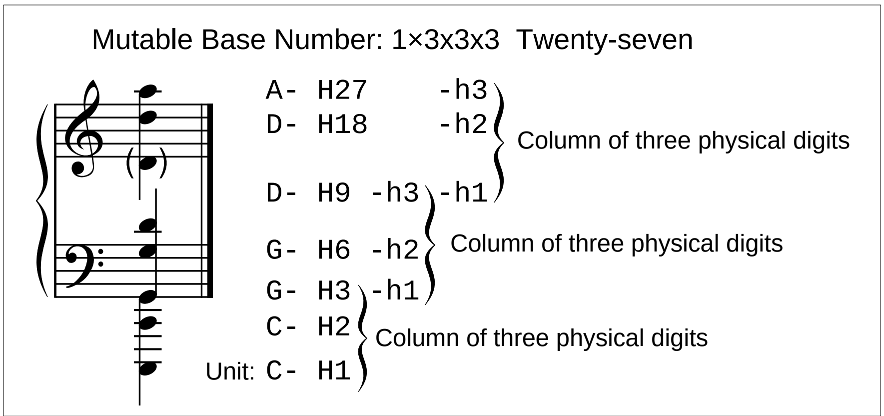

Figure 1.2 The mutable base number twenty-seven. Each column contains three physical digits (i.e. three notes), with the upper columns nested within the ratios of the fundamental series (H1 through H27). On the left is a chord containing the frequency relationships which physically embody the value twenty-seven.

Before going on to describe the mutable base positional number system more fully, it could be helpful to look in detail at the physical tokens, the digits, to be used – i.e. the ratios of the harmonic series.

THE HARMONIC SERIES

For traditional tonal music, the basic physical elements of harmony are derived from the natural modes, or default patterns of vibration, of a physical object. The great eighteenth century theorist and composer Jean-Philippe Rameau (see Chapter 8.12) used the term *corps sonore*, the sounding body, which he described as nature's gift to mankind. And it was a French scientist, Joseph Sauveur, who first described the harmonic series in 1697.

When a physical object is energised by some event, say a piano string struck by a felt-covered hammer, the energy transferred to the string by the hammer blow, causes the object, in this case a piano string, to oscillate. As the string oscillates to accommodate the sudden input of energy, what gradually occurs is a transfer of the energy from the string to the surrounding atmosphere. Thus the energy reaches our ears in the form of pressure waves thrown off by the vibrations of the string – we hear a note. Over time the note fades away, as more and more of the initial input of energy is lost to the air. Eventually, the string comes to rest, or equilibrium; it has no more excess energy to liberate, and the note falls silent. Significantly, after the initial strike of the hammer, the string settles down to sound one note and not a jumble of sounds. However, this one note that we perceive, contains many 'sub-notes' which we rarely separately distinguish, but apprehend as the timbre – the tone quality of the note. These 'sub-notes' are themselves not jumbled up either. The note we perceive (under normal circumstances) is the fundamental oscillation of the string. The sub-notes are a range of whole numbered multiples of this fundamental oscillation. Together they form a sequence of integer harmonics – a harmonic series, which is customarily written: h1, h2, h3, h4, ..., hn.

In this form, the harmonic series is described in terms of *frequency* – a timebased measure. The wavelength of the second harmonic is half that of the fundamental harmonic (Figure 1.3) and as both waves travel at the same speed, two 'h2-waves' will reach the ear in the time one fundamental 'h1-wave' passes. Over the time it takes for the fundamental tone to move through one complete cycle the second harmonic will complete two cycles. The second harmonic, h2, is twice as 'frequent', the third harmonic, h3, three times as 'frequent', the fourth four times, and so on.

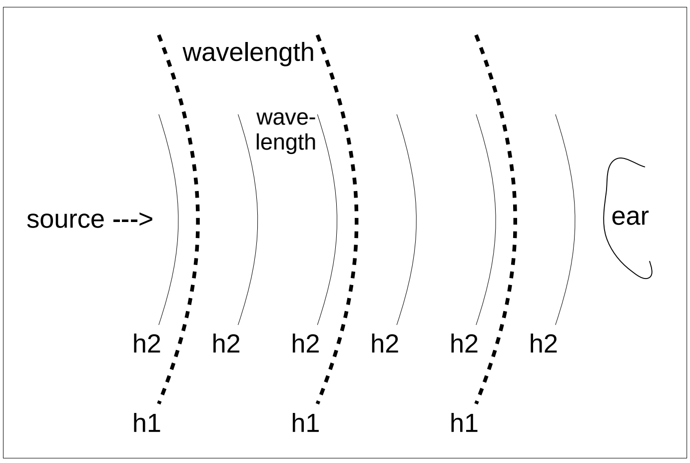

Figure 1.3 A schematic diagram of a sequence of waves, showing the wave peaks of frequencies h1 and h2.

Thus in the strictly mathematical sense, the overtone sequence is a harmonic series of wavelength relationships: one, one-half, one-third, one-quarter, etc., as illustrated in the introduction and Figure 1.3; but somewhat paradoxically, its normal written expression takes the form of an arithmetic series of whole number frequency relationships: one, two, three, etc.

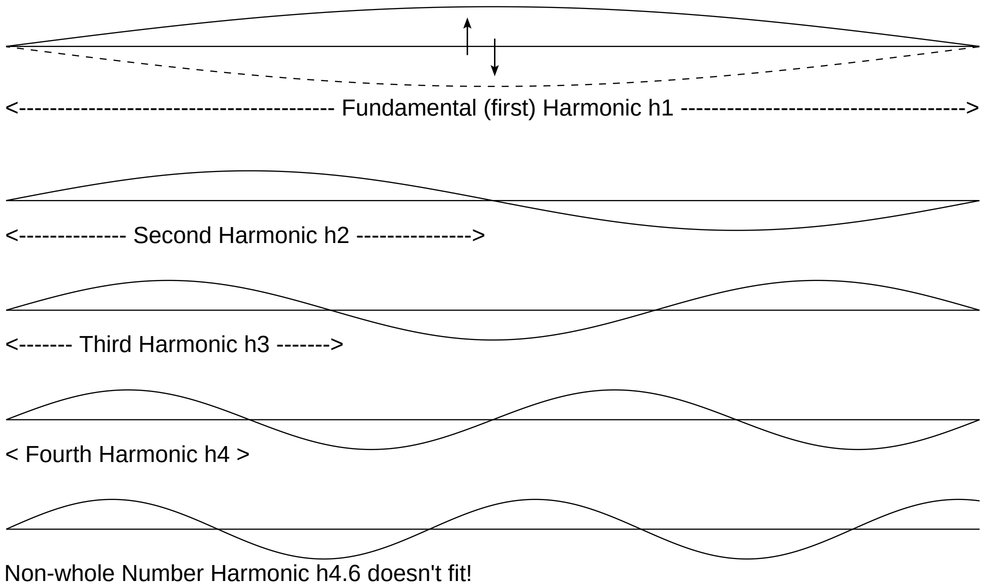

Figure 1.4 The first four natural modes of vibration of a sounding body: h1, h2, h3 and h4, followed by an impossible oscillation: h4.6 !

In Figure 1.4 the straight axis at the center of each vibrating string represents the string at rest – its two ends firmly fixed to the piano’s frame and bridge. The wavy lines represent distinct modes of vibration or oscillation which are possible for the string to perform as it stores and radiates energy to the surrounding environment. (There are some details concerning wave types and other technicalities being glossed over here, as they would needlessly complicate the discussion.) Because the string is fixed at both ends, all the possible modes of oscillation must cleanly start and finish at these two fixed points. The patterns of vibration which meet this criterion are termed *standing waves*; they are the waves which can maintain a permanent existence on the string, so long as energy remains. This restriction limits the string’s permissible modes to complete (whole) vibrations. Non-whole vibrations won’t fit, h4.6 the last example in Figure 1.4 illustrates this; h4.6, and any other non-whole numbers/waves, cannot survive the fixed ends and die away almost as soon as they form. The standing waves that can and do find permanent existence on the string are the oscillations of the harmonic series: h1, h2, h3, h4, ..., hn.

Indeed, it could be argued that in a sense the harmonic series is the *universal chord*, the statement, in ascending order, of every possible natural interval – the whole number ratio or relationship between two tones. Figure 1.5 illustrates the first sixteen ratios or ‘notes’ of the harmonic series. By firmly striking bottom C – h1 on the left of Figure 1.5 – all the other fifteen harmonic partials, and more, will also be sounded to various degrees. In western music, out of the extensive array of intervals aurally discernible, *nature’s gift* to use Rameau’s graceful expression, a few simple ratios from the beginning of the harmonic series form the basis for the harmonies (and rhythms) of tonal music. Chords that stretch beyond the first eight partials of the harmonic series (essentially seventh chords) become increasing difficult for the ear to decipher and interpret. As chords become more complex – that is harmonies involving relationships from extended series stretching out through h9, h10, h11, etc. – the ear gradually loses track of the relationships between the individual notes, particularly in regard to their connection with a fundamental tone: The unit frequency that allows other note frequencies to be ordered and made aurally intelligible.

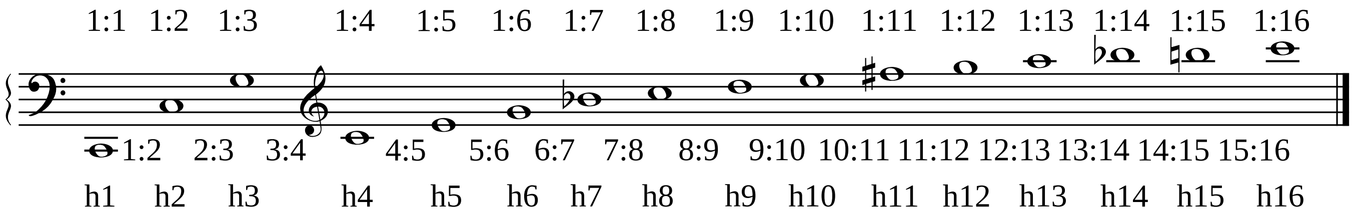

Figure 1.5 Harmonic Series: ratios of the first sixteen frequencies of the series are shown above the stave, with the ratios between adjacent harmonics marked below the stave and bottom the conventional shorthand ‘h1’ for the fundamental, to ‘h16’. (On occasions a capital H may be used to distinguish the lowest or most fundamental of a group of nested series – as in Figure 1.2 above.)

OSCILLATION AND NUMBERS

Now, having reviewed something of the nature of both number systems and the harmonic series, we are in a position to embark in earnest upon the construction of mutable numbers, first considered in terms of dynamical oscillatory structures (*factor format* mutable numbers) and then in the guise of a formal number system (*subscript format* mutable numbers).

In the Modulating Oscillatory Systems (MOS) model of an oscillatory structure, numbers are formed from patterns, and groupings of patterns, of integer related vibrations – the ratios of the harmonic series – and so every number, strictly, represents a division, rather than a multiplication of unity. Yet in essence, numbers are ratios: the relation of a set of magnitudes to a given unity. Numbers in MOS structures, that is chords considered as *configurations of partials* extended outward to include their enfolding harmonic series, partition their fundamental period into ever finer divisions (larger numbers) as they become more complex. Ultimately, at each discrete step or state of its evolution, a modulating oscillatory system (i.e. tonal compositions consisting of coherent sequences of chords) can be represented as a set of relationships embedded within a harmonic series of greater or lesser extent – a mutable number of greater or lesser magnitude. At least in principle, by enlarging the scope of such series, eventually, not only the harmonic progression of the objective notes, but also the higher frequencies of timbre and lower frequencies of temporal duration may be encompassed. Thus overall, the MOS model seeks to reduce the salient elements of a tonally organised composition to one set of integer relationships – one fundamental nesting harmonic series. Taken to its extreme logical application, this most fundamental series could be founded upon the period of the whole piece in performance. However, in practice, fundamental harmonic series founded upon (approximately) the composition’s pulse/beat, and extensive enough to encapsulate both the sense of key and harmonic flow of a composition, are sufficient. Although somewhat premature, Figures 5.18 and 13.8 give examples of this approach.

So, to begin at the beginning, the number one is h1 of a harmonic series consisting of one oscillator, two is h1 *and* h2, not h2 alone – h2 alone would be the number one, or h1, of a system based on double the frequency. To make coherent structures in the material world, each number must contain its own fundamental point of reference, as, unlike formal systems, there is no external agent to make sense of relationships in isolation. The number three is formed from h1, h2 and h3 together. The number four introduces a new element: four can be h1, h2, h3 and h4 of a harmonic series; but the Second Law of Thermodynamics decrees that physical systems will relax where possible to their ground state and so the physical number four is most often found in the form of h1, h2, nesting h4. That is, two entangled harmonic series, one nesting within the ratios of another, more fundamental, series – which in abstract mathematics might be expressed as the prime factors 2×2 (Figure 1.6).

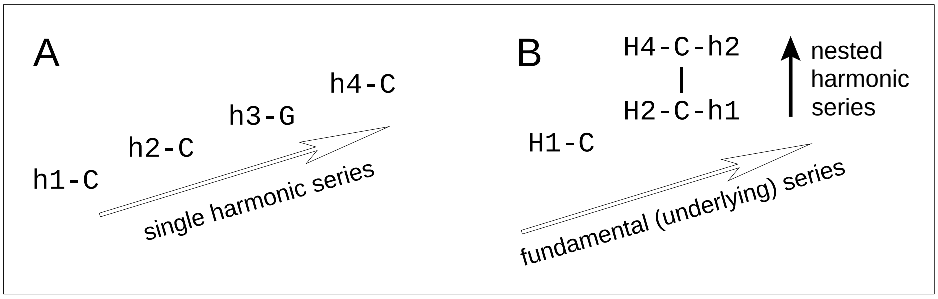

Figure 1.6 The two possible configurations of the physical number four, arrangement A contains one more component than B. Configuration A could be considered the 'physical' number four's prime/excited state while B represents its ground state. (The upper case H distinguishes the fundamental series from the nested series in B.)

In contrast to the physical number four, five like one two and three, cannot be broken down into any nested grouping and so takes the form of a single series: h1, 2, 3, 4, 5. Next, the physical number six, which could take the complex and energetic single series form of h1 through h6, again allows scope for nesting to occur: h1, 2, 3 nesting h6, or 3×2 in abstract math. Notice also that the ground state is 3×2, and not 2×3, which is a slightly more energetic arrangement (h1, 2 nesting h4, 6). Musically these two configurations could be expressed as time signatures/meters: 3/4 for the ground state (3×2) and 6/8 for the slightly more energetic arrangement of 2×3. See Figure 1.13 below. Though not pursued in this chapter, from the standpoint of the MOS model, harmonic and metrical elements of music are viewed as aspects of the single phenomenon – oscillation – and are equally amenable to interpretation as mutable numbers.

Each number, in whatever configuration, is taken to be composed of one or more complete harmonic series, that is, there are no gaps between the fundamental tone and the topmost frequency. For example, 3×3 (h1, 2, 3 nesting h6, 9) is the stable ground state of the physical number nine; but a configuration of h1, 2, 3 nesting h9, with h6 missing, is a unstable transitory arrangement that would collapse to h1, 2, 3, nesting h6 (3×2) the ground state of physical number six. Equally, h1, 2, 3 nesting h3, 6, 9 is untenable, as in a wholly relational system no two different things can occupy entirely the same relationships (i.e. two h3 ratios) – if they did they would be one thing, rather than two different things – h1, 2, 3 and h3, 6, 9 are two separate systems, both representing 3, the physical number three, the first one built on frequency f=1 and the other, f=3.

As the physical numbers of the MOS model's *mutable base number system* are made by patterns (or groups of patterns) of oscillations drawn from the ratios of the harmonic series, with the majority of numbers capable of forming nested structures. Most physical numbers will have a variety of forms, that is different nested groupings, each nominally of a differing energy level – all other factors being equal. One of these groupings will be the number's ground state or normal form. The ground state pattern of a mutable base number is distinguished by having prime oscillatory groups arranged (nested) in ascending order, from the most complex at the lowest frequency, through to the least complex at the highest frequency/energy level.

Thus the Mutable Base Number (MBN) forty-two in factor format:

Ground state - MBN forty-two = 7x3x2 - (h1, 2, 3, 4, 5, 6, 7 nesting h14, 21 nesting h42)
Excited state - MBN forty-two = 7x2x3 - (h1, 2, 3, 4, 5, 6, 7 nesting h14 nesting h28, 42)

The ground state employs h21 whereas the more energetic excited arrangement uses h28. Both configurations yield forty-two fluctuations per period. Such various groupings might be thought of as different hues, of a single, selfsame number. Most, but not all, physical numbers possess a broad palette of shades. Some physical numbers, however, stubbornly resist taking any less energetic form than one complete set of partials and have only a single vibrational configuration. These are the *prime oscillatory patterns* – immutable numbers one might say – for example, h1 through h13 or h1 through h71 – thirteen and seventy-one.

Prime state: MBN thirteen = 13 – h1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13.

The *factor format* introduced so far, essentially represents the dynamical aspect of modulating oscillatory systems; a formal representation, similar to other number systems, will be introduced below.

Prime Oscillations

Perhaps the most striking feature of a physical oscillatory number system, is that *all numbers possess prime vibrational patterns*; it's just very unlikely that the second law of thermodynamics will turn a blind eye for long to a system which could relax to a less energetic configuration. For example, returning again to the number six:

Prime state 6 –> Ground state 3x2
(h1, 2, 3, 4, 5, 6 –> h1, 2, 3 nesting h6)

Piling six oscillators on top of each other is a rather inefficient way of creating six fluctuations per period.

Prime state:	MBN six = h1, 2, 3, 4, 5, 6	or 1+2+3+4+5+6	- least efficient
Intermediate:	MBN six = h1, 2 nesting h4, 6	or 2x3	- quite efficient
Ground state:	MBN six = h1, 2, 3 nesting h6	or 3x2	- most efficient

The most efficient method of generating a physical mutable base number is by multiplication rather than addition. Building up layers of nesting in a mutable base number system represents multiplication, removing them division. This echoes of the fault line between the purely additive and the more efficient positional number systems, the latter building on the additive principle by introducing column shifts, taking the number system to a new dimension – multiplication. However, arbitrary operations of any sort would not be possible in truly self-organising material systems; addition, subtraction, multiplication and division are feasible only if done in accordance with the principles and mechanics of the system. Of course in a pseudo-physical system, like a tonal composition, the composer could do anything – and occasionally does! Music is only a partial example. A truly physical number system is a rather awkward and cumbersome object in comparison to the flexibility found in the abstract symbolic systems used by mathematicians.

The fact that all whole numbers can be expressed in single column additive form, in the mutable base number system, but less than all numbers can also take on the multi-column format that involves multiplication, would appear to suggest that the additive principle is primary or in some way more fundamental. In an ultra high energy environment, all physical mutable number structures would take the additive form, absorbing the maximum amount of energy possible. However, at much lower energy levels, the scarcity would encourage mutable number structures to husband their meagre energy load by distributing it amongst fewer ratios. This, most but not all numbers could achieve through the reconfiguration of their digit sequences, so as to form more efficient nested structures – the primes excepted. With the majority of numbers having a range of digit sequences available to them (i.e. 6, 2×3 or 3×2 for MBN six) often there would be scope to find a matching structure in balance with the surrounding environment.

From the perspective of these ‘physical’ (mutable number) digit sequences the relationship between the prime and non-prime integers of normal ‘abstract’ mathematics is somewhat different. As all physical numbers have a prime state digit sequence (a single series form), one might ask: What is wrong with the traditional primes; rather than what is right? What is wrong is that the traditional primes, viewed from the perspective of mutable numbers, cannot form internal nested patterns of one harmonic series fitting inside another. It is the traditional primes’ irregular oscillatory ‘shape’ which makes them awkward and irreducible and thereby unlikely to survive in a world ruled by the Second Law – except in low denominations, twos and threes in particular. Larger prime number systems could escape to lower energy levels by adding or losing a ratio and then go on to form a nested structure. It is interesting in this regard to note that highly compound numbers generally lie adjacent to the primes. This aspect – the entropy of mutable numbers – is further investigated in Chapter 3.

NUMBER PATTERNS

To digress a little from the main path, in this section a useful visual representation of nested structure, in the form of number patterns, is introduced. These patterns appear in illustrations throughout the book.

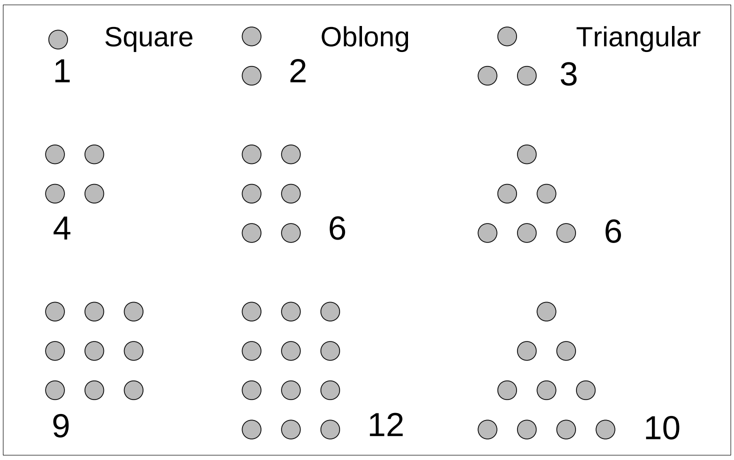

Figure 1.7 The first few square, oblong and triangular numbers.

Ancient forms of abstract mathematics would probably have involved the use of physical tokens to represent quantities and magnitudes, with pebbles, seeds, sticks, etc. among the most likely counters, beyond the ten finger-digits. In such presumably purely additive number systems, the arrangement of tokens can naturally fall into patterns. The early Greek mathematicians noticed and investigated many of these number patterns, in particular classifying some arrangements of counters as forming square, oblong and triangular numbers. Later mathematicians extended the idea of number shapes to pentagonal and hexagonal numbers, and eventually to the generalised concept of *polygonal numbers*. However, it is the square and oblong numbers which are of particular interest, in that they graphically encapsulate an aspect of nested oscillation; and to fully exploit their illustrative potential, the more general term *rectilinear* is useful to indicate number patterns forming fully filled out rectangular shapes in any number of dimensions. (Interestingly, the square and oblong patterns mimic the multiplication principle of positional notation and perhaps hint at its origin.)

Leaving aside the triangular and other polygonal arrangements, the rectilinear number format could be thought of in terms of coordinate dimensions, three of which can be represented on the page as the 'xyz' axes: vertical, horizontal and angled. And such a Cartesian approach illustrates a basic link between geometry and number patterns. In this classification all the prime numbers form one-dimensional rectilinear numbers (vertical), there are no multi-dimensional arrangements of these numbers which result in complete rectilinear configurations. Equally, after two, all even numbers will have at least one two-dimensional configuration consisting of two columns, e.g. 2×3, 2×5, 2×7, etc. Some numbers will possess patterns of filled rectangles in three or more dimensions, beyond the notional 1×1×1, the first of these is the cubic number eight (2×2×2) illustrated in the bottom right hand corner of Figure 1.9. Twelve also has three-dimensional configurations: 3×2×2 or 2×3×2 or 2×2×3, as well as the two-dimensional: 4×3, 3×4, 6×2, 2×6 and its one-dimensional prime state 12.

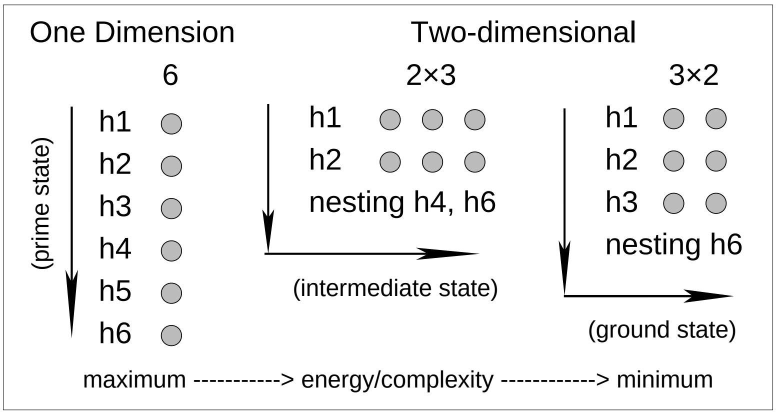

Figure 1.8 The number six has three possible rectilinear configurations: 1×6, 1×2x3 and 1×3x2.

These fully filled *rectilinear number patterns* mimic the possible nested configurations of whole number oscillatory combinations and mutable numbers, thus making helpful visual aids. For example, MBN six (formed from the factors 2 and 3) has two arrangements in two dimensions: 3×2 and 2×3. In oscillatory terms the 3×2 arrangement would be inherently the most efficient and so could be considered the ground state. Below in Figure 1.9, the vertical 'y' axis has been chosen to represent the first or fundamental layer in the system, the horizontal 'x' axis the second layer and the angled 'z' axis another level of nesting. In later chapters the 'xyz dimensions' will be used to represent three levels of nested structure: 1) the fundamental nesting series, 2) nested series and 3) aggregated series, respectively.

While all numbers have a one-dimensional rectilinear form, the prime numbers two, three, five, seven, etc. (seven not illustrated in Figure 1.9) do not have a regular rectilinear configuration in two or more dimensions; for example, three and five illustrated in two dimensions in Figure 1.9. In contrast, six has two two-dimensional arrangements and eight has two two-dimensional arrangements and also one three-dimensional form. All these arrangements are shown in Figure 1.9. Also illustrated on the right of Figure 1.9 is the dominant-seventh chord in a one-dimensional configuration, h1 through h8. Perhaps the ultimate source of the instability and motive energy it embodies might be attributed in some degree to the attraction of the 'energy well' represented by its ultra-efficient three-dimensional ground state.

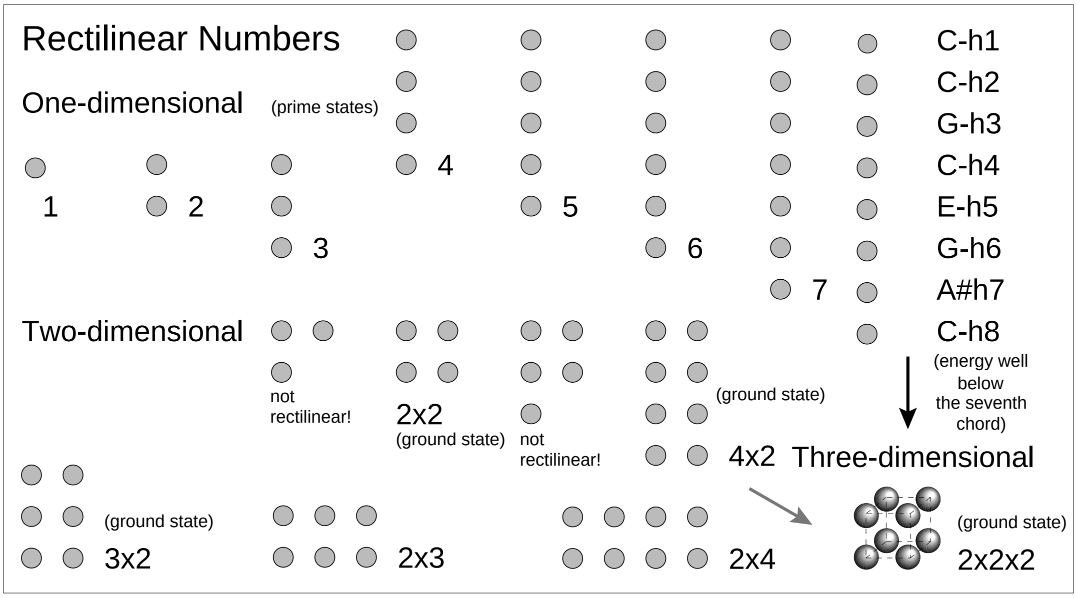

Figure 1.9 The first eight natural numbers expressed in 'rectilinear' format. (Three and five also illustrated in non-rectilinear two-dimensional arrangements.)

MUTABLE BASE NUMBERS

In the mutable base position-value number system used to describe and model tonal music in the form of 'physical numbers' expressed in musical sound, each harmonic series, in a more or less complex chain of nested harmonic series, can hold any number of integer ratio 'note-digits', and each of these harmonic series of variable extent equates to a *column* in the mutable base positional number being expressed.

By definition the initial column in a number system, the units column, is base one – it defines the unit. For mutable numbers the initial column defines the absolute fundamental frequency of the system, and, in the formal *subscript format*, encompasses the unit alone: 'MBN 0_1'. However, in the *factor format* notation – which more nearly mimics the physical systems being described – the unit may on occasion be omitted and taken as understood, thus for Fig.1.2 '1×3×3×3' could be written '3×3×3'.

As each succeeding column represents another level of nesting within a unit/fundamental harmonic series, the extent of any additional column/series contributes a share to the absolute value being expressed, with the cumulative product of earlier column bases being combined through to the last, most significant, column and the digit it contains. As illustrated in Figure 1.2, where the least significant column steps in units (H1, H2, H3), the middle column steps in base three (H3, H6, H9) and the highest column steps in base three-by-three (H9, H18, H27).

So for example, after defining the unit value, whether explicitly or not, if the second column is base five and its digit is six, the number thirty emerges: 1×5×6. However reversing this pattern, if the second column (after defining the unit value) is base six and its digit is five, the same number thirty would still emerge: 1×6×5. *Both these arrangements describe the same number, though by two different digit sequences* – illustrated in Figure 1.10.

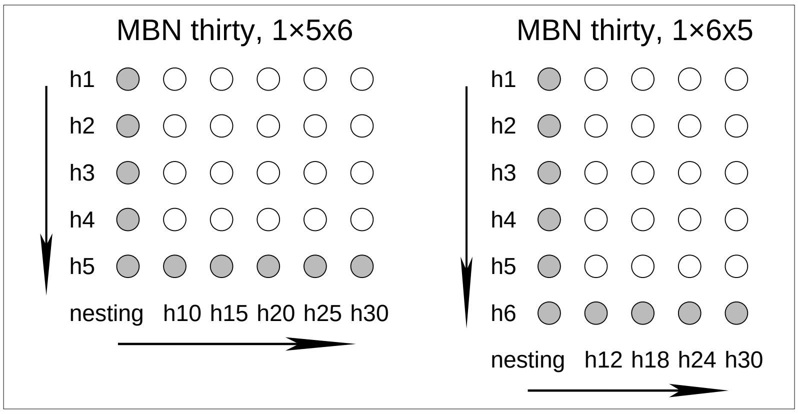

Figure 1.10 Number pattern illustration of MBN thirty as two different nested oscillatory structures or mutable base digit sequences. The fundamental nesting layer is represented vertically and the nested upper layer horizontally.

Thus the bases of mutable numbers (i.e. the nested harmonic series) are in no way fixed; any column/ nested series, after the initial column's unit base, can potentially have any base; and for many numbers, by rearranging the column bases, a different digit sequence can be arrived at, which still represents the same value as the former arrangement. For thirty, in factor format, apart from the digit sequences illustrated above, eleven other sequences are possible:

Groundstate: MBN thirty = 1×5x3x2 (h1, 2, 3, 4, 5 nesting h10, 15 nesting h30)
Intermediate: MBN thirty = 1×5x2x3 (h1, 2, 3, 4, 5 nesting h10 nesting h20, 30)
Intermediate: MBN thirty = 1×3x5x2 (h1, 2, 3 nesting h6, 9, 12, 15 nesting h30)
Intermediate: MBN thirty = 1×3x2x5 (h1, 2, 3 nesting h6, nesting h12, 18, 24, 30)
Intermediate: MBN thirty = 1×2x5x3 (h1, 2 nesting h4, 6, 8, 10 nesting h20, 30)
Intermediate: MBN thirty = 1×2x3x5 (h1, 2 nesting h4, 6 nesting h12, 18, 24, 30)
Intermediate: MBN thirty = 1×10x3 (h1, 2, 3, 4, 5, 6, 7, 8, 9, 10 nesting h20, 30)
Intermediate: MBN thirty = 1×3x10 (h1, 2, 3 nesting h6, 9, 12, 15, 18, 21, 24, 27, 30)
Intermediate: MBN thirty = 1×15x2 (h1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15 nesting h30)
Intermediate: MBN thirty = 1×2x15 (h1, 2 nesting h4, 6, 8, 10, etc. through h30)
Prime state: MBN thirty = 30 (h1, 2, 3, 4, etc. through h30)

In practice, mutable digit sequences amount to arrangements of the factors (not necessarily prime factors) of the value being represented, and this feature of multiple digit sequences is an expression of the commutative nature of factors: that is, the order in which the factors are multiplied doesn't change the end result – *but does change the path by which the end result is achieved*. Unsurprisingly, this form of representing physical mutable numbers is called *factor format*.

The Role of Zero

In the physical world a dynamical mutable base number structure has *no zero token* indicating an empty column and multiply by a predetermined base, indeed, it doesn't have tokens at all, rather it is constructed from real material relationships. In a physical system one might take the absence of structure and relationship to constitute zero but this would only be meaningful from the perspective of an external observer. Thus the digits in a physical number system have no terminating digit (unlike octal: 1, 2, 3, 4, 5, 6, 7, 0) marking the close of one column and an overflow into the next. Mutable digits continue without limit, starting from one: h1, 2, 3, 4, ..., hn. However, in the abstract formal representation of physical mutable numbers offered below – *subscript format* – a use will be found for the zero token.

A physical mutable base number system is essentially a chain of interlinked nested harmonic series. The system must store a positive non-zero number in each link/column – so as to determine the value of any succeeding link(s). The system stores the history of its own evolution in its columns, which it must do, in order to determine and preserve its structure – as its form, its column structure, is not predetermined or externally referenced.

For example, in the decimal numbers 203 and 243, the zero in the former digit sequence transmits, or stands in for, a predetermined ‘multiply by ten’, while the four in the latter, both transmits a ‘multiply by ten’ to the succeeding columns and adds ‘four by ten’ to the sum. In contrast, the (ground state) mutable base digit sequences, in factor format, for these two numbers (decimal 203 and 243) are: 1×29×7 and 1×3×3×3×3×3. Here, all the leftward column(s) simply transmit their ‘base’ multiple (like the zero does in fixed base systems) to the right-hand additive column; and this transmission occurs without columnar addition (as in the mid-column plus 0×10 in 203) . Thus while there are no ‘zeros’ in a physical mutable base number structure – every column/series has a non-zero oscillatory pattern (see Figure 1.2) enchained to its neighbours – the function of zero is present, in that whatever the column values happen to be, is transmitted on to the next column. For example, the five column links of the physical mutable number 1×3×3×3×3×3 transmit their multiplier values to the sixth, most significant column.

Factor Format: MBN 1×3x3x3x3x3 = (1×3x3x3x3) x 3 = Decimal 243

This may seem academic as the factor format just amounts to multiplying out factors, pure and simple. However, what must be remembered is that the factor format represents a physical (or pseudo-physical) system, and when the format is translated into a dynamical structure, as in Figure 1.2 for MBN twenty-seven, this way of organising the factors reflects the actual mechanics of the system. Every mutable base number translates to a physical structure of nested whole number relationships.

Now the zero-like function of passing on a multiplier effect in factor format model of nested harmonic series can be used to construct a representation of mutable numbers which is strikingly similar to other positional systems. Below the number three thousand in decimal and mutable numbers have their factors represented as column base subscripts.

Subscript Format	Decimal: $3_{x10}\,0_{x10}\,0_{x10}\,0_{x1}$	MBN: $2_{x2}\,0_{x2}\,0_{x3}\,0_{x5}\,0_{x5}\,0_{x5}\,0_{x1}$
Factor Format	Factors: 3 x 10 x 10 x 10 x 1	Factors: 2 x 2 x 2 x 3 x 5 x 5 x 5 x 1

Here the column bases are explicitly shown as (decimal) subscripts and the factor format mutable number is turned around to match other positional systems with the most significant digit to the left. Also in this formal *subscript format* a zero digit, transmitting the multiplier effect of the bases, fills all columns except the final most significant one. Although the format is rather cumbersome, it does have the virtue of familiarity and inclusivity – underlining the closeness of mutable numbers to the other positional number systems. Thus three thousand can be written:

$$\text{Mutable: } 2_2\,0_2\,0_3\,0_5\,0_5\,0_5\,0_1 = \text{ Decimal: } 3_{10}\,0_{10}\,0_{10}\,0_1 = \text{ Octal: } 5_8\,6_8\,7_8\,0_1 = \text{ Hexadecimal: } B_{16}\,B_{16}\,8_1$$

$$= \text{ Binary: } 1_2\,0_2\,1_2\,1_2\,1_2\,0_2\,1_2\,1_2\,1_2\,0_2\,0_2\,0_1$$

Perhaps the hexadecimal is the most unfamiliar amongst the fixed base numbers, but from Figure 1.1 we know that Hex B equals Decimal 11. Thus in decimal notation:

$$(11 \times 16 \times 16) + (11 \times 16) + 8 = 3000$$

Writing in the column bases in this way produces in effect a *generalised positional notation* as the above representations of magnitude three thousand, in a variety of positional formats, illustrates. (Though a purist might prefer the subscripts of binary, octal and hexadecimal numbers written in their own notation, this could add to the confusion as they would all be ‘10’ in their own systems.) In this universal positional format the useful information resides in the column subscripts/bases for musically descriptive mutable numbers, while for the fixed base systems it resides in the column digits. The mutable numbers needed for the harmonic analysis of tonal music, have repeated zero digits and interesting bases, while vice versa, fixed base numbers have repeated base subscripts and interesting digits. However, in normal usage, the factor format – 1×5×5×5×3×2×2×2 – for mutable numbers is perhaps to be preferred over the subscript format, both for simplicity and similitude to the physical oscillatory structures it seeks to describe. Here are a few example mutable numbers and equivalents:

Number...	Decimal...	Mutable...	Factor Format	Pseudo-physical System
	<-------	<---------	----------->	
Six	6_1	6_1	1x6	Common major chord
Six	6_1	$3_2\ 0_1$	1x2x3	Bare fifth chord
Eight	8_1	8_1	1x8	Seventh chord (major)
Eight	8_1	$2_2\ 0_2\ 0_1$	1x2x2x2	Bare octaves
Twenty-four	$2_{10}\ 4_1$	$8_3\ 0_1$	1x3x8	Dominant-seventh chord
Twenty-four	$2_{10}\ 4_1$	$6_4\ 0_1$	1x4x6	Tonic chord
Fifteen	$1_{10}\ 5_1$	$3_5\ 0_1$	1x5x3	Minor chord (see Figure 1.17)
Ninety-nine	$9_{10}\ 9_1$	$3_3\ 0_{11}\ 0_1$	1x11x3x3	Ground state
Ninety-nine	$9_{10}\ 9_1$	$11_3\ 0_3\ 0_1$	1x3x3x11	Intermediate state
Ninety-nine	$9_{10}\ 9_1$	$1_{99}\ 0_1$	1x99×1	Prime state (1 group of 99 units)
Ninety-nine	$9_{10}\ 9_1$	99_1	99	Prime state (99 units)

Mutable numbers with fractional values are discussed separately in Appendix B.

The Role of Columns

When viewed as a dynamic physical system, mutable number columns can and do change their value, as their digit sequence evolves. *The simple rule is that the final column is additive* – the most significant column in terms of magnitude – *and all the other columns are multipliers*. This most significant digit is generally constructed from the objective musical sound of the composition, the chords, while any multiplier columns contain information about underlying implied relationships (e.g. as shown in gray notes in Fig. 1.12) The additive, most significant, ‘leading edge’ column could be the only column, as in the last example above (ninety-nine), in which case there are no multiplier columns. The primes are always of a single additive column form – a single harmonic series. It is an interesting feature of the mutable base number system that it distinguishes prime from non-prime numbers by structure and mechanism – at relaxed energy levels. However, most musically interesting numbers will normally have more than one effective column and so contain one additive column and a number of zero-digit multiplier columns. In the subscript format these numbers take on a characteristic $N_z ... 0_c\ 0_b\ 0_a\ 0_1$ form.

Dynamically viewed, a column and the digits it acquires, most often begin life in an additive mode and then switch to a base multiplier mode in maturity – when the column ceases to be the leading edge of the whole structure. One might visualise mutable digit sequences as acquisitive mechanisms, capturing suitable ratios, by addition, in their most significant (outside) column, and then after due processing, storing the ratios away in multiplier columns: rather like patterns of organic growth seen in nature.

It is often convenient to stack mutable number columns (factor format digit sequences) in ascending order, when dealing with music in score format. (A tower format was the convention used in the Mayan system of counting.) This allows digit sequences to be written under the staff and read like a figured bass in a score. The ‘×’ separator can be left out in this format.

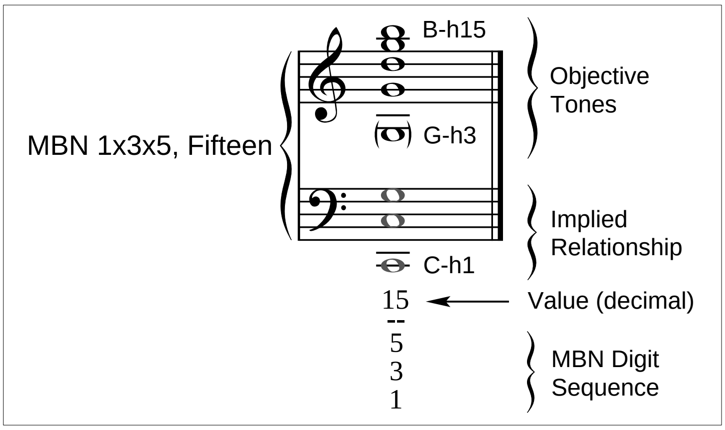

Figure 1.12 The mutable base digit sequence 1x3x5 below its prime state arrangement of 15_1 (which is equivalent to the value in decimal), written like a figured bass below the staff. The gray implied ratios in the bass clef forming an underlying column, would derive from the harmonic context leading up to the chord, e.g. C-major 1x2x6.

MULTIPLE DIGIT SEQUENCES

We have already seen in Figure 1.8 that the mutable base number six has three different distinguishable configurations: h1 through h6; or h1, 2 nesting h4, 6; or h1, 2, 3 nesting h6 – the first arrangement being the prime state, followed by an intermediate state and the latter arrangement being the ground state.

Prime state: 6 –> Intermediate state: 1x2x3 –> Ground state: 1x3x2
Prime state: 6_1 –> Intermediate state: $3_2\,0_1$ –> Ground state: $2_3\,0_1$

And to recap a little on the details of mutable number digit sequences: In the factors format the order of columns runs from left to right (like Intel reverse byte order) the least significant group of digits are to the left – opposite to familiar positional systems. Separating the columns are multiplication signs ‘×’, indicating nesting. Within columns, i.e. between the multiplication signs, it is convenient to use decimal numbers – often multi-column decimals. Thus, as seen above, decimal number 203 has a mutable base sequence of 1×29×7, with ‘29’ standing as a *single mutable digit in a single column*. Written into these and some other illustrations are initial ‘1×’ and trailing ‘×1’. Notionally, all physical mutable base numbers nest within a fundamental period – an initial ‘H1’ column of unit value – though not always explicitly written out or physically instantiated. Effectively, factor format ‘1×5’ and ‘5’ are loosely interchangeable. Also a leading, right-hand, ‘×1’ column entry does not change the value of a mutable base digit sequence – but does alter the status of the preceding digit (decimal number) by converting it into a column base. The right hand ‘×1’ is in a sense the ‘outside edge’ of a physical number, a potential nascent column waiting to be actualised, so establishing a new column. Here are some examples:

MBN 203	= 1×203	= 29x7	= 1x29x7	= 29x7x1	= 1x29x7x1	= decimal 203
MBN 203_1	= $203_1[0_1]$	= $7_{29}0_1$	= $7_{29}0_1$	= $1_7 0_{29} 0_1$	= $1_7 0_{29} 0_1$	= decimal 203

For each column in a mutable base digit structure, the ‘leading edge’ of the preceding column, the last digit in the column, is also the first digit in the next column, and as such, it defines the period of the new column, that is, the column base – illustrated in Figure 1.11.

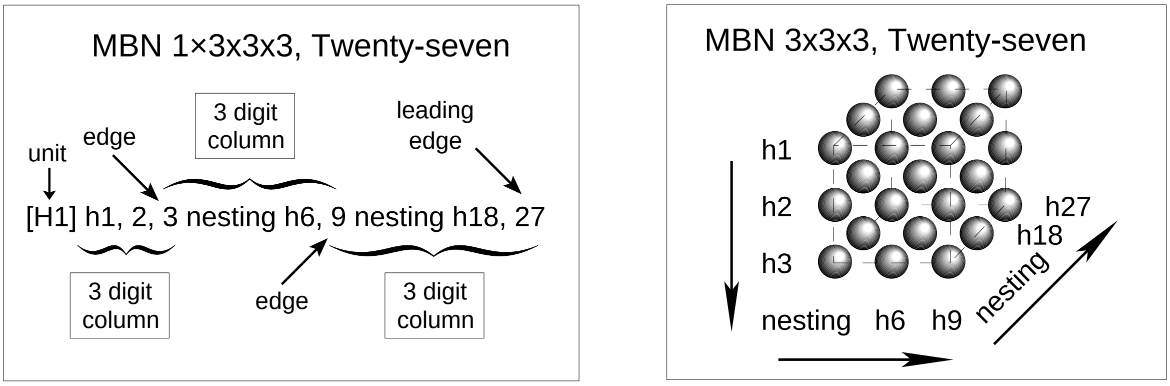

Figure 1.11 The columns of a mutable base number system are like the links in a chain, the columns clasp the first and last digit of their neighbours. (MBN twenty-seven is also featured in Figure 1.2.)

The definitive characteristic of a mutable base positional system is that most values can be accessed by more than one digit sequence. (Only the prime numbers or rather the prime oscillatory patterns, have unique digit sequences.) This feature lends a mutable base number system a particular richness and flexibility, in that a range of different digit sequences representing an unchanging value may be, more or less readily, interchanged. And because mutable numbers are no more than the formal face of physical oscillatory structures, mutable base numbers could be thought of in terms of material entities or systems, with pathways or *conjunctions* between their separate structural forms, formed by these equal overall sums[6]. In western tonal music these conjunctions – the *common value* of multiple digit sequences, expressed as a sustained frequency – which perhaps signals the commensurability of a chord progression to the ear. For example returning to the case of MBN six (Figure 1.13), there are three possible physical configurations or digit sequences – MBN 6_1, 2_30_1, 3_20_1. The mutable number six illustrates how an oscillatory position-value number can possess a range of different configurations which all yield the same magnitude (harmonic h6), each by a varying route..

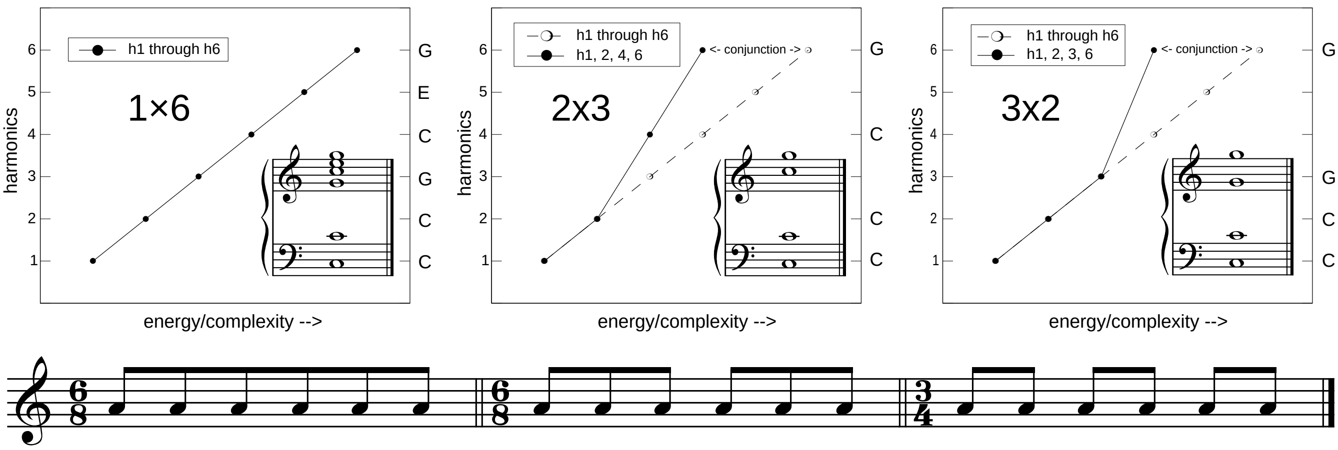

Figure 1.13 The three configurations of MBN six displayed in graph format, with harmonic interpretation inset and metrical interpretation below. All three arrangements yield interference patterns of six fluctuations per period, however, each internally groups the fluctuations differently, as illustrated by the meters.

Indeed, six is a member of the group of most fecund mutable base numbers, that is, the set of numbers possessing the largest range of distinguishable alternative digit sequences relative to their value,

the set of most fecund numbers begins: 2, 4, 6, 8, 12, 24, 36, 48, 72, 96, 120, 144, 192,...

These most fertile numbers are the product of the prime factors two and three, at first; later in the set five, seven, and the other primes enter the fray[7]. For example, while the mutable base number six has three distinct configurations, the number twelve possesses eight alternative, physically distinguishable, arrangements:

12, 6x2, 2x6, 4x3, 3x4, 3x2x2, 2x3x2, 2x2x3.

The fecundity of these numbers, and indeed most mutable base numbers excepting those with single prime oscillatory patterns, grows more or less rapidly, as the numbers advance.

It is, I think, a most attractive notion that a single number might contain a spectrum of differing shades or attributes. Indeed, as a mutable base number system reaches larger values, so the range of alternative configurations grows progressively ever richer. It is this range of alternatives, I suspect, that underlies the multiplicity of harmonic progressions found in tonal music. In essence, each chord or harmony represents one mutable base digit sequence (in relation to all the other chords/harmonies within the piece) and once the system – a tonal composition, in principle – rises above the most rudimentary harmonic level, it makes contact with a set of mutable numbers with a sufficiently rich palette of alternative digit sequences, so as to provide the necessary scope and variety required for the extended creative elaboration of harmony, while still being held within an overarching unity.

Each individual chord progression within a tonal composition, is none other than the reconfiguration of the digit sequence within a mutable number: two linked chords, representing in sound, two different mutable number digit sequences of one and the same value. For example, the full or perfect cadence progression of dominant-seventh and tonic chords, taken in isolation, can be viewed in terms of the transition between two intermediate state digit sequences, of the mutable base number twenty-four – illustrated in Figure 1.14.

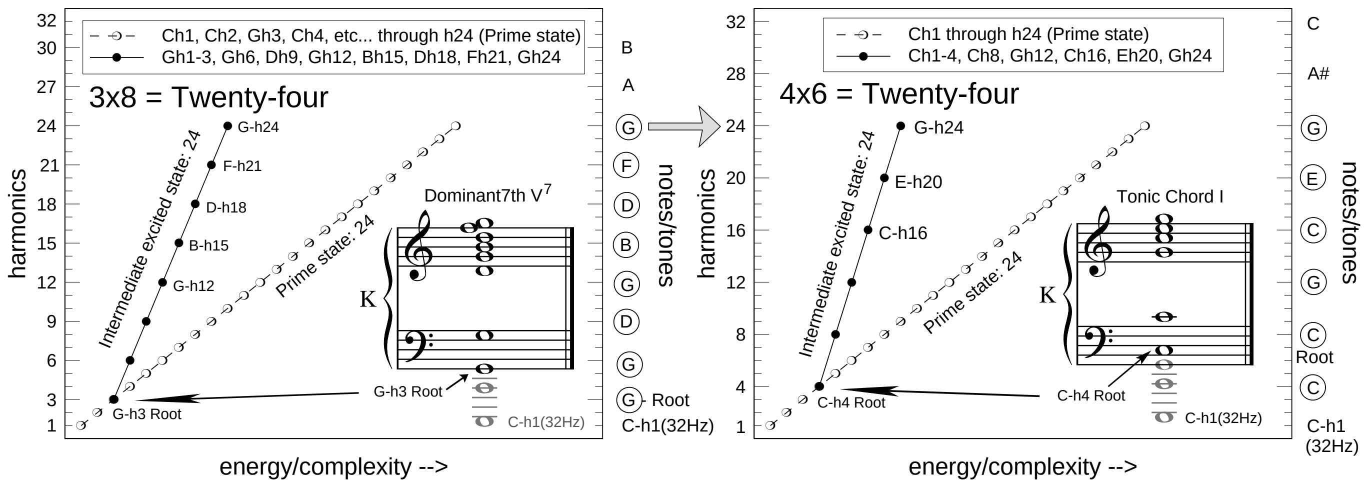

Figure 1.14 The full or perfect cadence chord progression (V^7- I) viewed as the transition between two digit sequences (intermediate states) of the mutable base number twenty-four.

h1, 2, 3 nesting h6, 9, 12, 15, 18, 21, 24 –> h1, 2, 3, 4 nesting h8, 12, 16, 20, 24.

Dominant7th –> Tonic Chord

MBN: $8_3\,0_1$ –> $6_4\,0_1$

And once the ear has stepped across the bridge of this single mutable number spanning the two chords, it listens-out for the next bridging number. Perhaps it will find yet another digit sequence of the same value to take it forward, or alternatively, be forced to seek out a new value – arrived at by an addition or subtraction applied to the leading edge additive column of the number which originally carried the ear to the chord. (Such additions and subtractions are counted in the units or groupings of that column.) So stepping, from digit sequence to digit sequence – number to number – a tonal composition proceeds with a continuous and inevitable numerical logic, from initial chord to final cadence.

In Figure 1.14 illustrating the harmonic progression from dominant G^7 to tonic C-major chord (left graph to right), the objective sounding tones are shown as black dots connected by a continuous line, and as black notes on the staff. The grey notes below the staff and the open dots connected by a broken line, fill in the missing ratios of the fundamental harmonic series implied by the chord progression. Ultimately, it is the fundamental period of C-h1(32Hz) which ‘contains’ and thereby relates the two objective chords to each other. The small anti-clockwise shift in angle between the continuous black lines in the two graphs represents the slight relaxation of energy and complexity in the transition from dominant-seventh to tonic chord. The digit sequence of the tonic C-major chord (4×6 or MBN $6_4\ 0_1$) is a little closer to the ground state (3×2×2×2 or MBN $2_2\ 0_2\ 0_3\ 0_1$) of the mutable base number twenty-four, than the somewhat more energetic G^7-major chord (3×8 or MBN $8_3\ 0_1$).

MBN twenty-four though relatively small is remarkable fertile in regard to distinguishable digit sequences – there are twenty in all.

3x2x2x2 - Ground state	3x2x4	4x6 - C-major chord (tonic)
2x3x2x2	2x4x3	8x3
2x2x3x2	2x3x4	3x8 - Dominant-seventh chord
2x2x2x3	6x2x2	12x2
4x3x2	2x6x2	2x12
4x2x3	2x2x6 - C-major chord	1x24 - Prime state
3x4x2	6x4	(24x1)

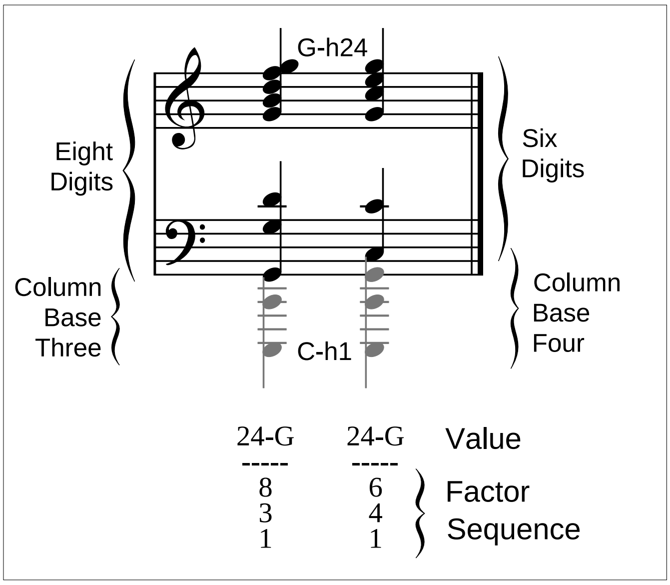

Figure 1.15 The two stave insets from Figure 1.14 enlarged and joined together. The full or perfect cadence chord progression of dominant to tonic (MBN $8_3 0_1$ –> $6_4 0_1$) with the mutable base number twenty-four in two of its twenty alternative digit sequences written below the staff – rather like a figured bass.

Though the perfect cadence illustrates the principle of applying mutable base numbers to music, it is rather minimalist in terms of a tonal composition, or even a tonal phrase. To encompass a broader span of chords, larger frequency ranges with more nested columns and deeper fundamentals are required. Often the written notes of the actual music form no more than a middle band within a wide spectrum of frequencies. The objective tones are the crucial element dictating the overall configuration of the digit sequence, that is, the sounding notes *with their harmonic spectra*. Probably the ear and aural cognition only partially and hazily penetrate the depths of these extended mutable number digit sequences; yet, it is entirely sufficient that they grasp what certainly does lie within their range: the notes themselves, a number of low order partials and the conjunction frequency (normally also a low order partial) that conjoins the chord progression. The acquisition of a common frequency linking two different integer related configuration of tones (i.e. two chords) by the ear, provides enough information to imply, if not deduce, the remaining deep 'tail' of the mutable number digit sequence.

Figure 1.16 contains a short phrase from J.S. Bach's setting of the chorale *Ach Gott und Herr* illustrating mutable numbers over a somewhat broader canvas. Other versions of this phrase are given in Chapter 6, Figure 6.15, and in the *Introduction to the MOS Examples and Analysis with Mutable Numbers* (Appendix F). In this example the mutable numbers are still confined to two columns (three-column numbers appear in later examples), and this allows the leading edge column (upper digit) to contain series which include all the principal notes of the harmony. This is not the predominant configuration for mutable number digit sequences describing tonal compositions. Indeed, it would be possible here in Figure 1.16 to break up the upper row of digits, into two lower energy columns; however, in doing this some of the principal harmonic notes would no longer fall in the leading edge column. This is in fact the normal disposition of notes in a MOS analysis; with most falling within the uppermost series but a few notes dropping through to the series below. The point is that at some level of extrapolation, all the notes of the chords in tonal compositions could be made to fall in the most significant additive column, but that such columns are likely to form extensive, complex series. Such extended series are prone to collapse – the Second Law again; and, in forming a variety of more stable nested structures, they also give rise to a great wealth of harmony – not least amongst which is the minor chord.

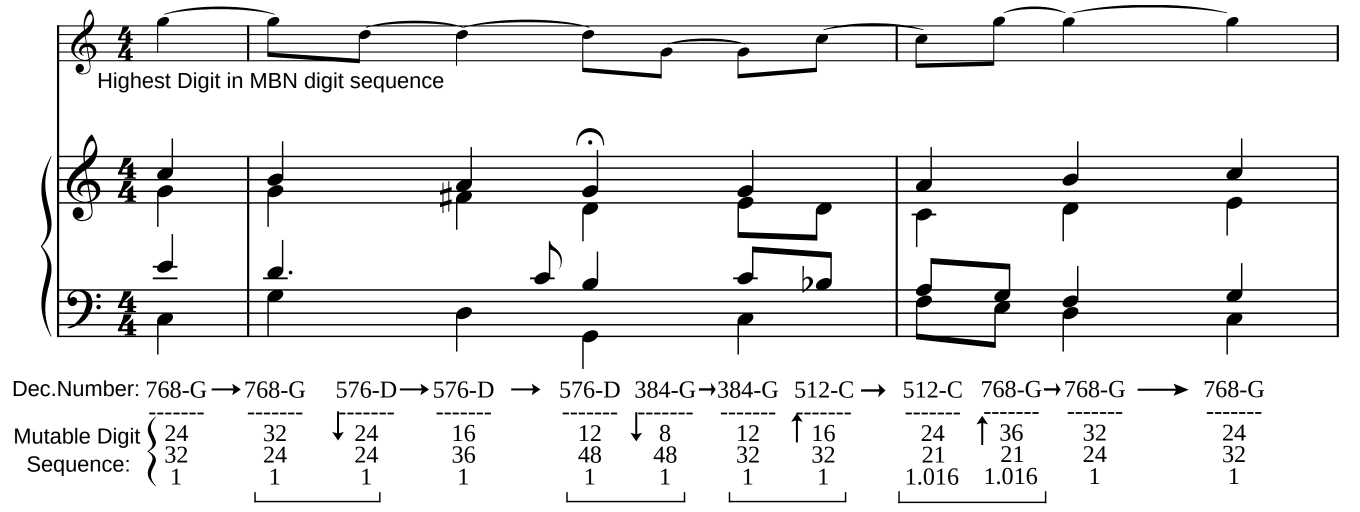

Figure 1.16 *Ach Gott und Herr* – harmony by J.S. Bach. Steps between digit sequences (chords) representing the same number have horizontal arrows; change of number, within a chord, is marked by vertical arrows and by horizontal connecting square brackets below. Passing notes though shown, are not considered in this example (but are in Appendix F), and the significance of the fractional unit 1.016 is explained on page 24.

THE MINOR CHORD

We have so far examined principally mutable base numbers with two, three and four digits in their lowest effective column, next comes mutable base numbers with five digit columns (i.e. MBN ...$_{5}0_{1}$), which have a particular significance when applied to harmonic analysis. Columns containing five digits are, principally, the digit sequences which allow the minor chords to be created out of what are essentially, the major relationships found at the beginning of the harmonic series. (The minor mode is explored in greater detail in Chapter 11.)

The ground state of the mutable number fifteen (factor format 5×3) contains a three-digit column built on the five digit base of its lowest effective column. The three digits of the upper column (E-h5, E-h10 and B-h15) provide the structural backbone of an E-chord without major or minor third. Given that the objective tones of a minor triad have the frequency relationship 10:12:15, the least complex single harmonic series of which all these tones can be a member, is the series founded on frequency 1, or C-h1 in this example, that is to say, the prime state series of the mutable base number fifteen, i.e. h1, 2, 3, 4, 5, etc., through h15.

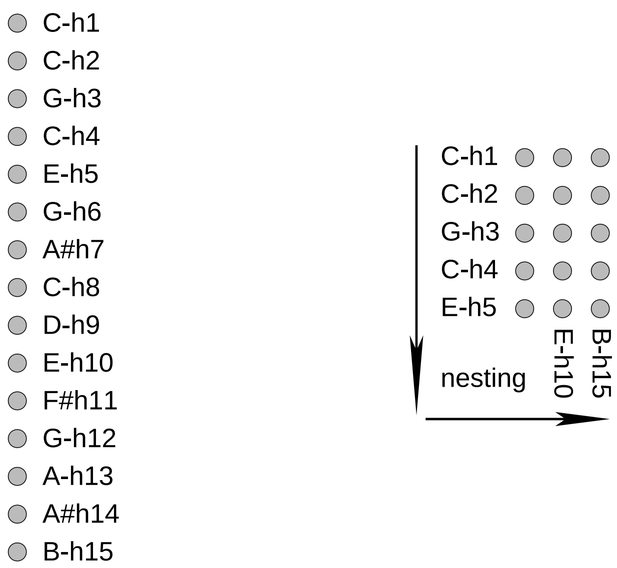

Figure 1.17 On the left, MBN fifteen in the form of a prime state harmonic series – C-h1 through B-h15, and on the right, the less energetic ground state arrangement of C-h1, 2, 3, 4, 5, nesting h10 and h15.

However, the extended and complex series C-h1 through h15 is highly prone to relax to a less complex, lower energy state, its ground state in particular. (Its only other option being 3×5 which is somewhat more energetic.) Fundamentally, nature tends to seek out the ground-state or lowest energy configuration of an entity or system – like a pencil balanced on end will tend to fall onto its side but not, contrariwise, jump up onto its end. Well not very often! Paraphrasing the extended prime state series, in the form of a nested structure, maintains the essential information content of the system, i.e. the value fifteen, while expressing it in a more efficient form. It is this highly attractive ground state configuration of digits (5×3), with the ‘foreign’ objective minor-third tone G-h12 from the underlying series superimposed upon the upper E-based series (E-h5, E-h10, B-h15), that constitutes the minor triad – viewed from the perspective of a mutable base number system. The foreground most significant column, of the digit sequence 5×3; the higher energy upper layer, is identified by the ear with the harmonic ‘meaning’ or sense of the chord (i.e. an E rooted chord) while the awkward objective minor-third G-h12, interjects from its position in the background fundamental series, as a non-dissonant colouring agent, within the harmonic gist of what is perceived to be a series built on E. A graph and number pattern representation of this digit sequence is given in Figure 1.18.

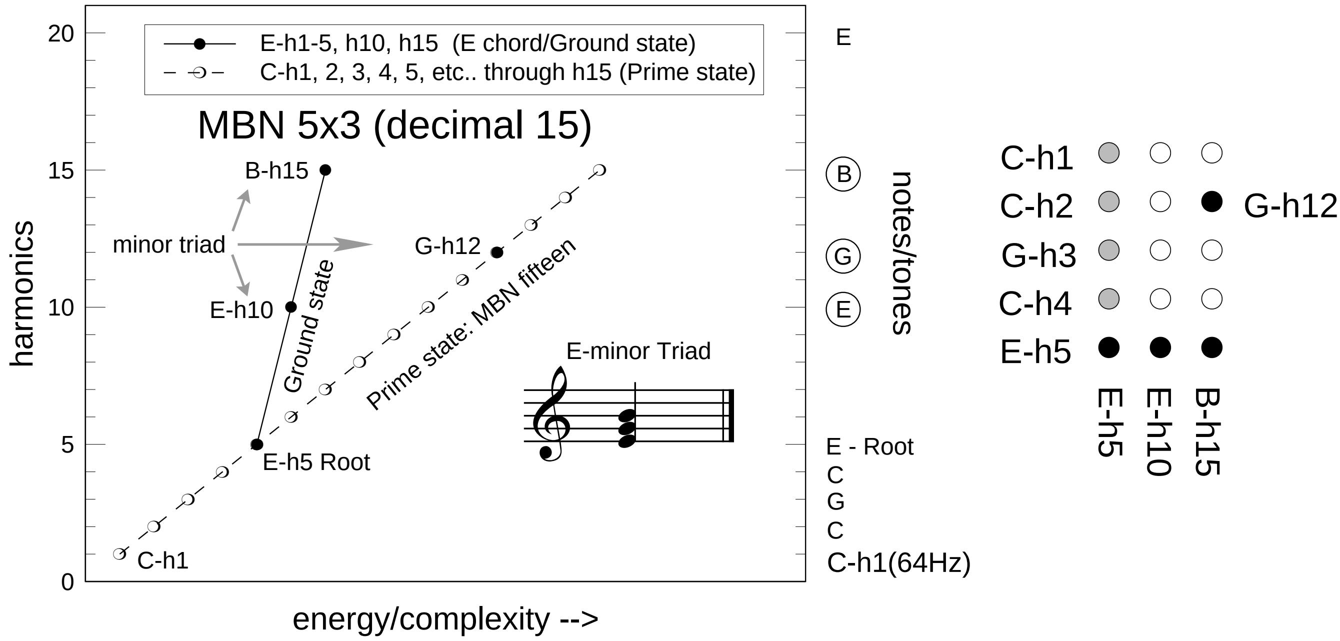

Figure 1.18 The minor triad: mutable base digit sequences with five digits in the column lying below the outer leading edge, most significant digits – MBN $3_5\,0_1$ – illustrated in graph format and on the right as a number pattern.

PRELUDE No.1 - J.S. BACH

Finally, we arrive at a point where it is possible to undertake a little mutable number analysis. Printed below is a mutable base number analysis of a few measures of the first prelude from book one of the Well-tempered Clavier, often described as the definitive example of the extent, or area, of a key. A complete and more detailed version of this analysis can be found in Chapter 12 and Example S.

In Figures 1.21a/b the transitions between digit sequences which equate to the same value are connected by horizontal arrows, and where the ear must seek out a new connection by addition or subtraction there are vertical arrows, upward pointing for addition and downward pointing for subtraction. The re-configurations of columns and digits, within single mutable numbers (connected by horizontal arrows), might perhaps provide a mechanism by which the objective chord successions of tonal music become an aurally intelligible harmonic language. A significant attribute of these reconfigurations of digit sequences representing a single value – i.e. chord progressions – is that the upper tone of the system, lying comfortably within the audible range of hearing, *remains constant*. The upper tone of the mutable number, the value of the system, conjoins the two chords in the progression, thus providing the ear with a clear signal of the chord progression’s commensurability.

A range of other features of MBN analysis are also illustrated in Figures 1.21a/b. Firstly, in measures 1 to 8, the conjunction values (in decimal/prime state mutable) with their various mutable base digit sequences appended below, are applied to the music on the basis of the objective tones being fixed, just as one would expect of a keyboard instrument. Thus the mutable base number values are ‘frozen’ at their home ‘C’ tonal center levels. This is done so as to produce a more ‘readable’ analysis, where the fundamental value varies around, but remaining focused on, the unit. However, inherent in any dynamical tonal scheme based on the ratios of the harmonic series, is the relational curve of the spiral (cycle) of fifths – the problem that equal temperament was invented to overcome; well mask, rather than solve. So as the steps charted by the MBN system from chord to chord are calculated in terms of true harmonic series and whole number relationships, some account must be taken of the ‘flexing’ of structural relationships which can arise from this math (e.g. the downward perfect fifth, C-h32 × 2/3 = F-h21.333..., rather than the whole number F-h21). Music, as a pseudo or quasi-physical system, driven ultimately by the composer’s choices and interpreted by complex mental processes, must be allowed a few liberties. The tolerance of the ear appears sufficiently accommodating to allow such flexing to be accepted. The processes of aural cognition, no doubt both innate and part learnt, constantly adjusting to make the best sense of changing objective musical stimuli. In these first measures, from 1 to 8, this necessary adjustment is shown in the flexing of the fundamental period about the mean value of C-h1(1Hz).

From measure 9 to 11 inclusive, the vice-like grip of the pitch grid has been relaxed, and in these three measures the mutable base number system is illustrated tracking the relationships as if they were truly self-organizing. Here the system has been set free to compute its own destiny, moving freely about in ‘pitch-space’. (This is the style of MBN system used in the first section of the full analysis in Chapter 12 and Example S.) Also, as remarked above, number operations are only possible if supported by the mechanics of the underlying physical system – unlike abstract mathematics, which allows a greater freedom of action. These restraints are reflected in the delimited, though still quite extensive, range of chord progressions available to the tonal musician. Indeed, many arbitrary chord progressions do not result is satisfying and meaningful tonal music: composers over the centuries since the dawn of the tonal era, have been busily exploring which operations are supported, and in what order.

Lastly, the illustration cuts to measures 20 to 24 of the prelude, where again the pitch relationships are frozen and the relational flexing accounted for in the fundamental period, as in measures 1 to 8; here the harmony is at its most complex and intense, with two diminished-seventh chords sandwiched together.

$$\mathrm{I}^7 - \mathrm{IV}^7 - \mathrm{dim}^7 - \mathrm{dim}^7 - \mathrm{V}^7$$

The intervallic relationships within the diminished-seventh chord don’t allow, or at least don’t encourage, a nested structure to develop; there is no common ground between the frequency ratios of the notes upon which to build a nested arrangement. Figure 1.19 illustrates the situation in measure twenty-two of the prelude. The lack of aurally discernible structure generated by the diminished chords is reflected in the loss of the top column (layer of nesting) from the mutable base digit sequence, with the column below extending upward by purely additive means to fill the gap, that is, up to the mutable base number carried over from the previous chord (rather like Figure 1.16, except that there the chords do have common relational ground). Although the mutable base number system itself can handle many diminished-seventh chords in succession, the ear soon loses its bearings without the firm ground of rooted chords. The source of the ‘rootedness’ of chords – fairly simple upper layer nested structures/series – disappears in these harmonies, resulting in a perception of no discernible root – see Figure 1.19. From the view point of the mutable base number system, the root is still there, but at a depth far too great for the ear to unravel, sitting at the bottom of extended harmonic series of 38 and 20 ratios, respectively in measures 22 and 23. The ‘missing’ layer of structure is indicated by two asterisks ** in the score – Figures 1.21a/b.

Considered as a sequence of mutable numbers, the prelude is in essence a computation – an example of number processing by means of musical sound. The precise sums computed are given in Figure 1.20.

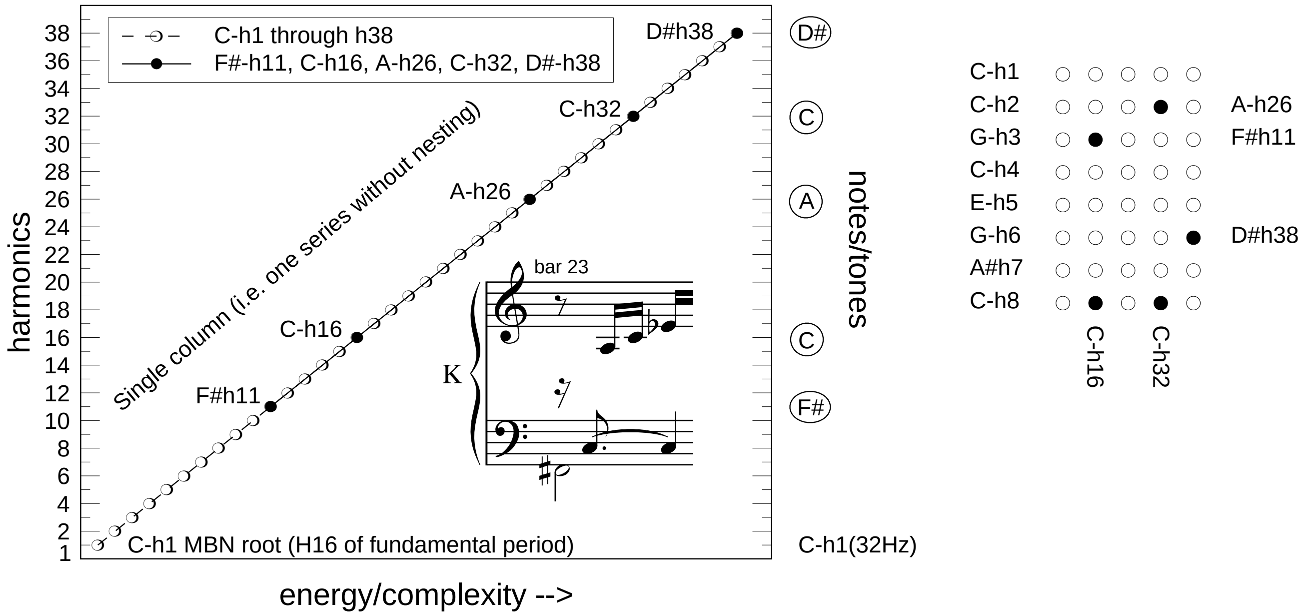

Figure 1.19 The intervallic configuration of the diminished-seventh chord allows little scope for nested structures to develop as the only common relational ground between the frequencies' prime factors of 2, 11, 13 and 19 is two, which coincides with the fundamental of the underlying series built on C-h1, rather than distinguishing the chord from it.

In Figure 1.20 notice that the values computed in the prelude only change by addition or subtraction within measures where the harmony is static and remain constant between measures where the harmony itself changes –i.e. where digit exchanges representing the same value occur. Also the overall trend from start to finish (excluding a final flourish of activity in measures 34 and 35) is downward in value, as the composition (a pseudo-physical system, in principle) releases energy, gradually and satisfyingly approaching equilibrium with its surrounding environment.

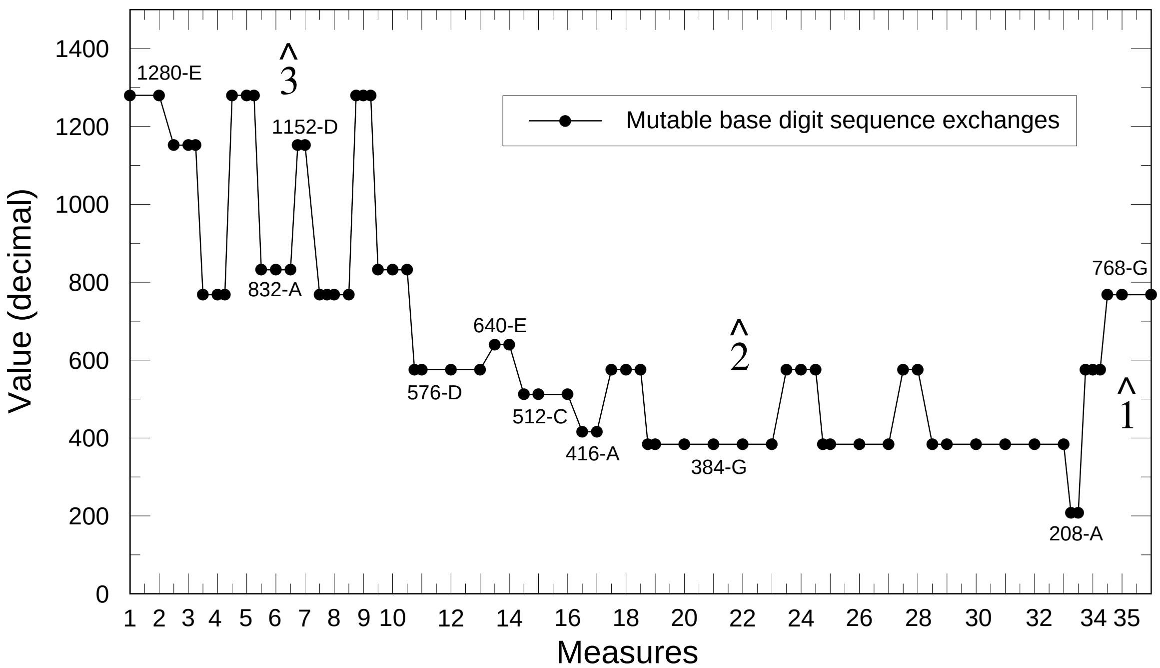

Figure 1.20 Number processing in Prelude No.1, with the mutable base digit sequence exchanges for all values computed, drawn as points.

Figure 1.21a Measures 1 to 8 of Prelude No.1 in C-major, from the Well-tempered Clavier by J. S. Bach, with mutable base numbers and digit sequences appended below each chord.

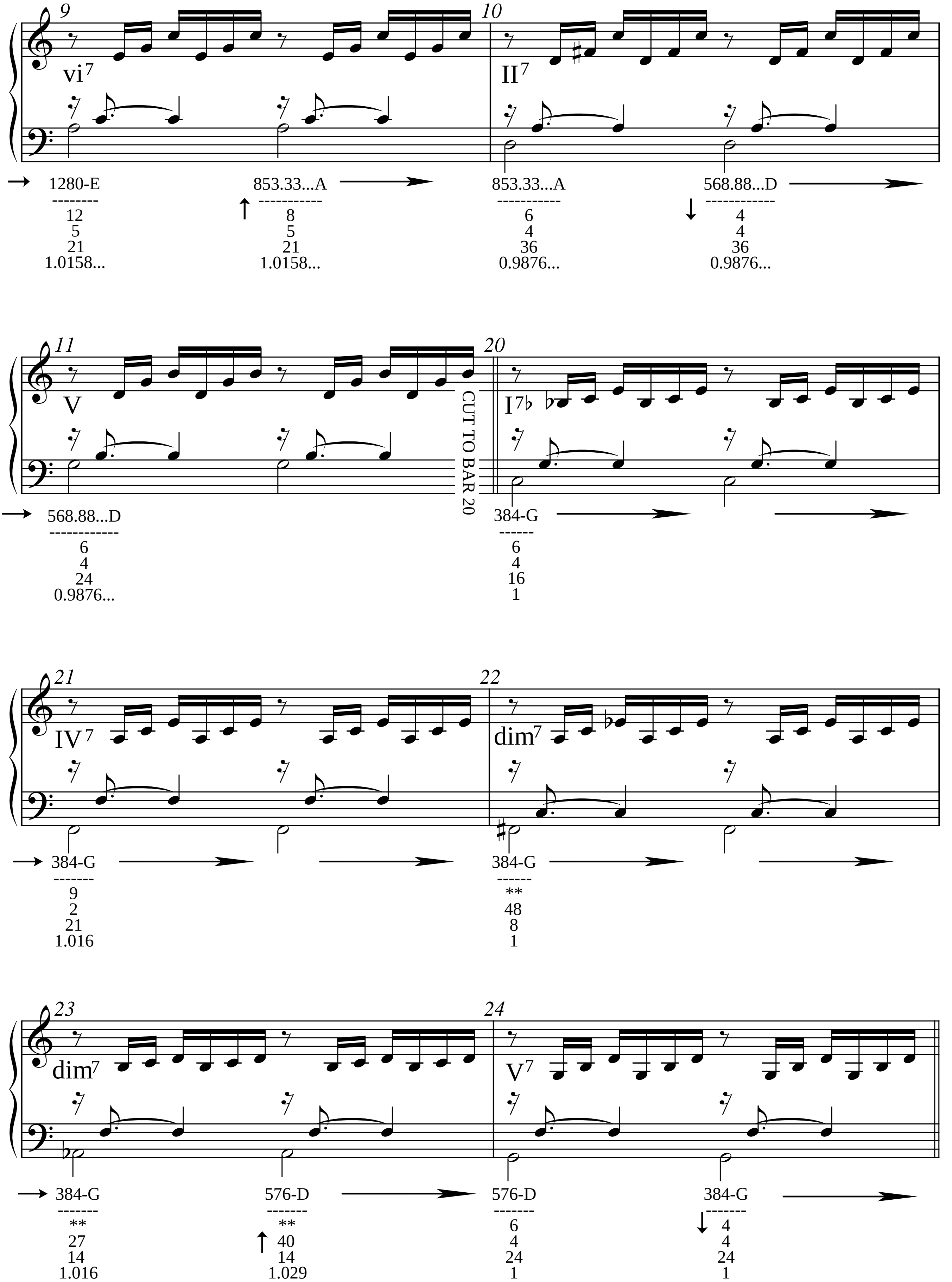

Figure 1.21b Measures 9 to 11 and 20 to 24 of Prelude No.1 in C-major, from the Well-tempered Clavier by J.S. Bach, with decimal numbers and mutable base digit sequences appended below each chord.

CONCLUSIONS

A truly useful music theory should, hopefully, enlarge our understanding; shedding light both on the intrinsic qualities of the music itself, and on its historic development. Additionally, if the theory also aspires to be thoroughly fundamental and elemental in its description of the art, it will inevitably be mathematical in character. Bearing these criteria in mind, the principal conclusions which may be drawn from the identification of an integral number system embedded within tonal music are:

1) The canon of tonal music is what composers, musicians and most importantly the audience have selected over generations. To a considerable extent this body of music can be viewed as an external manifestation of inner aural and mental processes. Indeed, the very pattern of nerve pulse encoding in the auditory pathway recorded by investigators (Rose[8] et al 1967, Kiang[9] 1969, as given by Beament[10]) is analogous to the scheme of mutable numbers, considered as a meta-level re-implementation, in that the frequency value being signalled, is the lowest common multiple of the actual extant nerve impulses. This algorithm of the hearing mechanism, transferred to tonal music interpreted in terms of mutable numbers, would map these lowest common multiples of nerve impulses to conjunctions – the mutable number value – and nerve impulse streams to notes (pitch-frequency).

As there is nothing more ordered than a number system, and as the fundamental structures of tonal harmony reveal themselves to form such a system, music making and listening may ultimately be interpreted as a source of low entropy organisation, made available to human beings in a form digestible by the processes of aural cognition – nourishment for the mind, and depending on your point of view, perhaps also the soul.

2) The history of the tonal era may be redrawn in terms of the powerful attraction of this source of low entropy: its gradual discovery, its long zenith from Corelli to Schubert, its decline and eventual rejection by many composers toward the end of the nineteenth century... Wagner to Schoenberg. However, as the composers of the new art music moved away from mutable numbers in their compositions, so the vast majority of the audience, many musicians and some composers deserted the avant-garde in favor of the historic repertoire and style – beginning with the Mendelssohnian revival of Bach in the mid-nineteenth century. From about the same time, popular domestic and commercial music making grew exponentially; the household piano, brass/military bands, choirs, jazz, vaudeville, etc. began to fill the void also, though often with rather more humble tonal offerings. In more recent years, after a century of turbulent experimentation, classical art music is showing some signs of a return to the tonal fold.

3) Approached from the view of a number system embedded within tonal harmony, the music of common practice may be construed as a form of computation. In digital electronic computers a formal number system (binary numbers) is united with a physical representation of digits as two levels of electrical potential – 'on' and 'off' representing one and zero. Similarly, formal mutable numbers meet their physical representation in sounding tonal music. A music score may thus be interpreted, in principle, as a stored program: software which runs upon the hardware architecture provided by the particular circumstances of each musical performance. (An overt example is given in Chapter 13 and Example R, where the logic sequence of a computer program written in the BASIC language is reinterpreted in mutable base numbers as a tonal composition, a piece for violin and piano: *The Divisors of Seventy-two.*)

4) In view of the elemental nature and neutral quality of mutable base numbers, as well as providing a mathematical underpinning to standard Roman numeral analysis, the MOS model offers the prospect of fruitful combination, or perhaps even integration, with other theoretical systems; in particular, those approaches that stress the dynamic and organic aspects of tonal music, for example, Schenkerian analysis.

Commentary

There is, potentially, a great deal going on under the surface in tonally organised music. Indeed, the written notes of a composition are perhaps just the visible tip of an iceberg of aural relationships and cognitive processes. Through the broader context of the modulating oscillatory systems model of nested harmonic series, the objective chords of tonal compositions and different metrical proportions may be connected together and interpreted as being (parts of) mutable number digit sequences, written in explicit, and in places implicit, frequency relationships. However, great care is required: In the late nineteenth century a number of music theorists were lead into difficulties pursuing theories about sub-harmonics or ‘undertones’ which proved to be contrary to experience and acoustics. The wider context of mutable numbers, the submerged body of the iceberg, if it has existence beyond that of a formalism applied to scores, it is to be found within the mechanisms of the human ear and mind. This is largely an uncharted ocean, particularly the higher levels of cognition, which hopefully one day will be laid bare. Until that time firmly grounded fact yields to hazy speculation. However, a significant point to note is that *the interpretation of tonal music provided by the MOS model requires only that the ear apprehends a conjunction tone lying well within the range of normal hearing*. The aural perception of this ‘still center’ (either note or low order harmonic) connecting adjacent chords is sufficient; the ‘deep tail’ of the mutable base numbers is implied by this conjunction frequency, though perhaps never explicitly deciphered by aural cognition. Yet undeniably, music in the tonal era largely developed through the agency of human choice, made on the basis of what delights the ear and aural understanding. So it would appear, if the mutable number approach proves to be valid and coincidence is discounted, that whatever else tonal music may be, it is also an exercise in mathematics. Such a conclusion is perhaps not surprising given that ratio, and generally simple ratios at that, form the foundation of the art. What is surprising though, is that the nature of tonal music should be so precisely mathematical at its heart, that it produces the structure of a positional number system: and one wonders what this intensely numerical character might imply for not only aural cognition but the mechanisms and structures of the mind in general. This however, I must stress is speculation, and should it turn out to be the case that mutable numbers have no connection at all with the processes of aural cognition, the MOS idea simply falls back to what it essentially is and always has been, *a mathematical model applied to traditional tonal western music*, and hopefully, still a useful analytical tool.

In Chapter 12 and Example S, Prelude No.1 from the first book of *The Well-tempered Clavier* is analysed in full using mutable base numbers; and other example mutable number analyses are also given in the EXAMPLES folder. Generally, sophisticated pieces and those of greater extent require larger mutable numbers to encapsulate their complete harmony. Values in the range from three hundred to three thousand, expressed as three-column (or more) mutable numbers, are typical.

Historical Perspective

From an overview perspective of world music cultures, the development of western tonal music during the four hundred years from 1500-1900, was atypical. Overall, music cultures throughout the world have tended to evolve structural models based on a vibrant use of pitch and rhythmic elements, often associated with ostinato motives and dance rhythms, creating semi-continuous textured heterophonic strands, over or through which melodic features of song or chanting are interwoven. The melodic contours frequently include expressive pitch inflection and microtonal intervals lying beyond the whole-numbered ratios of diatonic harmony, as for example is used in classical Arabic or Indian musics. Typically, in the majority of traditions, often complex rhythmic structures are worked out, in the foreground, yielding lively, engaging, energetic music styles – exemplifying a close connection between physical movement, dance and musical sound – as for example is found in many African cultures. Western art music was to develop along a rather different path: with these foreground rhythmic complexities, perhaps one might even call them 'computations of rhythm and meter', being transferred, to a significant degree, to the rarefied frequency level of harmonic progression. Fundamentally similar processes were at work in both areas – each equally amenable to MOS/MBN analysis – only the frequency domains were completely different. The focus of activity in the western tradition was sublimated from the frequency range of beats per minute to that of cycles per second. In western music, as the 3:2 hemiola rhythmic proportion became an increasingly rare, though still powerful device, the 3:4 pitch ratio of the dominant-tonic chord progression was to become all pervasive. Of course, the differences are of degree and emphasis, not absolute or exclusive. However, the atypical evolutionary route of western musical practice (engendered by the development of polyphonic music) across a landscape of high frequency harmonic organization, was to open up a significantly wider range of relational exchanges and transitions than it is possible to access at the coarser-grained, low-frequency level of rhythm and meter. With these enhanced relational resources, western musicians were, intuitively, to forge an extensive and logically consistent system of musical organization – the tonal system. *That this music system is also effectively a number system, is the burden of this document*.

In Summary:

1) chords, through extrapolation to related sets of nested harmonic series, form the digit sequences of mutable base position-value numbers;

2) chord progressions are reconfigurations of mutable number digit sequences representing the one, selfsame, value; and,

3) tonal compositions are successions of mutable numbers (i.e. digit sequences) which, in principle, amount to number processing in sound.

Notes

1. In contrast to the situation of music – poised between symbolic glyphs and material existence (written scores and live performance) – mathematicians have not generally needed to give ‘physical utterance’ to their calculations and proofs employing ‘abstract’ symbolic number systems (decimal, sexagesimal, duodecimal, binary, etc.) and so have not been as constrained as musicians by physical restrictions in the range and types of their number processing.

2. Elements of Music?, Archive Nov.03 Vol.17.2 - June.05 Vol.18.9. Some of the ideas and concepts used in this book were developed by studying objective physical systems (e.g. the periodic table of elements and the solar system) from the perspective of musical/tonal principles of organisation. Though this approach proved fruitful and illuminating, particularly in regard to the development of concepts and tools for the subsequent analysis of underlying structure in tonal compositions, I must stress that the propositions outlined in the articles were of an exploratory and highly speculative nature.

3. Webster’s and Oxford dictionaries define ‘digit’ as "a finger or a toe" and "any of the numbers from 0 to 9", amongst other things. In this document both these meanings are adopted with the context determining which, or indeed if both are intended; with the former definition extended to include "or a harmonic/partial/ratio of the harmonic series". Thus the number formed by raising three fingers might be described as three digits, as similarly might the three physical harmonic frequencies h1, 2, 3; both of which signify the same value as the one written digit 3. Though occasionally a little thought and care may be needed to distinguish one or other meaning, as often as not, it is both the physical relationship of ‘real’ digits and the symbolic written digit that is being jointly referenced.

4. For a fuller account see: Barrow, J.D., *The Book of Nothing* (Vintage/Random House, London, 2001) and H. Midonick, (Ed) *The Treasury of Mathematics: 1*, (Pelican Books, London, 1968) p249.

5. Midonick, H., (Ed) *The Treasury of Mathematics: 1*, (Pelican Books, London, 1968) p249.

6. Essentially mutable base numbers and the modulating oscillatory systems they represent, considered as a general relational theory, are expressive of the two most fundamental characteristics of phenomena in the material world, those of identity and events: what ‘things’ are and how ‘things’ change.

Hall, A.R., *From Galileo to Newton* (Wm. Collins 1963/Fontana, London, 1970) p33. "... Throughout the ages the problem faced by natural philosophers has been, in broadest terms, the duality of identity and change. The whole universe, the materials composing it, the events that occur in it, are in one sense always the same, in another always different."

7. Lists of the number of arrangements (i.e. digit sequences) of mutable base numbers from 1 through 5000 and 1 through 20,000 can be found in the EXTRAS folder/directory: files 1-5000.TXT and 1-20K.TXT. The sequence of most fecund numbers are marked in the lists. Interestingly the difference between any two adjacent fecund numbers is also a fecund number.

8. Rose, J., Hind, J., Anderson, D., Brugge, J., *Response of Auditory Fibers in the Squirrel Monkey.* (Journal of Neurophysiology 30, 1967)

9. Kiang, N.Y.S., *Discharge Patterns of Single Auditory Fibers,* (MIT Research Monograph 35, Cambridge MA, 1969)

10. Beament, J., *How We Hear Music,* (Boydell Press, Woodbridge, UK, 2005)