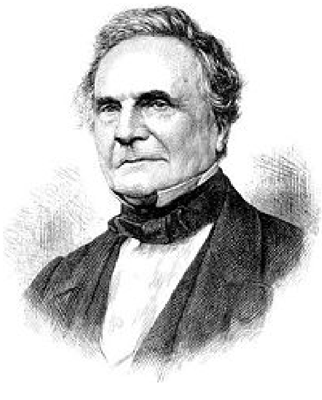
COMPUTATION WITH MUTABLE NUMBERS

Generally throughout these documents[1], the argument is advanced that tonal music, stripped down to an elemental structure founded upon nested harmonic series, forms an implicit sequence of values – *mutable base numbers* – written in an acoustic physical notation.

In support of this contention, that chords in tonal music may be interpreted in essence as being positional numbers written in sound (–i.e. may be construed as forming parts of much broader harmonic series h1 through hn) a demonstration ‘composition’ or ‘tonal procedure’ is presented below, that performs the overtly numerical task of finding the divisors of any given positive whole number. Judged on its merits, in purely musical terms, the piece is trivial, consisting as it does of repetitive arpeggios and scale passages. However, this procedure does result in a ‘composition’ of sorts which is both recognisably tonal music and practical mathematics. The example given below seeks out the divisors of seventy-two, though equally the procedure could be applied to any number that lies within the range of musical instruments, and in theory could be applied to any positive whole number. Further below, the same sequence of chords is ‘repackaged’ in a rather more palatable form in the piece *The Divisors of Seventy Two* to demonstrate that this really is tonal music.

The numerical procedure embodied in this ‘composition’ is also given below in the parallel form of a computer program. This program is written in the almost readable prose of the BASIC programming language. This procedure/program requires a digital electronic computer to function. That is, a physical device capable of handling positional binary numbers by means of representing the digits zero and one as the absence or presence of a defined level of electrical potential within the computer’s circuitry. Going back in time to the nineteenth century, the mathematician Charles Babbage designed somewhat similar devices: the mechanical difference and analytical engines. Though never finished in their own day, these were likewise physical devices, but machines that used cogwheels and cylinders to represent the digits of positional decimal numbers – rather than binary numbers as in electronic computers or the mutable numbers used the musical example below. Theoretically, there is little to distinguish between the modern computer and Babbage’s engines, beside the technicalities of operation (and of course a huge speed differential). Interestingly, Ada Lovelace the daughter of Lord Byron, who supported and collaborated with Babbage on the project and is the author of the fullest account of the analytical engine’s true potential, suggested that among other things the device might: “compose elaborate and scientific pieces of music”.



Picture courtesy Wikipedia

Charles Babbage (1791–1871) was born in London into a prosperous banking family with land holdings at Teignmouth in Devon. Babbage was educated at home with tutors and at a variety of schools and private academies, and in 1810 went up to Trinity College, Cambridge. At Cambridge he meet and became friends with many of the coming generation of scholars and scientists such as John Herschel, the son of the great astronomer. With others of this circle he founded the Analytical Society in 1812 and in the same year transferred to Peterhouse College were he felt the teaching of mathematics was superior. Though a good scholar himself, he was to leave Cambridge with only an honorary degree conferred without examination. His somewhat awkward academic career was to be a story oft repeated throughout his life. Babbage was a man of proud character with a talent for alienating friends and colleagues; he held on to, not to say nourished, differences and arguments with a rare passion. Ultimately, this trait would lead to him failing to realise the great potential of his principal contribution: the development of serious mechanical computation. The stimulus for this invention came from the difficulties encountered in producing reliably accurate astronomical, navigational and mathematical tables by human hand. Babbage saw that the basic but mind-numbingly repetitive arithmetic could be better done by machine, and he set about designing a device capable of performing such a task: the difference engine. The history of mechanical calculation is long, going back to ancient methods of reckoning involving fingers, pebbles, etc., devices such as the abacus, and many later inventions, through to work by Leibniz on automated arithmetic – which Babbage had read at Cambridge. Building on this knowledge, Babbage came up with the design of his difference engine, for which he was able to secure government funding. All looked well for the project at first and much was made of his plans at the intellectual gatherings he regularly hosted at his London house. One visitor was Augusta Ada Byron, daughter of the poet Lord Byron, a keen amateur mathematician. Indeed, Ada's mother, also a mathematician (a "Princess of Parallelograms" in Byron's cutting put-down) encouraged her study in the hope that it might counteract any inherited traits of her father's character! Ada was fascinated by Babbage's invention and requested the plans for closer study. Though their relationship was never more than platonic, Babbage liked Ada, helped her to further her mathematical studies and respected her for perceiving the true nature of his invention. Perhaps crucially, being a woman she didn't provoke or challenge his pride. Remarkably Ada probably grasped the wider implications of Babbage's engine more fully than he did himself, and she is now considered to be the first ever computer programmer. Later, Ada, Countess of Lovelace was to render Babbage a great service by translating the account of his more advanced analytical engine, which he gave to Italian mathematicians (having alienated most English colleagues), adding substantial material of her own. It is this document above all which preserved Babbage's work, and nearly a century later, communicated it to Alan Turing the father of the modern electronic computer. Despite all Ada's enthusiastic interest and support, there were delays and eventually total failure in the construction of the difference engine. This was partly due to domestic tragedies in Babbage's life, ill health, a lack of focus on one single design, and as always, personal animosities. However, Babbage did make contributions over a wide area of engineering, science and mathematics; he held the post of Lucasian Professor of Mathematics at Cambridge from 1828 to 1839. Charles Babbage died in London on the 18th October, 1871.

Similarly, in seeking to use positional mutable numbers as the basis of operation for computation, an appropriate physical device is required, that is, a physical device specifically designed to match the particular characteristics of its operational number system. For mutable numbers (–i.e. chords in tonal music) an appropriate physical device is a musical instrument or ensemble, though one could imagine far more powerful oscillatory processors, with frequency ranges and sensitivities far in excess of that required for the pursuit of music. However, the instruments we have and use to make music are adequate for the demonstration of tonal computation in sound. As indeed is exemplified in the performance of any piece from the period of common practice.

First the mathematics. An underlying mechanism for finding the divisors of a given number is described in *Journey to the Heart of Music* Chapter 7, in the section on *bow waves* in the Table of Harmonic Series – or Sieve of Eratosthenes. (In regular mathematics it is called the difference of two squares.) There the formula $N = n^2 - s^2$ was derived and described, for any odd number N. A slightly more complicated formula was found for where N is an even number. However, a further development of the simpler odd-number relationship, allows the inclusion of even numbers within a single algorithm. The best way of seeing how this algorithm works is visually, by picturing the numbers as areas – that is squares and rectangles. Figures 13.17 and 13.18 illustrate this extension of the $N = n^2 - s^2$ relationship from odd to even numbers with two examples. The odd number N = 133 is examined first and then the even number N = 82.

Because the same 'remainder' relationship holds true for both odd and even numbers, with the two gray leftover rectangles and one square (Figures 13.17–18) necessarily taking integer values when whole numbers divide N (despite n and s themselves not always being whole numbers), this characteristic allows a simple algorithm to be devised: whenever a perfect whole numbered square (–i.e. gray square below), equal to or less than the given number ‘N’, is subtracted from ‘area N’, leaving over two rectangular areas; then, two whole numbered divisors of N will be: The root of that perfect (gray) square, and the sum of that root, plus the area of the two rectangles divided by that root (–i.e. the sum of the non-root sides of the rectangles).

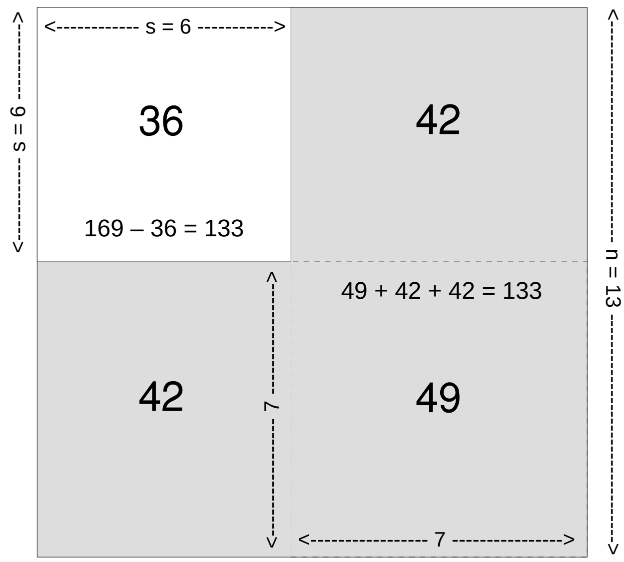


Figure 13.17 Number = 133. By subtracting s^2 from n^2 the number N is deduced (–i.e. 13 x 13 minus 6 x 6 = 133). Looking at the squares of n and s, superimposed, reveals that N 133 (the gray area) is composed of the square 49, plus two identical rectangles, thus 7 x 7 + 42 + 42 = 133. The gray area may be combined into one rectangle 7 × 19. Therefore the divisors of 133 are 7 and 19.

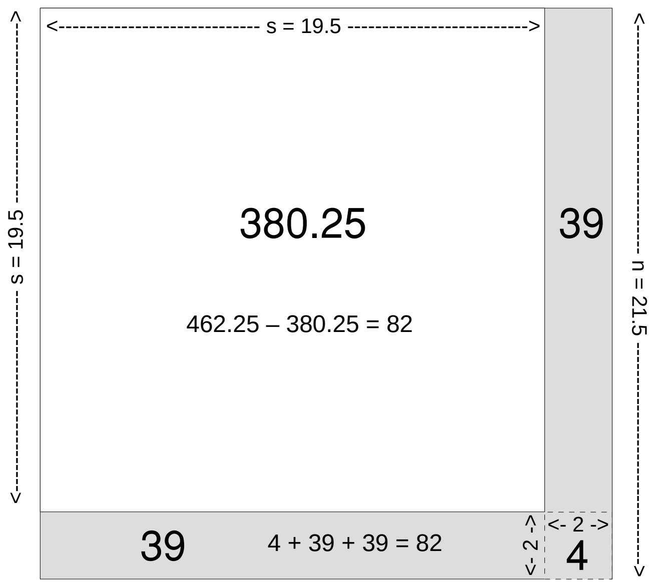


Figure 13.18 Number = 82. The same relationship as illustrated in Figure 13.17, holds true for even numbers too, in spite of n and s being fractional: 21.5 x 21.5 minus 19.5 x 19.5 equals 82. The imposition of s^2 upon n^2 again reveals another square and two identical rectangles: 2 x 2 + 39 + 39 = 82 which may be combined into the rectangle shown in Figure 13.19. Therefore divisors are 2 and 41 (i.e. 2 + (39 + 39)/2).

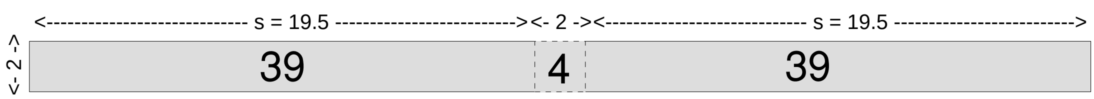


Figure 13.19 Rotating one leftover rectangle perhaps makes things clearer: area N = 82 = 2 x (19.5 + 2 + 19.5)

Thus when presented with any number ‘N’ for which one wishes to find the divisors, first calculate the largest perfect square equal to or less than N and then proceed in whole numbered steps downward from this square, testing each descending square in turn against the algorithm. Whenever the procedure produces an integer result for the ‘leftover’ square and rectangles, two divisors of N have been found. Essentially, the algorithm anchors the largest (gray) perfect square that will fit within ‘area N’, in N’s bottom right corner, and sequentially compresses this (gray) square, in whole number steps, to one.

In the BASIC programming language (BBC Basic V) this procedure could be written out as three steps: First, acquire the number to be divided; second, find the largest perfect square that is less than or equal to it; and third, check each perfect square from this largest down to the unit square, in integer steps, for leftover rectangles with whole number areas. Whenever the result meets this criterion, print out the whole numbered divisors found.

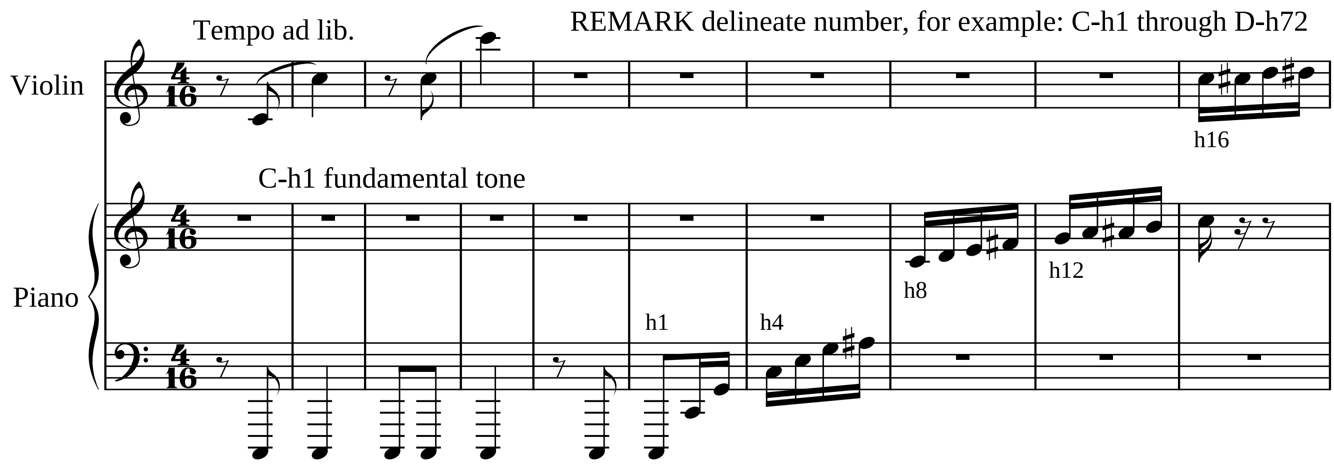
```
REMARK delineate number, for example 72.
PRINT "Please specify whole number to be divided"'
INPUT note_number

REMARK Loop 1. find largest perfect square equal to or less than note_number.
sqrt = 0
REPEAT
  sqrt = sqrt + 1
  square = sqrt * sqrt
UNTIL square >= note_number
IF square > note_number THEN sqrt = sqrt - 1

REMARK Loop 2. work down from value of sqrt to 1 in whole steps.
WHILE sqrt >= 1
  square = sqrt * sqrt
  difference = note_number - square
  result = difference / sqrt
  REM test if result is a whole number.
  IF result = INT(result) THEN
    divisor_1 = sqrt
    divisor_2 = sqrt + result
    PRINT "Divisors: "; divisor_1; " x "; divisor_2
  ENDIF
  sqrt = sqrt - 1
ENDWHILE
END
```

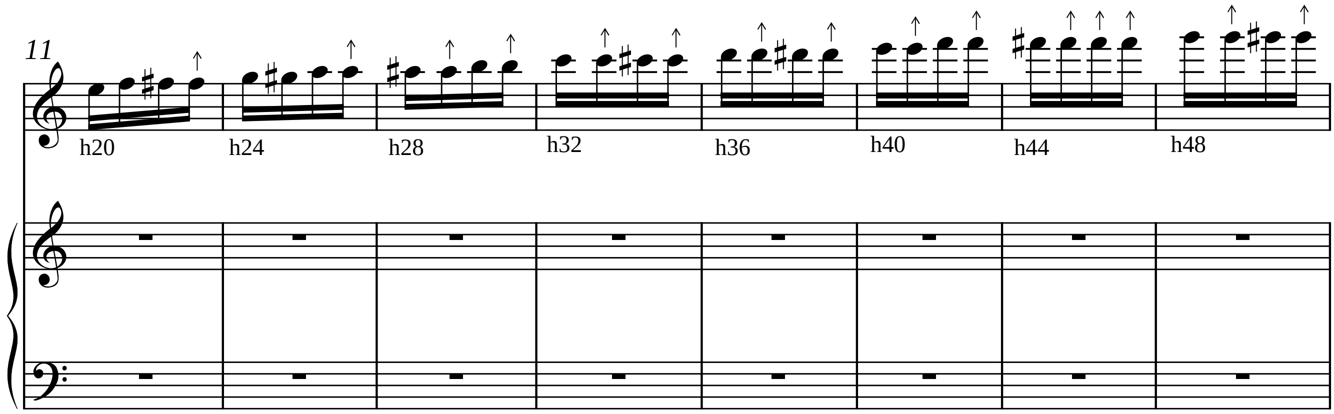
Applying the selfsame procedure as given in the above BASIC program, but using mutable base numbers operating upon the ‘physical devices’ that we call musical instruments (and writing out the progress through each loop exhaustively, thus labelled Loop 1.1, Loop1.2, etc...), produces the score given in Example R for the input number seventy-two. Below the first three pages are reproduced for quick reference. The score requires microtonal notes to be played in the upper part (violin), indicated by small arrows above the notes where one staff note covers a range of two or four harmonics. For example, the written top C may stand in for four frequency inflections: C-h64, C-h65, C-h66 and C-h67.

Notes

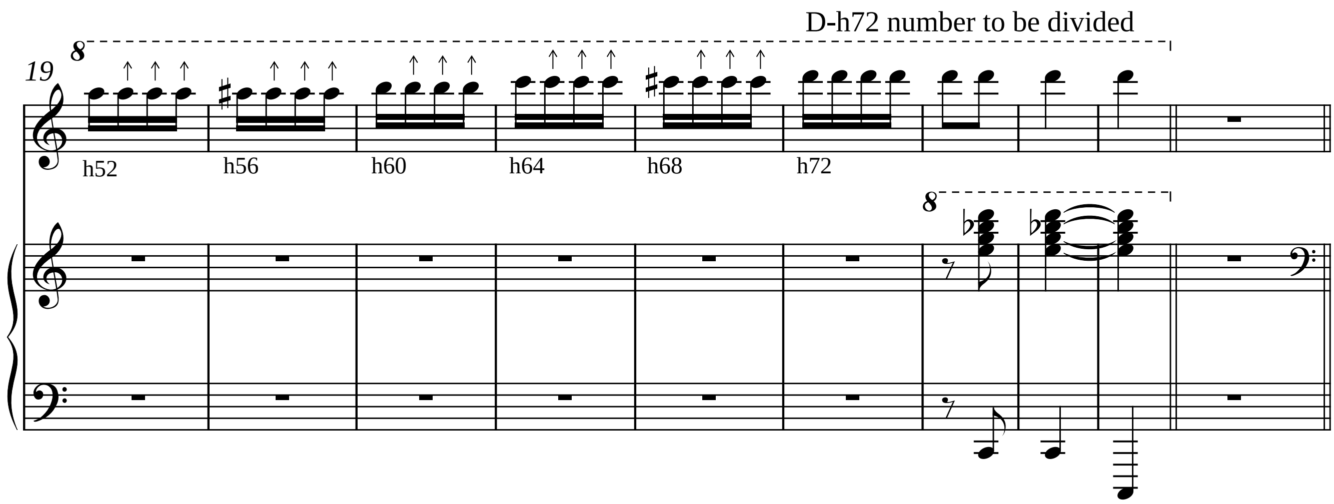
1. This document has been extracted from Chapter 13 of *Journey to the Heart of Music*, thus the Figure numbering and references.

An Explicit Demonstration of Tonal Computation in Mutable Base Numbers





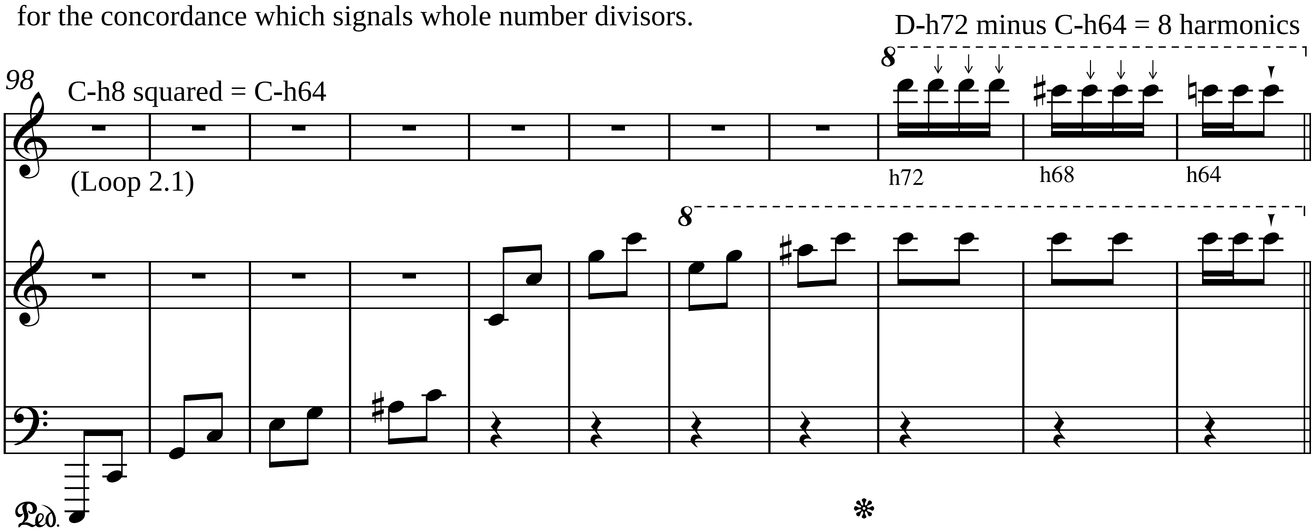
[Where more than one harmonic of the fundamental tone C-h1 is represented by a single note, e.g. F♯h22 and F♯h23 above, arrows (↑↓) are used to distinguish between them. The arrow symbol indicates roughly an eighth-tone, quarter-tone or three eighth-tones as appropriate.]

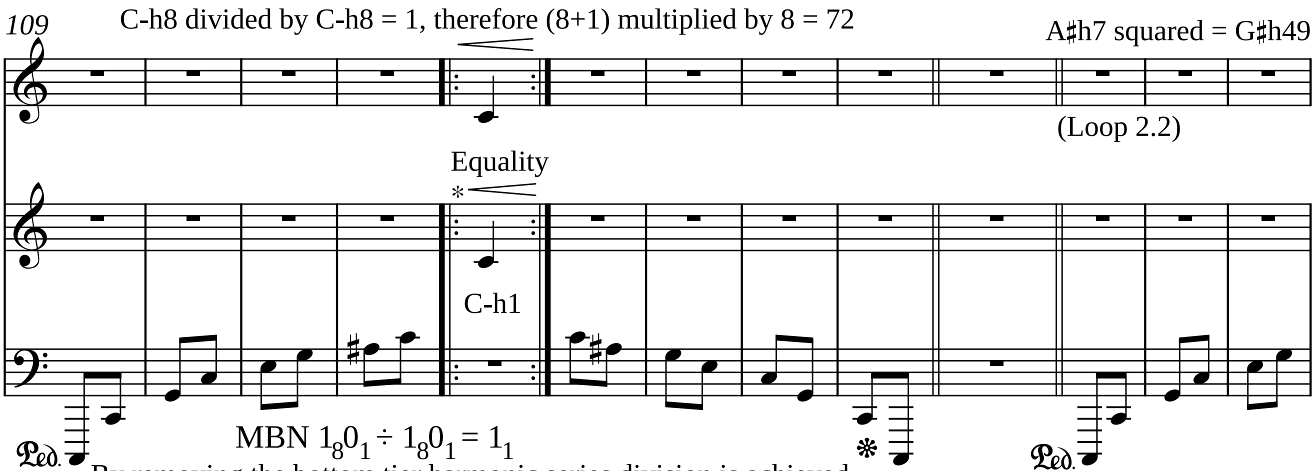


REMARK Loop 1. Find largest square number equal to or less than D-h72

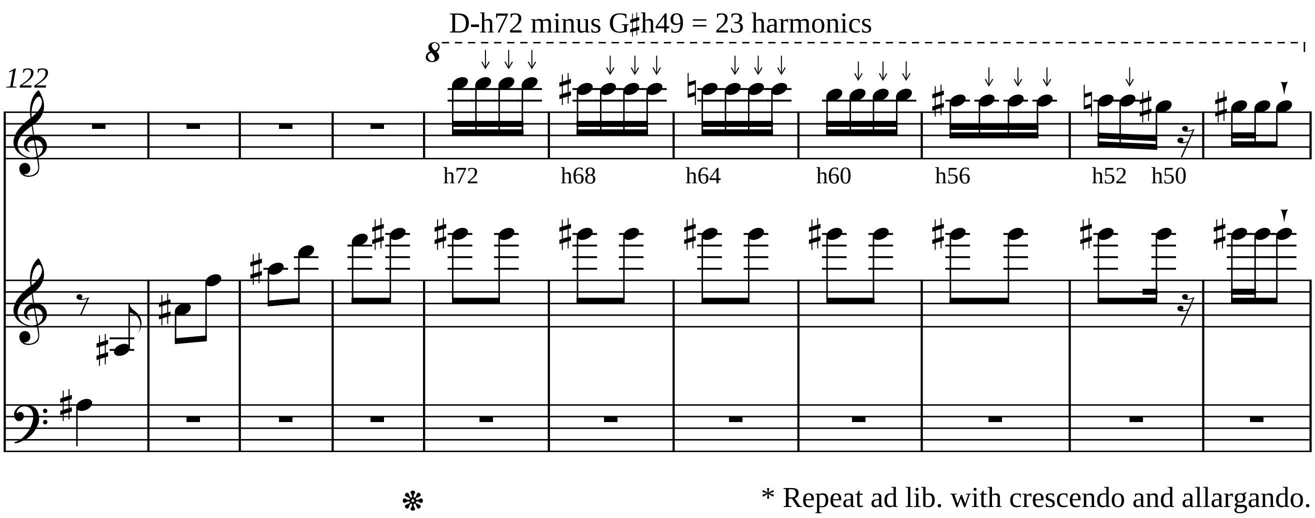

66
9 squared
(Loop 1.9)
8
64 < 72
Ped.
79
8
therefore 8 squared is largest
square number equal to or
less than D-h72
9 squared, E-h81 is greater than D-h72
8
81 > 72
8
81 > 72
Ped.
88
8
64 < 72
8
64 < 72

REMARK Loop 2. Check each square number in descending order from C-h64 for the concordance which signals whole number divisors.





By removing the bottom tier harmonic series division is achieved, here division by eight leaves C-h1 of the second tier nested series. –i.e. Decimal 8 ÷ 8 = 1



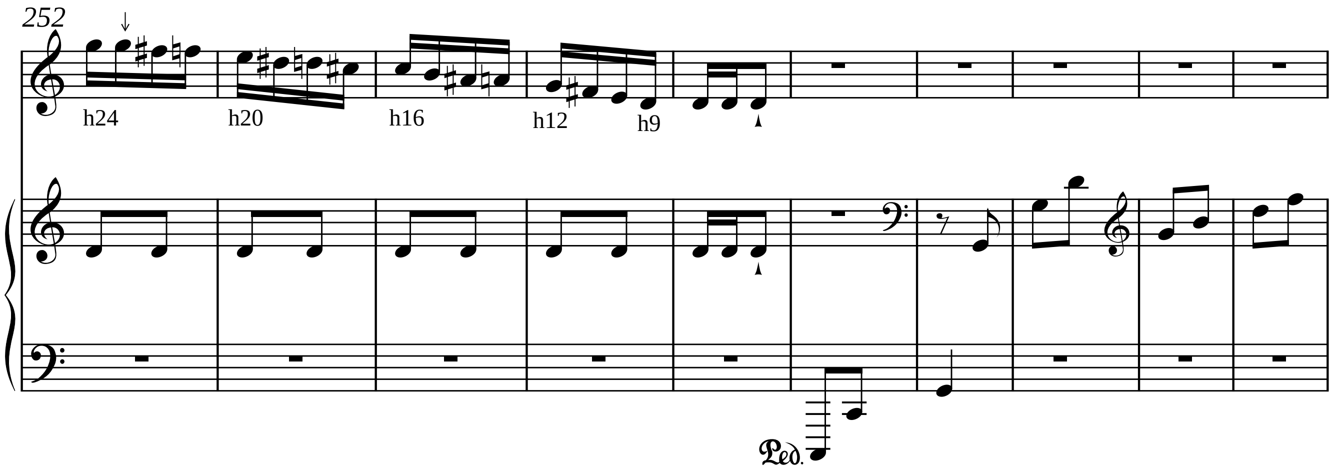
* Repeat ad lib. with crescendo and allargando.

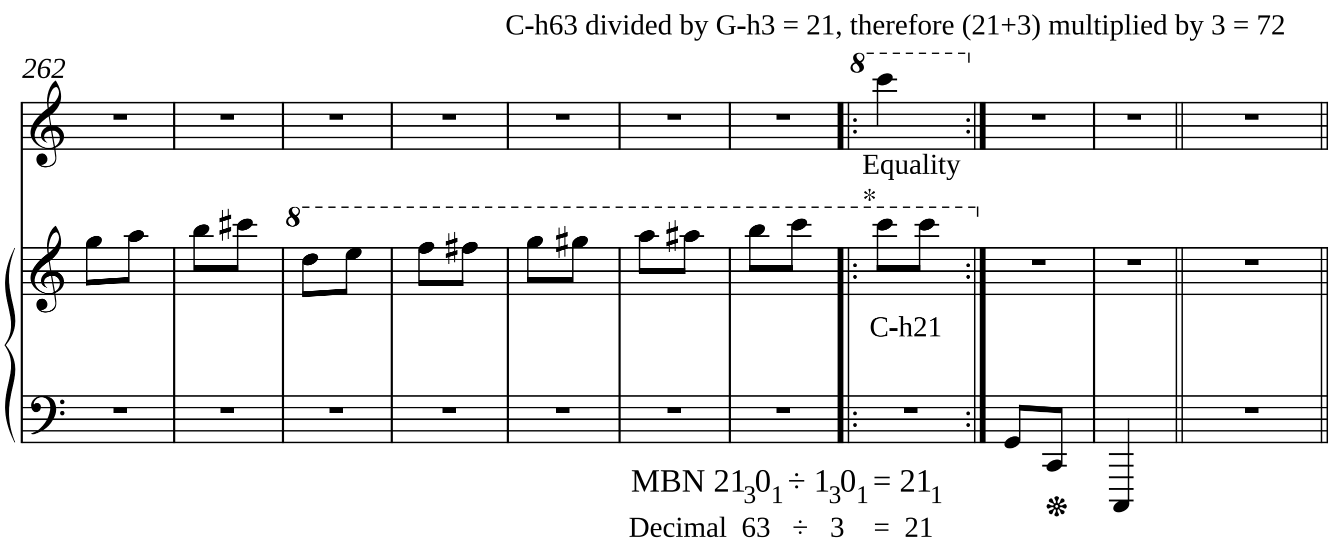
* Repeat ad lib. with crescendo and allargando.

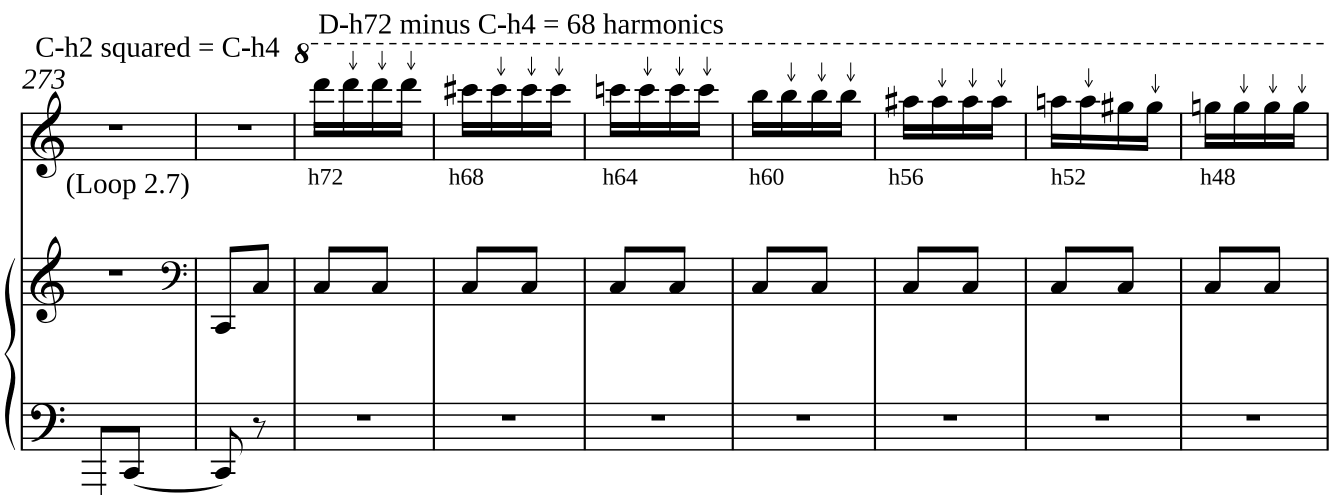
* Repeat ad lib. with crescendo and allargando.

* Repeat ad lib. with crescendo and allargando.

* Repeat ad lib. with crescendo and allargando.







* Repeat ad lib. with crescendo and allargando.

MBN $34_2 0_1 \div 1_2 0_1 = 34_1$

Decimal 68 ÷ 2 = 34

* Repeat ad lib. with crescendo and allargando.


C-h1 squared = C-h1
314
(Loop 2.8)
h72 h68 h64 h60 h56 h52 h48 h44
D-h72 minus C-h1 = 71 harmonics
323
h40 h36 h32 h28 h24 h20 h16 h12
331
h8
h4
h1
h1 h2 h4 h6
h8 h10 h12 h14
Ped. h1

* Repeat ad lib. with crescendo and allargando. (The piano note C♯ plus three eighths of a tone may be obtained by retuning down the unused D♯ above.)

The Divisors of Seventy Two

For Two Bass Recorders (with F♯ keys) and Treble or Sopranino

In ‘Journey to the Heart of Music’ Chapter 13, an example of computational number processing is provided both in the form of computer code and tonal sound. The music below is an attempt to illustrate that the ‘dry’ chord progressions of this rather theoretical example could be incorporated in to a piece of normal common practice tonal music: thereby exemplifying the contention, that all tonal music is, fundamentally, arithmetic.


19
25
31
37


43
49
molto rit.
accel.
The Cmaj-9th chord in the key of C major
expresses the relationship of C-h1 to D-h72.
55
poco meno Tempo primo
Loop:1.1-C
1.2-Cmaj
1.3-Gmaj
1.4-Cmaj
1.5-Emaj
61
1.6-Gmaj
1.7-B♭maj7th


67
allargando
molto allargando
1.8-Cmaj7th
1.9-Dmaj9th,
(E-h81 > D-h72)
Therefore C-h64 is greatest square less than D-h72
73
accel.
Tempo primo
79
84


89
rit.
poco meno
95
101
107


113
119
124
Loop:2.1-C7th
2.2-B♭7th
2.3-Gmaj
131
2.4-Emaj
2.5-Cmaj
2.6-Gmaj


138
meno
2.7-Cmaj
2.8-Cmaj
Thus the divisors of seventy-two are:
8×9, 6×12, 4×18, 3×24, 2×36 & 1×72.
144
149
rit.
154
rall.
molto allargando