Bach's Calculation

Prelude No.1 in C major from the Well-Tempered Clavier, analysed in Mutable Base Numbers – Example S3

To date this is the third MOS/MBN analysis of J.S. Bach's Prelude in C major from the *Well-tempered Clavier, Book I*. I suspect it may not be the last!

(MOS = Modulating Oscillatory System,   MBN = Mutable Base Number.)

The reason for returning again to this iconic piece is that my understanding of the model has developed further, with its application becoming more refined and precise, this has been principally brought about through the discipline of placing Mutable Base Numbers in the context of Group Theory – explored in the document: *The Mathematics of Tonal Music and Mutable Numbers*.

First, a crucial definition: the meanings attached to, and relationships between, the terms nested and nesting. A diagram, Figure S3.1, will save many words.

```
                                    B-h5 -- H15-B
                                         -- H14-A#
                                         -- H13-A
                                    G-h4 -- H12-G
                                         -- H11-F#
                                         -- H10-E
                   Nested Series    D-h3 --- H9-D       Nesting Series
                                         --- H8-C
                                         --- H7-A#
                                    G-h2 --- H6-G
                                         --- H5-E
                                         --- H4-C
Nested Fundamental Tone (h1)  --->  G-h1 --- H3-G
                                         --- H2-C
                        Unit  --->       --- H1-C  <---  Nesting Fundamental Tone [H1]
```

Figure S3.1 An illustration of one harmonic series (h1–hn) nested within another underlying series [H1–Hn]. Grasping the distinction between a nested series denoted by the lower case 'h' and an enfolding nesting series denoted by an upper case 'H' is crucial to understanding the ideas set forth below.

This document aims to be a stepping-stone between the earlier style(s) of analysis typified by Examples S1 and S2 **and the current technique and style** – of which Example Y is typical. Thus the analysis presented below is something of a hybrid embodying elements of both the earlier 'summary' styles and the later 'absolute' approach. The essential difference between the two approaches is that in the earlier methods the uppermost nested series – which encapsulates the harmonic progression of the composition – is placed, nested, within the lower reaches of its enfolding fundamental nesting series (somewhat as in Figure S3.1); whereas in the later analyses this upper nested series resides at a vastly higher position in this underlying nesting series – for example 'G-h1 -- H19131876-G' (bar 3, page 11) rather than 'G-h1 --- H3-G' above.

The contrast between these lower and higher placings is stark. The difference between nested (h1) fundamental tones linking with ratios in the underlying nesting series of summary analyses is of the order of tens/hundreds, compared to billions/trillions with the currently used method! The justification for this development in favour of vastly extended underlying fundamental series is mathematical rigour: Such extensive underlying nesting series allow the upper nested h1-based series (encapsulating the harmony) to match precisely with whole numbered ratios in the enfolding fundamental nesting series throughout the composition; and consequently, the 'absolute' nesting (H1) fundamental tone, *the unit*, is fixed and unchanging from beginning to end.

Luckily, the numerically broad gulf between these earlier analyses and the current approach can be bridged, it is really little more than the difference between using an approximation in real numbers in place of the precise cumbersomeness of whole numbers. **Indeed, the earlier 'summary' styles of analysis amount to an approximation to the latter 'absolute' technique.** The significant difference being that the ratios of the upper nested series in these earlier summary analyses do not precisely match all the whole numbered elements in their underlying series; and so, because the upper nested series embodies the real observable facts of the composition and thus 'drive' the analysis, they force the ratios in the underlying nesting series to flex in accommodation, with the inevitable consequence that the ultimate unit [H1] is not fixed but shifts about. This characteristic of an unstable [H1] unit is obviously a somewhat infelicitous element in summary analyses when considering them in terms of mutable base number processing; however the ability to view these earlier analyses as a 'shorthand' form of the later absolute whole number method resolves this problem.

The great virtue of the summary analyses is that they present a concise and more or less digestible story to the eye and mind, with the added benefit of relating the observable facts of the music to a meaningful (though somewhat unsteady) unit of the scale. While the great virtue of an absolute whole number analysis is rigor, it is simply the whole story; though one told in numbers so large it is better suited to the computer than a human being. Thus both forms of analysis fulfil a purpose, have their place.

So what has changed?

A clearer focus on the music, that is, on the objective sound, both notes and partials. These are viewed as dynamic facts driving the system from the top, rather than using the more bottom-up approach pursued originally. Also earlier analyses were given inner layers of nesting that to some degree amounted to arbitrary additions, rather than (as below) presenting only that which emerges from bare acoustic or mathematical necessity – a single nested series.

There are practical arguments too in favour using of a single nested series, arising from the huge size of the whole numbers produced in absolute solutions. MBN: $18_{255091680}0_1$ the first complete mutable number computed in the analysis below possesses 728,842,241 separately distinguishable digit sequences, a rather alarming 2,872,569,447,246,608,640 digit sequence permutations in total! and in its 'ground state' forms a twenty column mutable number.

MBN: $2_2 0_2 0_2 0_2 0_2 0_3 0_3 0_3 0_3 0_3 0_3 0_3 0_3 0_3 0_3 0_3 0_3 0_3 0_3 0_3 0_1$ $(2^5 . 3^{15})$

Trying to handle such detail could be an interesting challenge (perhaps one for a mathematician or computer programmer) but might add little to the gist of the analysis. Indeed, the scope for intermediate level nested series to roam, presented by such an abundance of digit sequences, is prodigious. Notwithstanding, by using only the one, upper, nested series in the analysis, much mind-bending detail is avoided, plus this single nested series may be represented musically in the score – as is done in the analysis.

Strict adherence to the symmetry of exchanges. In the past I have on occasion been swayed by the configuration of notes and partials into disregarding the precise symmetry of some exchanges in order to better reflect what appeared to be the essence of the harmonic progression or dominance of a particular connecting frequency. An example of this is given in measures one and two of the analysis below: Where three alternative exchanges are provided, out of which the least obvious conjunction frequency (D-h18) has been selected on mathematical grounds above the more visceral C-h16 or delicate E-h20. This swerve toward mathematical consistency introduces the possibility that some level of divergence can occur between what tonal composers have written and what tonal mathematics demands. This is only to be expected: Musicians working in the tonal idiom may create whatever sounds they wish, whatever music fits their purpose; and while tonal compositions will broadly, overall, conform to the principles of tonal number processing there will always be scope for individual preference and invention to stretch the music somewhat beyond the underlying logic of its mathematical underpinning. A matter to be rejoiced of (e.g. Example V, page 5).

Providing an 'absolute' mutable number analysis through setting the music higher, often very much higher, in the fundamental series, as discussed above. This is where an absolute interpretation means an analysis in which every nested series frequency/tone in the composition is a whole numbered ratio in the underlying fundamental harmonic series – and preferably the lowest one, set within the lowest sequence possible. A few of these absolute values are given at the foot of the analysis (e.g. MBN: $18_{25509168}0_1$ or 459,165,024 in decimal) where space permits. The complete whole number analysis for Bach's prelude is given in a table on page 28, and this format combined with a summary analysis attached under a score constitutes the current method for mutable number analyses.

For any extended, harmonically interesting, tonal composition the absolute numbers are likely to be enormous. In order to facilitate finding these solutions a Perl script is provided, `procalc.pl`, capable of handling large integer numbers (see page 27).

Unit 'decoupling' in absolute analyses. Following on from the previous points: In an absolute analysis, where all nested series rest upon whole numbered ratios in an all encompassing fundamental nesting series, the distance numerically between unit and the nested series' encapsulation of the composition will be large, often very large. Under these circumstances the unit will almost certainly lose any direct root-like or key-like relationship to the composition. This is because of the inevitable drift in pitch associated with the application of Just relationships (to the harmonic progression) becomes 'hardwired' into the whole numbered connecting points between the nested series and its underlying nesting series. Consequently the underlying nesting series – the enfolding 'space' and unit – becomes detached from the composition in terms of containing or expressing any particular tonal 'direction' or preference.

A broader interpretation of the role of the underlying fundamental nesting harmonic series by way of adopting equally all ratios in the underlying series – an unlimited range encompassing all the whole numbers. Developing further the comments above, the fundamental nesting series could be regarded as being an expansive 'space' within which the composition plays out its life, the arena within which it evolves under the direction of its own internal relationships from opening chord to final cadence. That is, taking all the fundamental harmonic series' extended resources as potential anchor points for scale degrees, rather than being obliged to tie down nested series to some fixed and lowly position within an underlying nesting series: With the scope of the fundamental nesting series thus expanded, the nested series, the embodiment of the composition, may manoeuvre without constraint from whole number to whole number, in steps of Just relationships.

Although at its outset the harmonic series can only awkwardly accommodate the scale degrees. It is as if the series in its lower ratios is akin to a new born creature, clumsy, angular and gauche; yet able with further extrapolation, in maturity, to develop into a scheme capable of arbitrary levels of refinement and detail. To begin with, between the ratios one and two, H1 and H2, the prime and octave, there is nothing, no detail. However, between two and four there is some structure, the fifth, H3. Between four and eight, the Just intonation major-third and minor-seventh emerge, H4, H5, H6 and H7. Continuing on beyond eight, the major-second and ninth are found, H9; and further on all the other scale degrees come into view – though some ratios (both before and after eight) are more ill-aligned than others with the degrees of the scale. But this is only the beginning, the awkward birth of a system containing, potentially, limitless detail and smooth transition. At every upward power of two advance the available options double. Each succeeding power of two generation in the harmonic series is doubly richer in relationships.

Thus it is possible to set a nested series within an encompassing fundamental nesting series to any desired level of refinement, in respect of the placement of scale degrees. Naturally, the effect of this is, that the system will produce progressively larger mutable numbers while simultaneously causing progressively declining amounts of disturbance to the fundamental unit. Until at some point of extrapolation, every ratio in the sweep of the nested series through a composition matches precisely some ratio within its underlying nesting series.[1]

To illustrate the point, Figure S3.2 provides a diagrammatic representation of a range of potential anchor points for setting nested series within any enfolding fundamental series. The diagram presents the first ten power-of-two generations or 'slices' in a fundamental nesting series; tracing the series from the bare octave between H1 and H2, up to the level of H1024 where a fair approximation of the equal-tempered scale may be obtained. Horizontal dashed grids define each generation of the series, labelled left and right, while the vertical dashed grids mark an equal-tempered scale labelled top and bottom in cents. The upper labels give the precise cent values for the note-ratios presented between H512 and H1024. The black points on the diagram represent note-ratios in a fundamental series and are labelled with their note-ratio number. Grey points are power-of-two multiples of note-ratios occurring earlier in the sequence. An interesting side-effect of opening up to later, more prolific, generations in the harmonic series is that it becomes feasible to make mutable number analyses of tonally organised music based on a broad range of other scales and modes.

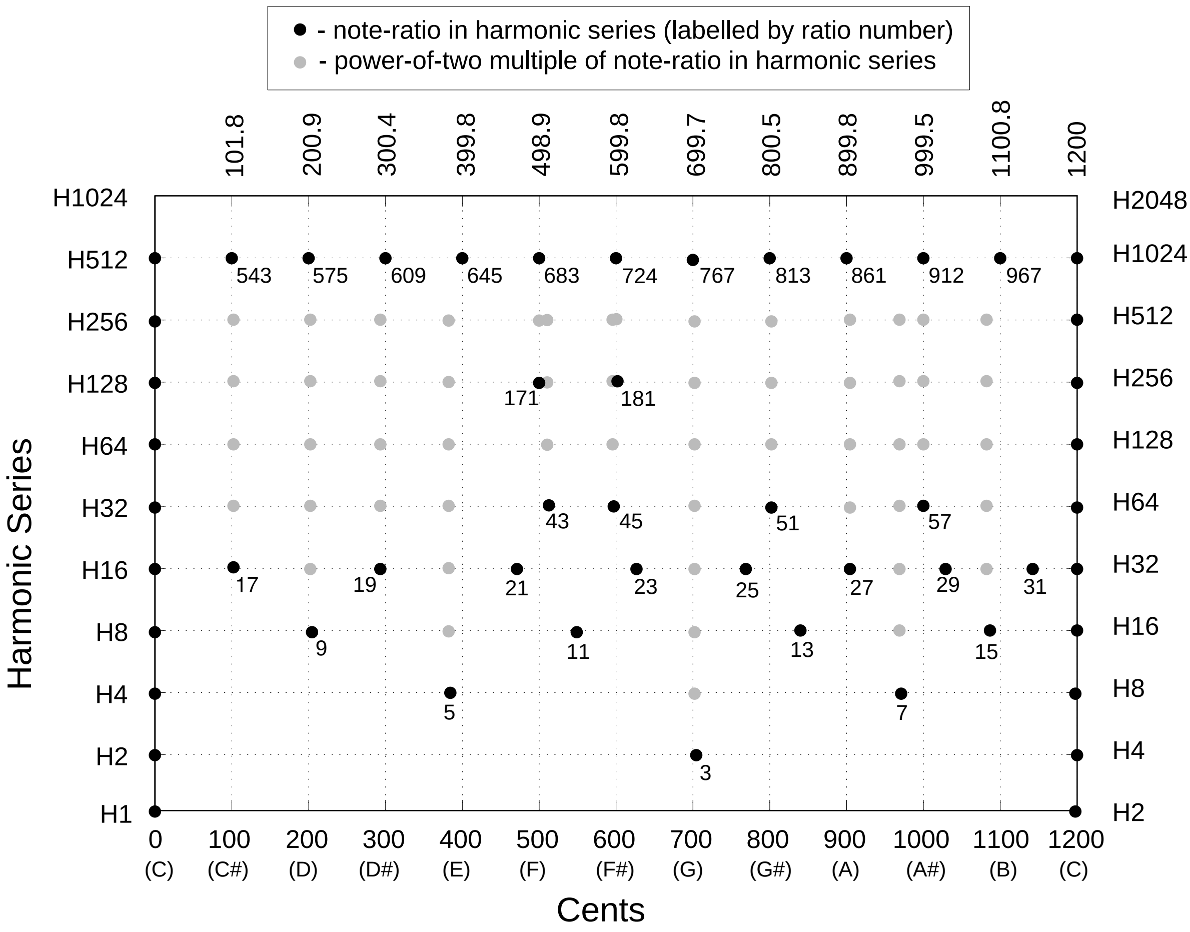


Figure S3.2 Schematic representation of ten power-of-two generations of the harmonic series with note-ratios labelled by their number and positioned within their generation.

Ultimately all the note-ratios of the harmonic series are available for use in the formation of mutable numbers – for whatever purpose. Some choices of note-ratios, as nested (h1) fundamental tones, will disturb their underlying series and unit more than others, and some will produce larger magnitudes than others. Numerically speaking there is no limit to the level to which a nested series may be raised within its enfolding parent; and thus, ultimately, any tonal composition moving in Just proportions can find a position within its all-encompassing nesting series cast from start to finish entirely in integer values. At this point of complete integration between nested and nesting series, where the motion of the composition through the 'space' provided by its underlying fundamental nesting series meets no impediment of mismatched ratios, no unit perturbed, the music is solely the product and expression of its own internal relationships – relationships encapsulated by Mutable Base Numbers.

Harmonic Series from the Top Down

The ideas presented in the preceding paragraphs and illustrated in Figure S3.2, concerning the relationship between a nested and nesting series, make this the right moment for something of a digression about the advantages of looking the wrong way through a telescope! That is to say, viewing the harmonic series from a top down perspective, which is the way tonal compositions are treated as 'driving' mutable base numbers from their most significant column and digit. Such a top down approach allows the structure of a 'fundamental' series to be viewed as being made up of innumerable interwoven threads consisting of each and every possible nested series – of which the prime ratio based nested series are particularly significant. An idea explored a little in *The Math of Exchange* with the aid of the Perl scripts `ths.pl` and `ths2.pl`. The output from these programs illustrate the way in which the individual strands of nested series, emanating ultimately from each prime ratio, intermingle and 'hybridise'.

For example H15, the first appearance of natural the major seventh interval in the harmonic series. From the point of view of G-h1[H3] in the underlying series B-H15 looks like B-h5 (Figure S3.1) and equally viewed from E-h1[H5] it appears to be B-h3. But turning the telescope around and looking downward from H15 the view becomes that of two nested harmonic series sharing the common connecting frequency H15. That is H15 is a point of conjunction, the first of many (e.g. H30, H45, H60,...), between these two nested series.

Extending this example further is illuminating. The awkward tritone relationship (C–F# in Figure S3.2) is ill-served by the opening note-ratios of the harmonic series at F#H11 and/or F#H23. However, one generation later at F#H45 the extrapolation of the nested series built on G-H3 and E-h5 amalgamate to provide a better placed solution: which is equivalent to the Just $^{45}/_{32}$ interval and approximately 590 cents. Carrying the process along a further two generations in Figure S3.2, these two nested series again united at F#H180, next to which lies the prime ratio H181, quite a close approximation to the equal-tempered F#, should that be required. ($^{181}/_{128} = 1.4140625$)

Similarly for the sub-dominant tone first set at the uncomfortably flat F-H21 position and often viewed as a particularly difficult interval to derive from the harmonic series. However given a more extensive underlying nesting series, H3 and H19 conspire together in the production of a well placed, near-tempered fourth at F-H171 – illustrated in Figure S3.2. ($^{171}/_{128} = 1.3359375$ equal-tempered fourth $= 1.33484$ approx.) While the nested series built on H5 and H7 come together at F-H85 (and F-H170) to yield a workable sub-dominant tone a little below the natural fourth. ($^{170}/_{128} = 1.328125$ pure fourth $^{4}/_{3} = 1.333...$)

Thus the awkward mismatch of some scale degrees with the opening ratios in the harmonic series are gradually eroded away by ever larger mutable base numbers.

The Example Analysis

A description of the mutable number analysis follows, each element in the analysis is introduced below and the elements are described in sequence as they occur from top to bottom of the pages.

Example: the Numbers

Ranging across the top of the pages are 'summary' Mutable Base Numbers encapsulating the tonal computation being conducted in the music immediately below them. The harmonic progression from chord to chord, in relationships of simple Just proportions, drives the number processing forward by impelling addition, subtraction or equality in order to satisfy the symmetry of the exchange being conducted. That is, to bring the exchange between adjacent nested harmonic series into balance with the harmonic progression of the chords they enfold (e.g. V–I, 3:2 or 3:4) the preceding series in the exchange may (or may not) be forced to acquire or shed notes/ratios/digits in order to find a point where their number is proportionally commensurable. The necessity of finding this 'conjunction' – in order that the symmetry of the exchange is respected – lies at the heart of tonal number processing by means of mutable base numbers. And though the numbers computed are relatively trivial, it is of course the beauty and order of the sound that we esteem.

Example: the Staves

Toward the top of the example are two double staves. The upper contains the score of Bach's Prelude. The lower double staff displays the nested harmonic series which embody this music. Highest lie the conjunction frequencies in diamond-headed notes, and below the conjunctions the body of the nested series encompassing the harmonic progression of the piece. These nested harmonic series are not fully populated with every note-ratio in their respective series. They contain all the note-ratios from their nested fundamental frequency (h1) up to the bass note of the composition – written in whole notes; thereafter, only note-ratios corresponding to the written music are notated (in half-notes) and then finally these nested harmonic series are crowned with their diamond-headed conjunction note-ratio. This conjunction note-ratio is either a top written note or a low order harmonic of any one, or more, of the written notes. These conjunction frequencies are scientifically observable facts as much as any of the written notes, indeed they will normally possess as much or more energy than the written note frequencies themselves.

However in contrast, while there may be some hints of the frequencies below the written bass notes of the Prelude, perhaps generated by the formant of instruments and accidents of acoustic resonance; for the most part what lies below this level is a mathematical contrivance implied by the observable fact of the music itself – but not physically present. The 'frequency' values of this underlying domain of mathematics is represented on the lower bass staff in whole notes. For the first ten measures the values are written out fully as 'chords' and afterward, from measure eleven, abbreviated. In measures five and nine a trailing placement of natural signs is employed to indicate 'notes' that appear both in their natural and sharpened form.

Example: the Analysis

The opening pitch of the piece is set at the slightly lower scientific pitch – middle C equal to 256Hz. Throughout the analysis, all subsequent numbers are determined, sequentially, with Just intervals (to a level of accuracy) leading to a consequent instability in pitch.

The first numbers encountered below the two double staves are labels attached to the nested fundamental tones e.g. C-h1(H256). The note letter identifies the nested fundamental tone (h1) and the number in parentheses gives its position with in the encompassing summary nesting series, e.g. (H256). Mid-measure a Roman numeral (Weberian) chord identifier is given – taken from R.F. Goldman's analysis presented in his insightful book *Harmony in Western Music*. Each succeeding measure is thus labelled.

Immediately below these labels the frequency levels of the various conjunctions are drawn out in grey bands across the page. Within these bands are written the conjunction-ratios to which the bands correspond e.g. D-h18*, with their frequencies in hertz enclosed in brackets e.g. (1152Hz). Large figures in square brackets also appear above or below these grey conjunction bands relating to the absolute values of the mutable numbers when emanating from a fixed (composition length) unit. More on this absolute interpretation below.

Flowing down from these grey conjunction bands and explicit frequencies are columns delineating the nested harmonic series (matching the notes on the second double staff) in letters and numbers. The highest note-ratio(s) in these columns represent the conjunction frequency, marked by asterisks for harmonics (of written notes) and a tilde for an objective written note. All lower written notes are also marked with a trailing tilde.

Continuing downward into the lower reaches of the nested harmonic series relating to each chord. Here the note-ratios are listed without asterisk or tilde and in a fainter type to indicate there non-objective status. These note-ratios correspond to the whole notes written on the lower bass staff, for measures one to ten they are shown in full but on many subsequent pages these non-objective note-ratios are abbreviated to chains of ‘root’ octaves with arrows, for want of space and convenience.

At the base of the nested series list the nested (h1) fundamental tone is set out again as in the label first encountered, e.g. C-h1(H256) plus a further square bracketed number. In parentheses below this nested (h1) fundamental tone, its summary frequency is given in hertz, e.g. 64Hz for the opening chord. The square bracketed number represents the ‘note-ratio’ in an absolute enfolding nesting series (e.g. [H25509168]) capable of nesting the entire composition in whole numbers from a single, unchanging, ‘super unit’.

Next below is the proportion of exchange between the current and succeeding chord/nested series. For example, ‘ --16:9 --> ’ which will carry the opening C major chord to that of D minor7th in measure two. The precise multiple of each party to the proportion, e.g. 16:9 or 8:9 or 8:18 etc., is predominantly determined by constructing individual nested harmonic series in the shortest form capable of expressing all the written notes of the chord it encompasses – following Occam’s principle. Similarly, the strict adherence to simple Just intervals is based on the premiss that these are the relationships extracted by human aural cognition from the approximations of equal-temperament and variability of performance. (There are also some issues around the minor third and occasionally some other intervals, see *Math of Exchange,* page 5.)

Finally, at the bottom of the list is the unit: H1, with its summary frequency in hertz, in the first chord F-H1 (0.333...Hz): The fundamental tone of the fundamental nesting series. In a practical ‘summary’ mutable number analysis this unit is not fixed, but subject to some degree of disturbance by the motion of the objective music as it negotiates the proportions of exchange between chords. It is an inevitable upshot of the process of applying Just intervals of exchange between the objective chords set within confined limits.

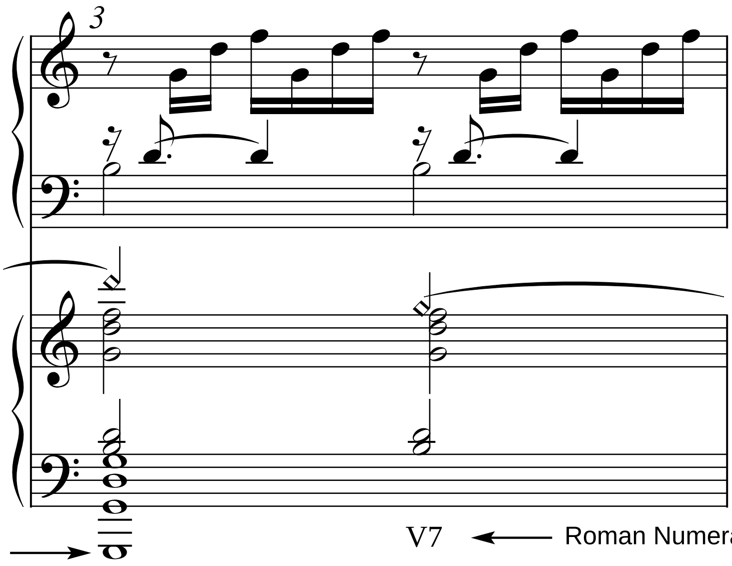
Interestingly here the unit is founded on the sub-dominant, on F-H1, rather than the tonic tone. Occasionally when making summary analyses it becomes apparent that the lowest viable sequence of positions for a nested series (encapsulating the music) to occupy within its enfolding fundamental series, is obtained by setting the summary unit (H1) to a scale degree other than tonic. My first and natural assumption in making these analyses, in general, has been to set the unit to the tonic frequency. However, regardless of initial preferences, the lowest practical summary analysis of this piece selects the sub-dominant as its unit. Thus in the analysis below of Bach’s Prelude in C major, each list begins with the summary unit: F-H1.

Alternatively, as discussed above, it is possible to extend the the fundamental series to such a degree, that every nested series in a composition is rooted on a whole number ratio within its enfolding fundamental nesting series. Consequently the unit may then be fixed throughout, with the mutable numbers generated becoming often extremely large. As for example in the exchange between the first and second chords in the Prelude producing, as noted above, the number MBN: $18_{255091680_1}$ which is near enough half a billion, 459,165,024 decimal. This is a mutable number with a unit possessing a period of approximately 20 days, 5 hours, 27 minutes and 6 seconds! If one craves the absolute (which I suppose I do) here it is.

(There is a further break down of these large mutable numbers at the foot of this document, pages 28–29.)

Example: Key to Analysis

Number processing (Summary Format) → (MBN) $= 24_{144}0_1 - 8_{144}0_1 = 16_{144}0_1$



Nested Harmonic Series

Nested Fundamental Tone → G-h1(H144)

V7 ← Roman Numeral Analysis

Objective Harmonics (black type & asterisk) & logical frequency 'hn'

Number processing

[whole ratio interpretation] Absolute Values → [459165024]

```
[459165024]
(1152Hz)  D-h24*  -8   [306110016]
          G-h16* --->     (768Hz)
          F-h14~
          D-h12~
          G-h8~
          D-h6~
          B-h5~
          G-h4
          D-h3
          G-h2
          G-h1(H144)[19131876]
          (48Hz)
          -- 3:2 -->
          F-H1
          (0.333...Hz)
```

Conjunction Frequency (numbered grey band)

Objective frequency in hertz (E.g. middle C = 256Hz)

Objective Notes & logical frequency 'hn' (black type & tilde)

Summary Nesting Ratio

Absolute Nesting Ratio

Implied underlying mathematical logic (in grey type)

Nested Fundamental Tone (h1) → G-h1(H144)[19131876]

Nested Fundamental Frequency → (48Hz)

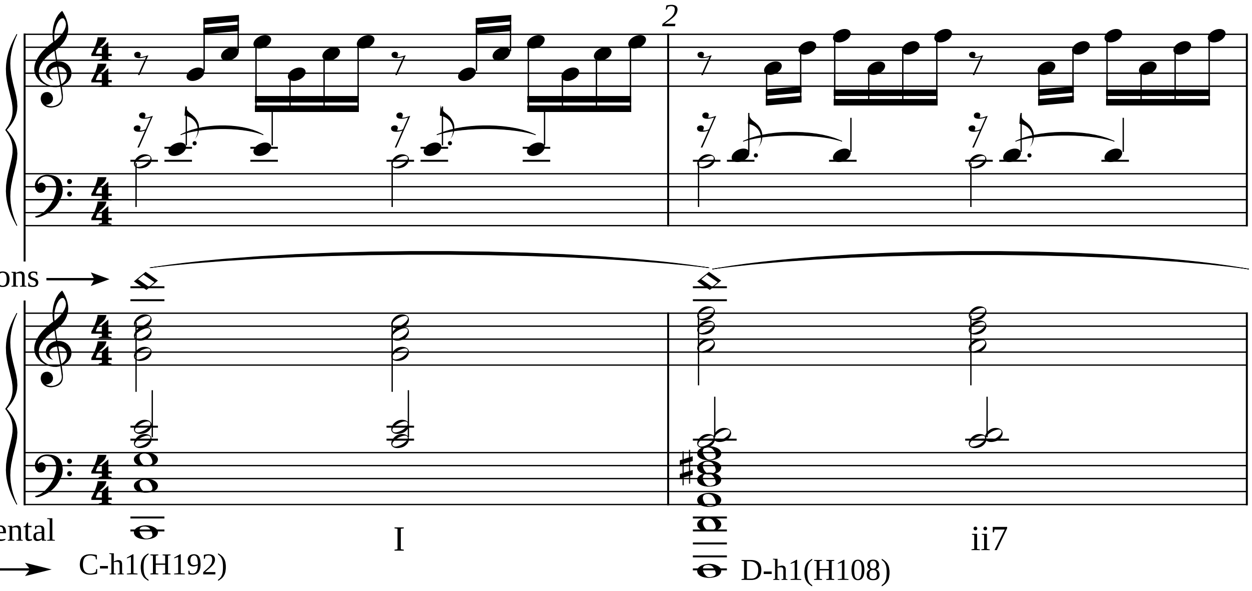
Proportion of Exchange → -- 3:2 -->

Summary Nesting Fundamental: the Unit F-H1

Summary Unit Frequency → (0.333...Hz)

Absolute Format MBN: $24_{19131876}0_1 - 8_{19131876}0_1 = 16_{19131876}0_1$

Summary Format Number → MBN: $18_{192}0_1$ = $32_{108}0_1$



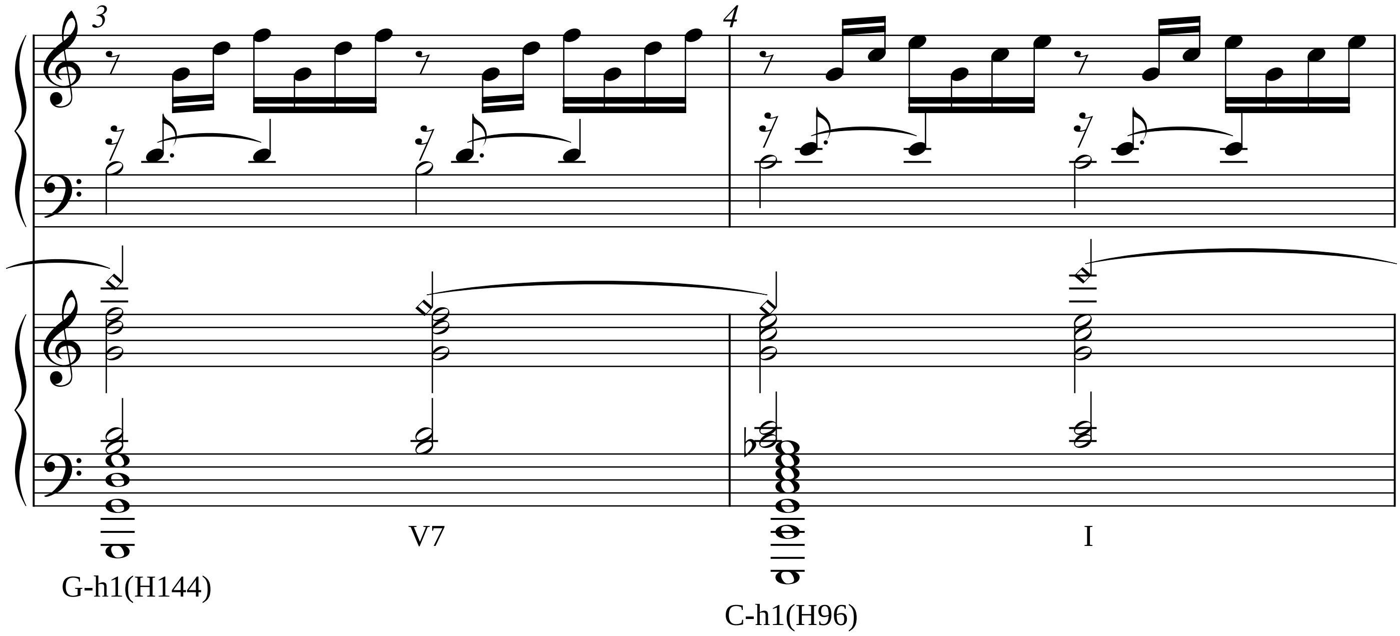
	[459165024]	
Conjunction:	D-h18* ---> (1152Hz)	D-h32* ---> (1152Hz)
	E-h10~	F-19~
	C-h8~	D-h16~
	G-h6~	A-h12~
	E-h5~	D-h8~
	C-h4~	C-h7~
	G-h3	A-h6
Nested	C-h2	F#h5
Fundamental	C-h1(H192)[25509168]	D-h4
Freq →	(64Hz)	A-h3
Proportion:	-- 16:9 -->	D-h2
Unit →	F-H1	D-h1(H108)[14348907]
Freq →	(0.333...Hz)	(36Hz)
		-- 3:4 -->
		F-H1(0.333...Hz)

Absolute Number → MBN: $18_{25509168}0_1$ ---------------> $32_{14348907}0_1$ --------------------->

Alternative Exchanges

	(1170.285714Hz)	(1280Hz)	(1137.777...Hz)
(1024Hz)	D-h32* --->	E-h20* -------->	E-h36* -4
C-h16* -------->	C-h28* +4	E-h10~	D-h32* --->
E-h10~	F-h19~	C-h8~	F-h19~
C-h8~	D-h16~	G-h6~	D-h16~
G-h6~	A-h12~	E-h5~	A-h12~
E-h5~	D-h8~	C-h4~	D-h8~
C-h4~	C-h7~	G-h3	C-h7~
G-h3	A-h6	C-h2	A-h6
C-h2	F#h5	C-h1(H192)	F#h5
C-h1(H192)	D-h4	(64Hz)	D-h4
(64Hz)	A-h3	-- 18:10 -->	A-h3
-- 14:8 -->	D-h2	F-H1(0.333...Hz)	D-h2
F-H1(0.333...Hz)	D-h1(H108)		D-h1(H108)
	(36.57142857Hz)		(35.555...Hz)
	-- 3:4 -->		-- 3:4 -->
	F-H1		F-H1
	(0.338624339Hz)		(0.329218107Hz)

$= 24_{144}0_1 - 8_{144}0_1 = 16_{144}0_1$ $= 24_{96}0_1 + 16_{96}0_1 = 40_{96}0_1$




```
[459165024]                                                [510183360]
 (1152Hz) D-h24*   -8   [306110016]     E-h40* --->       (1280Hz)
          G-h16* --->     (768Hz)       G-h24* +16
          F-h14~                        E-h20~
          D-h12~                        C-h16~
          G-h8~                         G-h12~
          D-h6~                         E-h10~
          B-h5~                         C-h8~
          G-h4                          A#h7
          D-h3                          G-h6
          G-h2                          E-h5
          G-h1(H144)[19131876]          C-h4
          (48Hz)                        G-h3
          -- 3:2 -->                    C-h2
          F-H1(0.333...Hz)              C-h1(H96)[12754584]
                                        (32Hz)
                                        -- 12:5 -->
                                        F-H1(0.333...Hz)
```

---------> $= 24_{19131876}0_1$

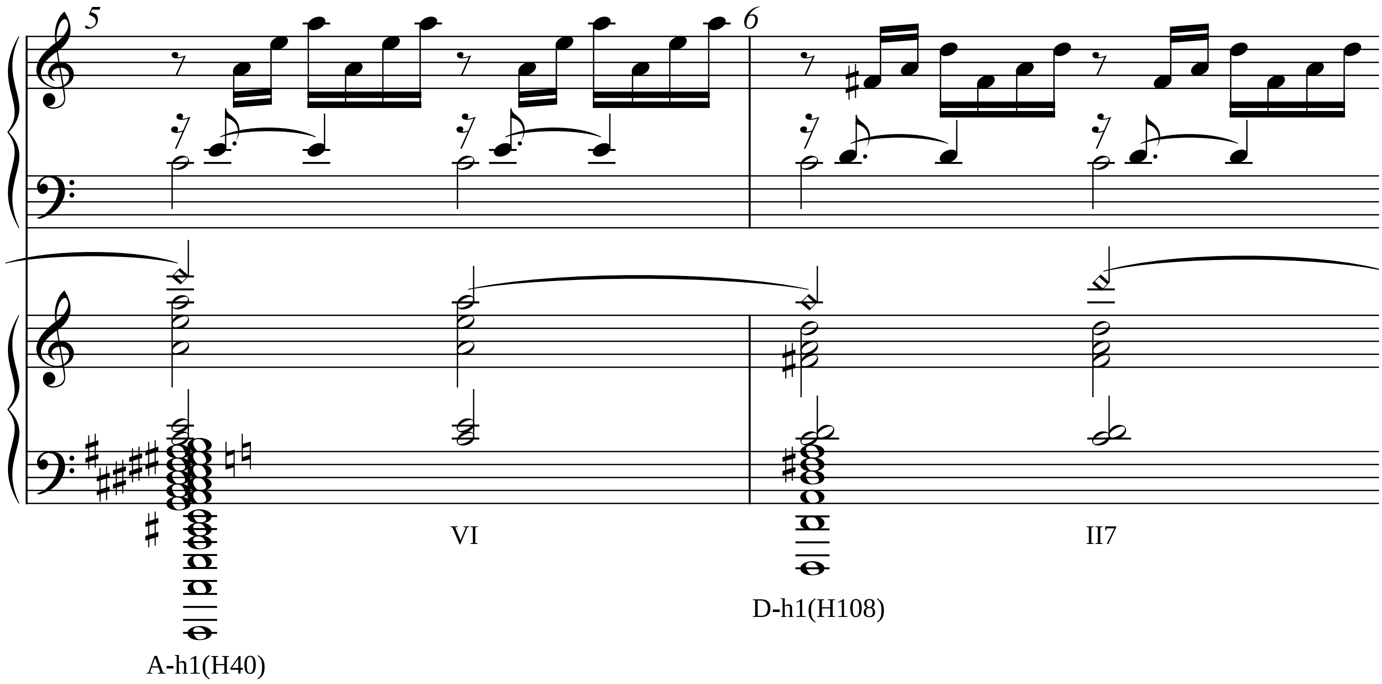
$- \ 8_{19131876}0_1$

$= 16_{19131876}0_1$ -------------> $= 24_{12754584}0_1$

$+ 16_{12754584}0_1$

$= 40_{12754584}0_1$ --------------->

$= 96_{40}0_1 - 32_{40}0_1 = 64_{40}0_1$ $= 24_{108}0_1 + 8_{108}0_1 = 32_{108}0_1$



[510183360]
(1280Hz) E-h96* -32 [340122240]
A-h64~ ---> (853.333Hz)
E-h48~
A-h32~
E-h24~
C-h19~
B-h18
A#h17
A-h16
G#h15
G-h14
F#h13
E-h12
D#h11
C#h10
B-h9
A-h8
G-h7
E-h6
C#h5
A-h4
E-h3
A-h2
A-h1(H40)[5314410]
(13.333...Hz)
-- 3:8 -->
F-H1
(0.333...Hz)

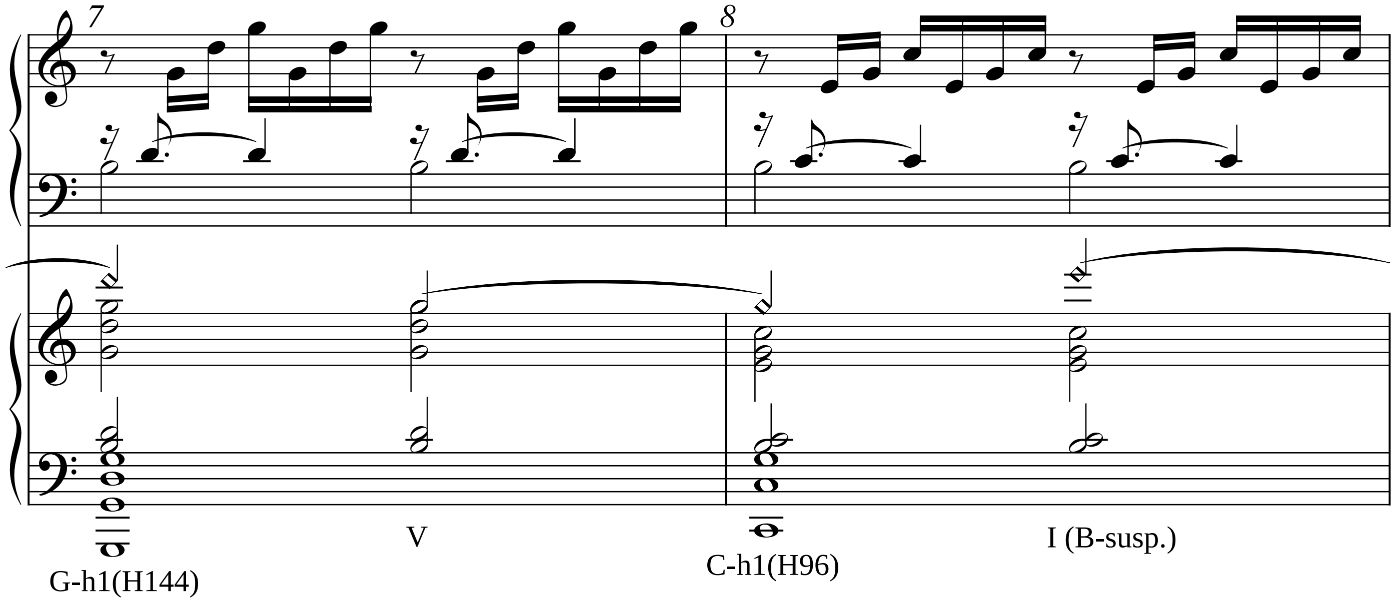
[453496320]
D-h32* ---> (1137.777Hz)
A-h24* +8
D-h16~
A-h12~
F#h10~
D-h8~
C-h7~
A-h6
F#h5
D-h4
A-h3
D-h2
D-h1(H108)[14171760]
(35.555Hz)
-- 3:4 -->
F-H1
(0.329218107Hz)

----------> $= 96_{5314410}0_1$ $= 32_{14171760}0_1$ ------------------->

$- 32_{5314410}0_1$ $+ 8_{14171760}0_1$

$= 64_{5314410}0_1$ -----------------> $= 24_{14171760}0_1$

$= 24_{144}0_1 - 8_{144}0_1 = 16_{144}0_1$ $= 24_{96}0_1 + 16_{96}0_1 = 40_{96}0_1$



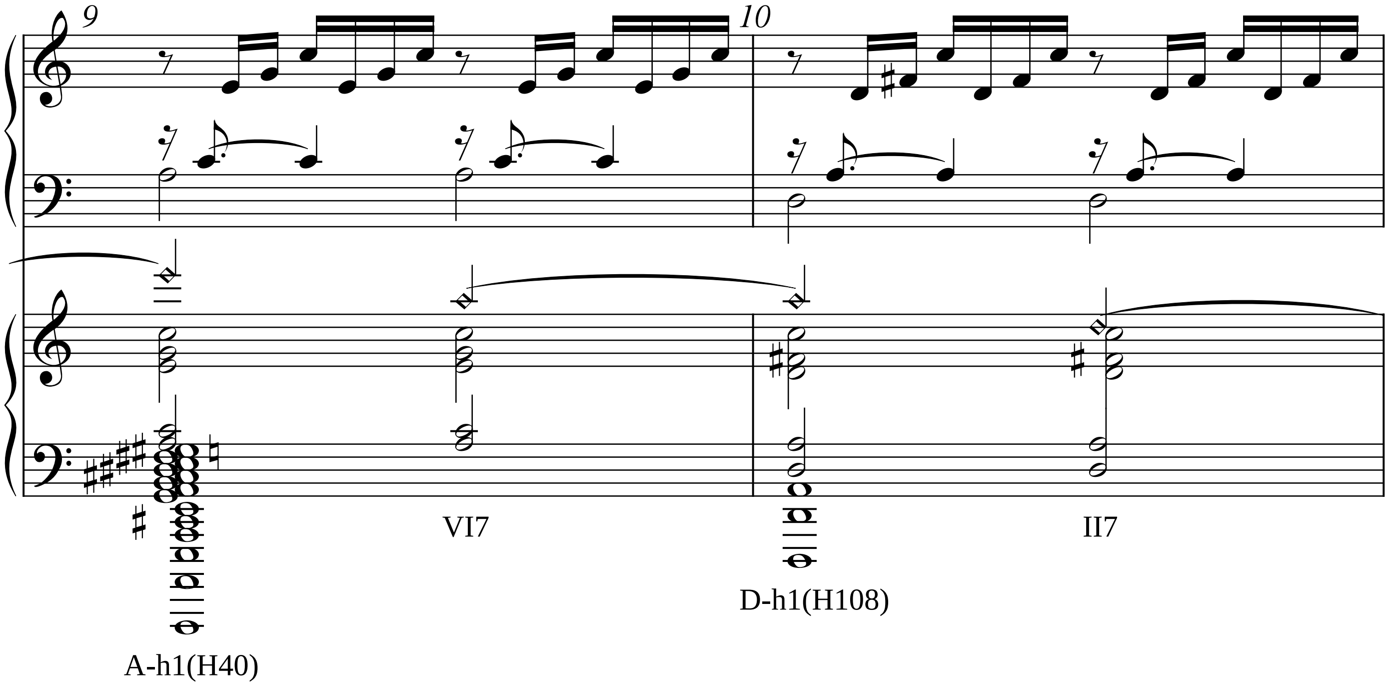
[453496320]
(1137.777Hz)

D-h24* -8 [302330880]
G-h16~ ---> (758.5185184Hz)
D-h12~
G-h8~
D-h6~
B-h5~
G-h4
D-h3
G-h2
G-h1(H144)[18895680]
(47.4074074Hz)
-- 3:2 -->
F-H1
(0.329218107Hz)

[503884800]
E-h40* ---> (1264.197531Hz)
G-h24* +16
C-h16~
G-h12~
E-h10~
C-h8~
B-susp.
A#h7
G-h6
E-h5
C-h4
G-h3
C-h2
C-h1(H96)[12597120]
(31.60493827Hz)
-- 12:5 -->
F-H1
(0.329218107Hz)

---------> $= 24_{18895680}0_1$
$- \ 8_{18895680}0_1$
$= 16_{18895680}0_1$ ----------------> $= 24_{12597120}0_1$
$+ \ 16_{12597120}0_1$
$= 40_{12597120}0_1$ ------------------>

$= 96_{40}0_1 - 32_{40}0_1 = 64_{40}0_1$ $= 24_{108}0_1 - 8_{108}0_1 = 16_{108}0_1$

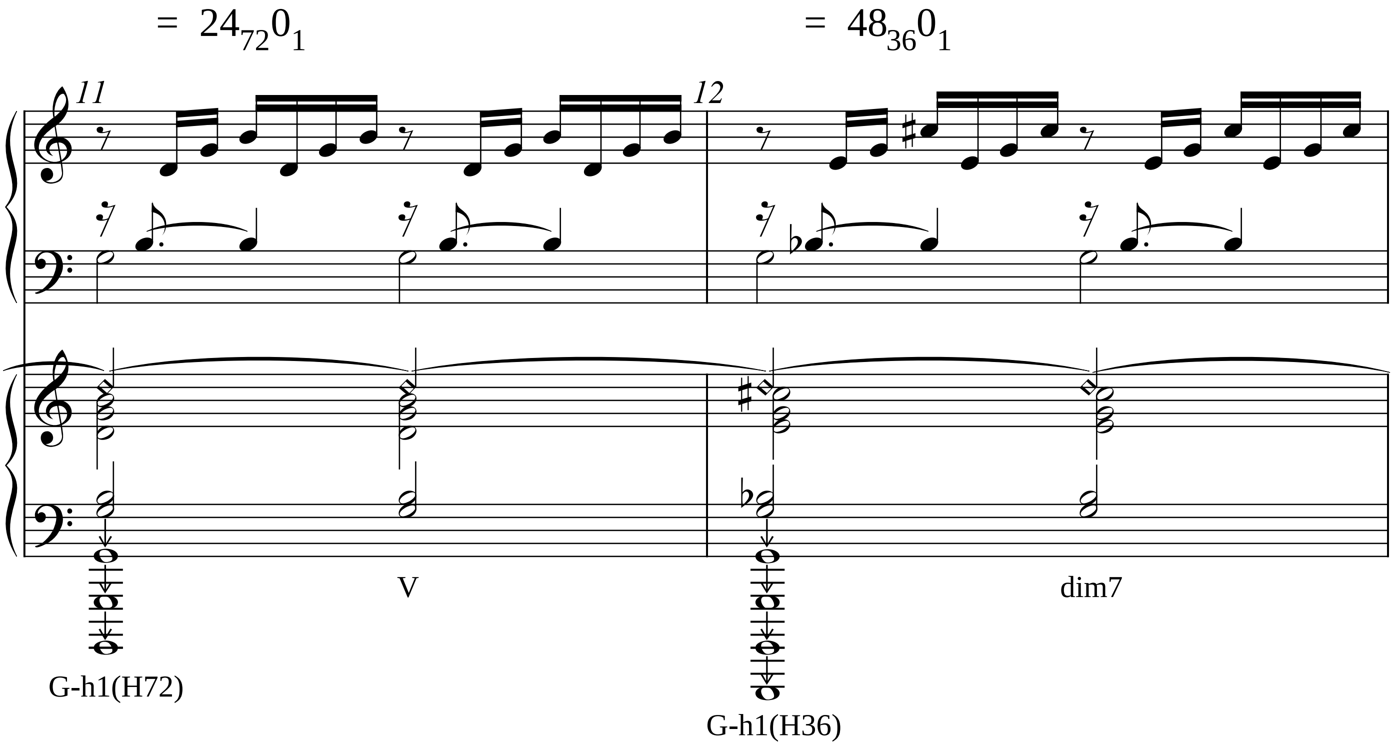



```
[503884800]
(1264.197531Hz)  E-h96*  -32     [335923200]
                 A-h64~  --->  (842.7983539Hz)  A-h24*   -8    [223948800]
                 C-h38~                         D-h16* ---> (561.8655691Hz)
                 G-h28~                         C-h14~
                 E-h10~                         F#h10~
                 C-h19~                         D-h8~
                 A-h16                          A-h6~
                 G#h15                          D-h4~
                 G-h14                          A-h3
                 F#h13                          D-h2
                 E-h12                          D-h1(H108)[13996800]
                 D#h11                          (35.11659807Hz)
                 C#h10                          -- 3:2 -->
                 B-h9                           F-H1
                 A-h8                           (0.325153686Hz)
                 G-h7
                 E-h6
                 C#h5
                 A-h4
                 E-h3
                 A-h2
                 A-h1(H40)[5248800]
                 (13.16872428Hz)
                 -- 3:8 -->
                 F-H1
                 (0.329218107Hz)
```

--------> $= 96_{5248800}0_1$

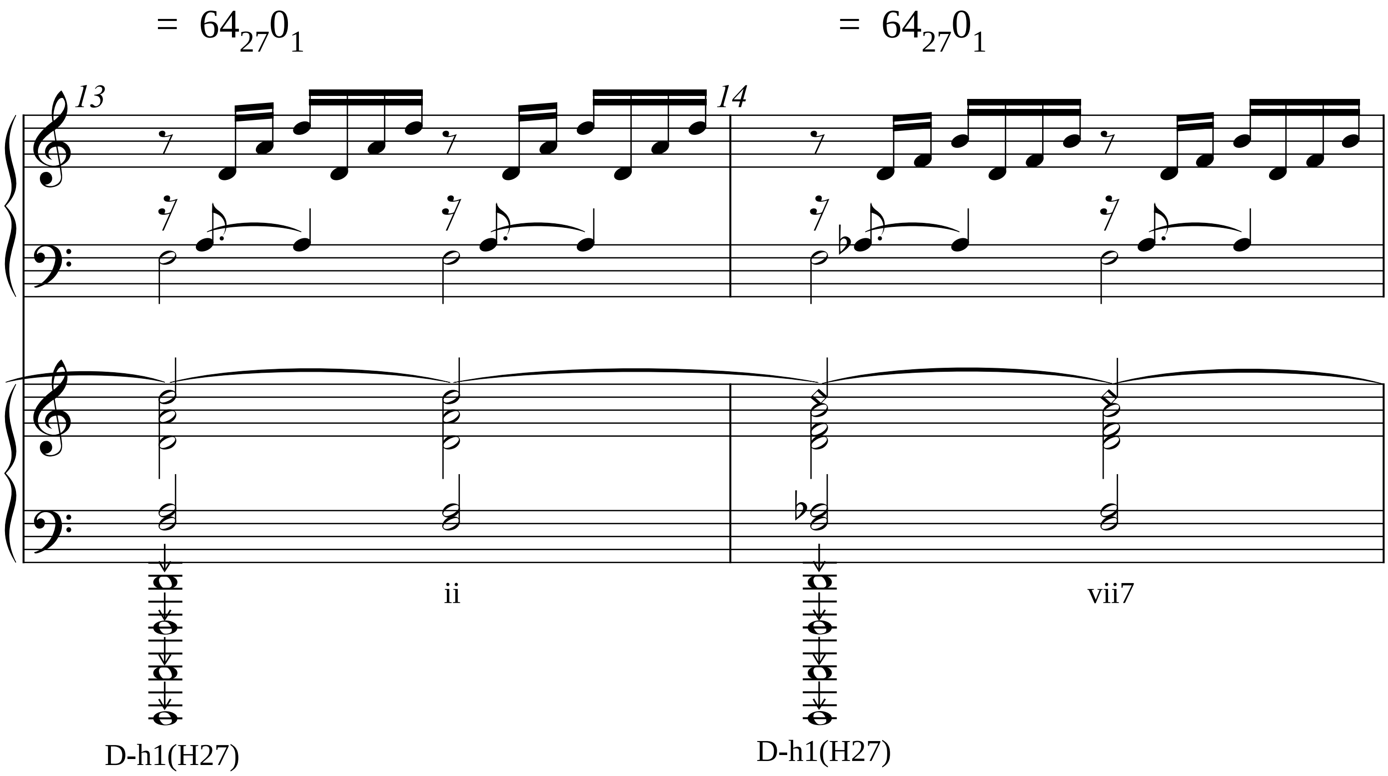
$- 32_{5248800}0_1$

$= 64_{5248800}0_1$ ----> etc... (see page 28 for the complete whole number analysis)




```
D-h24* --->     [223948800]
                (561.8655691Hz)
B-h20~
G-h16~
D-h12~
B-h10~
G-h8~
F-h7
D-h6
B-h5
G-h4
D-h3
G-h2
G-h1(H72)[9331200]
(23.41106538Hz)
-- 2:1 -->
F-H1
(0.325153686Hz)
```

```
D-h48* --->     [223948800]
                (561.8655691Hz)
C#h44~
G-h32~
E-h26~
A#h19~
G-h16~
F#h15
F-h14
E-h13
D-h12
C#h11
B-h10
A-h9
G-h8
F-h7
D-h6
B-h5
G-h4
D-h3
G-h2
G-h1(H36)[4665600]
(11.70553269Hz)
-- 4:3 -->
F-H1
(0.325153686Hz)
```



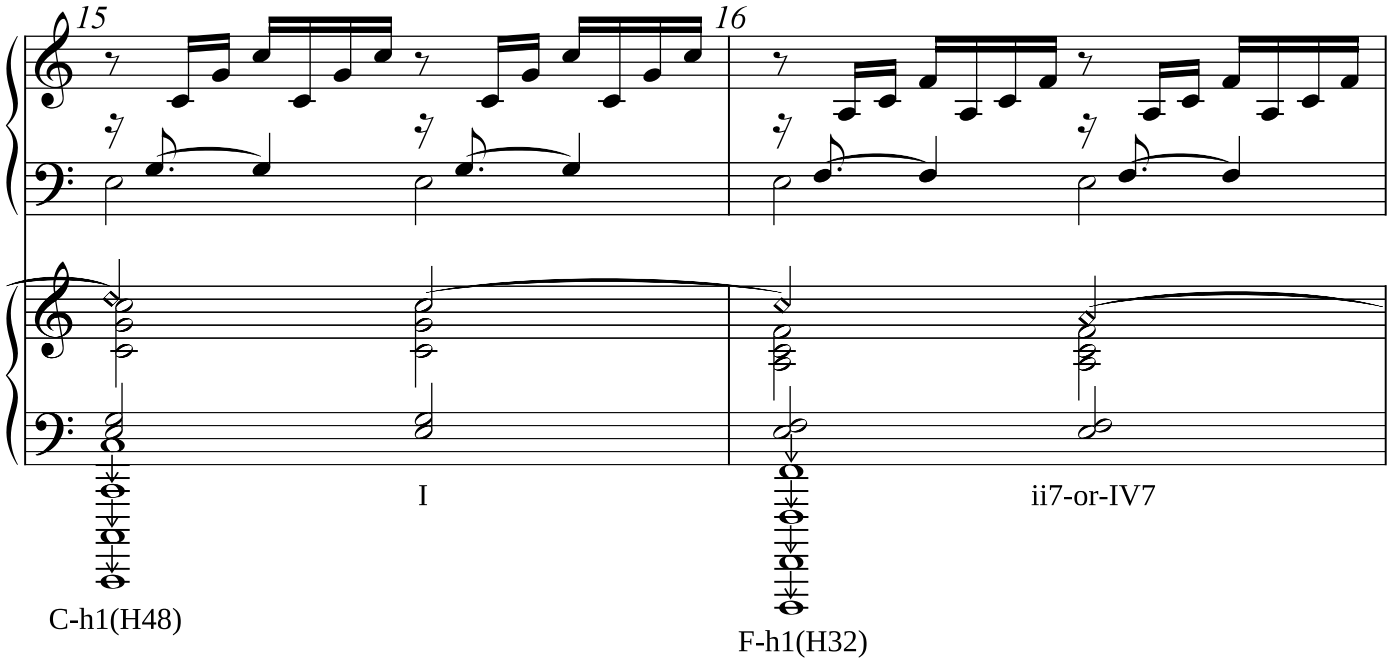
[223948800]

D-h64~ ---> (561.8655691Hz)
A-h48~
D-h32~
A-h24~
F-h19~
E-h18
D#h17
D-h16
C#h15
C-h14
B-h13
A-h12
G#h11
F#h10
E-h9
D-h8
C-h7
A-h6
F#h5
D-h4
A-h3
D-h2
D-h1(H27)[3499200]
(8.779149518Hz)
-- 1:1 -->
F-H1
(0.325153686Hz)

[223948800]

D-h64* ---> (561.8655691Hz)
B-h52~
F-h38~
D-h32~
G#h23~
F-h19~
E-h18
D#h17
D-h16
C#h15
C-h14
B-h13
A-h12
G#h11
F#h10
E-h9
D-h8
C-h7
A-h6
F#h5
D-h4
A-h3
D-h2
D-h1(H27)[3499200]
(8.779149518Hz)
-- 9:16 -->
F-H1
(0.325153686Hz)

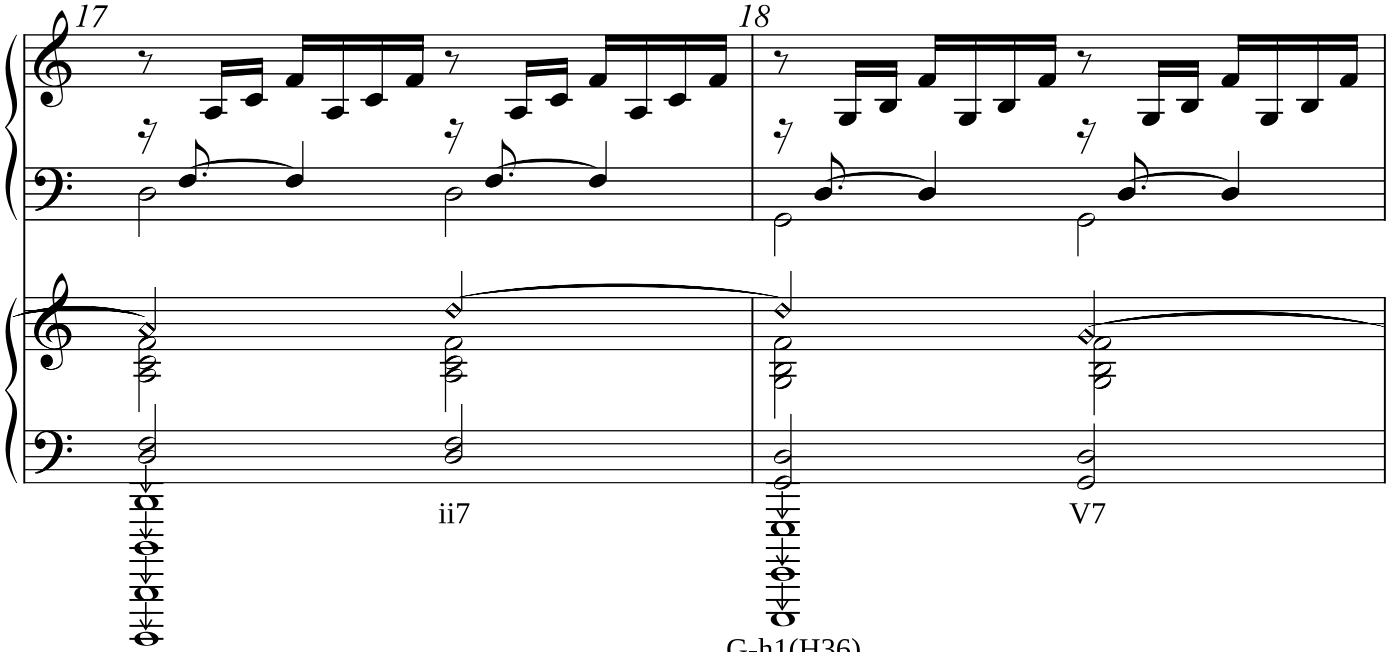
$= 36_{48}0_1 - 4_{48}0_1 = 32_{48}0_1$ $\qquad$ $= 48_{32}0_1 - 8_{32}0_1 = 40_{32}0_1$



[223948800]
(561.8655691Hz)
D-h36* -4 [199065600]
C-h32~ ---> (499.4360614Hz)
G-h24~
C-h16~
G-h12~
E-h10~
D-h9
C-h8
A#h7
G-h6
E-h5
C-h4
G-h3
C-h2
C-h1(H48)[6220800]
(15.60737692Hz)
-- 3:2 -->
F-H1
(0.325153686Hz)

C-h48* -8 [165888000]
A-h40* ---> (416.196719Hz)
F-h32~
C-h24~
A-h20~
F-h16~
E-h15~
D#h14
D-h13
C-h12
B-h11
A-h10
G-h9
F-h8
D#h7
C-h6
A-h5
F-h4
C-h3
F-h2
F-h1(H32)[4147200]
(10.40491795Hz)
-- 6:5 -->
F-H1
(0.325153686Hz)

$= 48_{27}0_1 + 16_{27}0_1 = 64_{27}0_1$ $\qquad$ $= 48_{36}0_1 - 16_{36}0_1 = 32_{36}0_1$



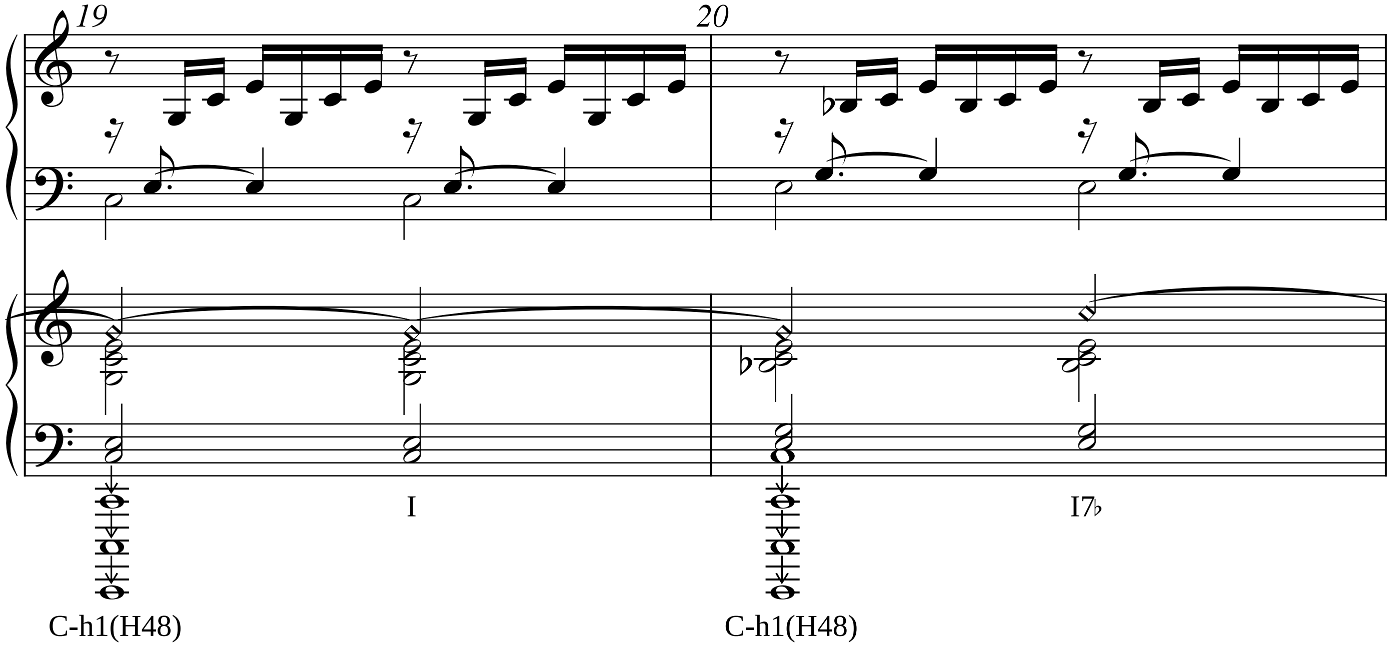
[165888000]
(416.196719Hz)

[221184000]
D-h64* ---> (554.9289573Hz)
A-h48* +16
F-h38~
C-h28~
A-h24~
F-h19~
D-h16~
C#h15
C-h14
B-h13
A-h12
G#h11
F#h10
E-h9
D-h8
C-h7
A-h6
F#h5
D-h4
A-h3
D-h2
D-h1(H27)[3456000]
(8.670764958Hz)
-- 3:4 -->
F-H1
(0.321139443Hz)

D-h48* -16 [147456000]
G-h32* ---> (369.9526382Hz)
F-h28~
B-h20~
G-h16~
D-h12~
G-h8~
F-h7
D-h6
B-h5
G-h4
D-h3
G-h2
G-h1(H36)[4608000]
(11.56101994Hz)
-- 3:4 -->
F-H1
(0.321139443Hz)

$= 24_{48}0_{1}$

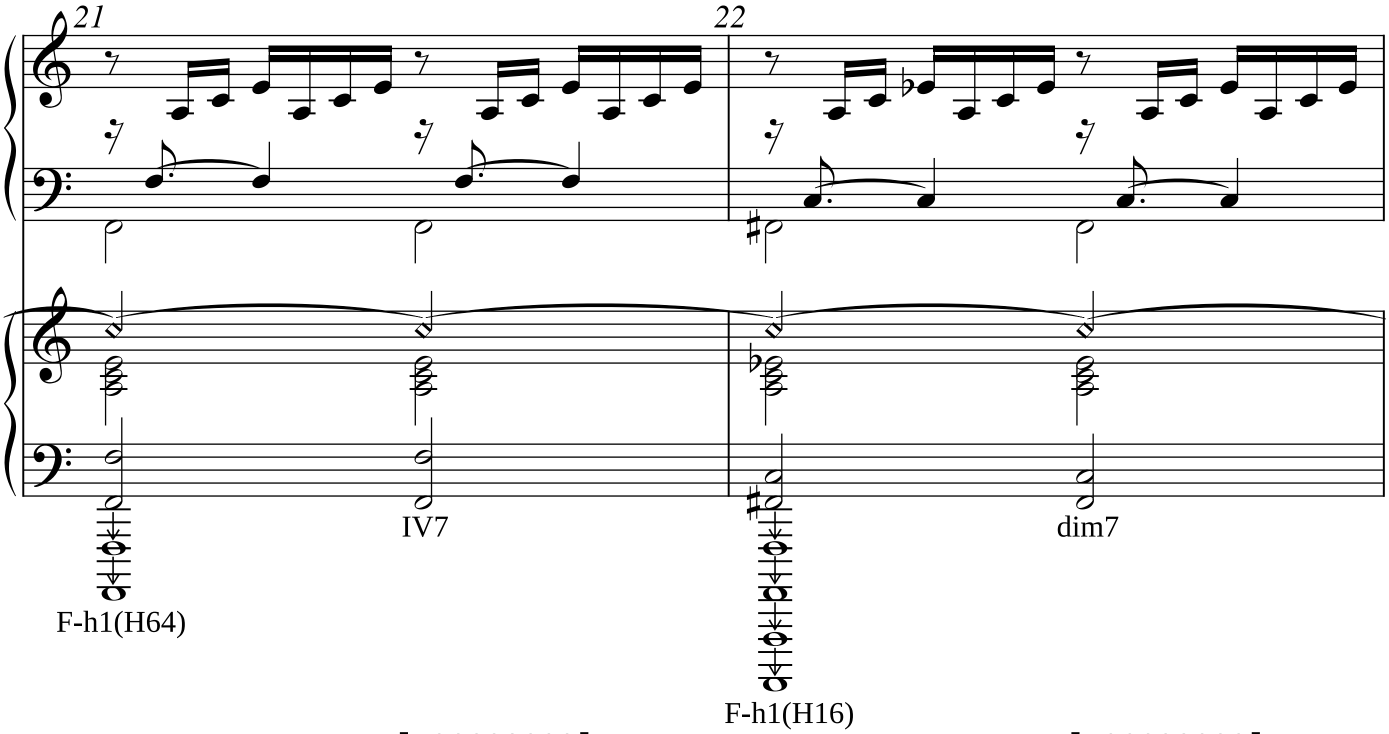
$= 24_{48}0_{1} + 8_{48}0_{1} = 32_{48}0_{1}$



[147456000]
G-h24* ---> (369.9526382Hz)
E-h20~
C-h16~
G-h12~
E-h10~
C-h8~
A#h7
G-h6
E-h5
C-h4
G-h3
C-h2
C-h1(H48)[6144000]
(15.41469326Hz)
-- 1:1 -->
F-H1
(0.321139443Hz)

[196608000]
C-h32* ---> (493.2701841Hz)
G-h24* +8
E-h20~
C-h16~
A#h14~
G-h12~
E-h10~
D-h9
C-h8
A#h7
G-h6
E-h5
C-h4
G-h3
C-h2
C-h1(H48)[6144000]
(15.41469326Hz)
-- 3:4 -->
F-H1
(0.321139443Hz)

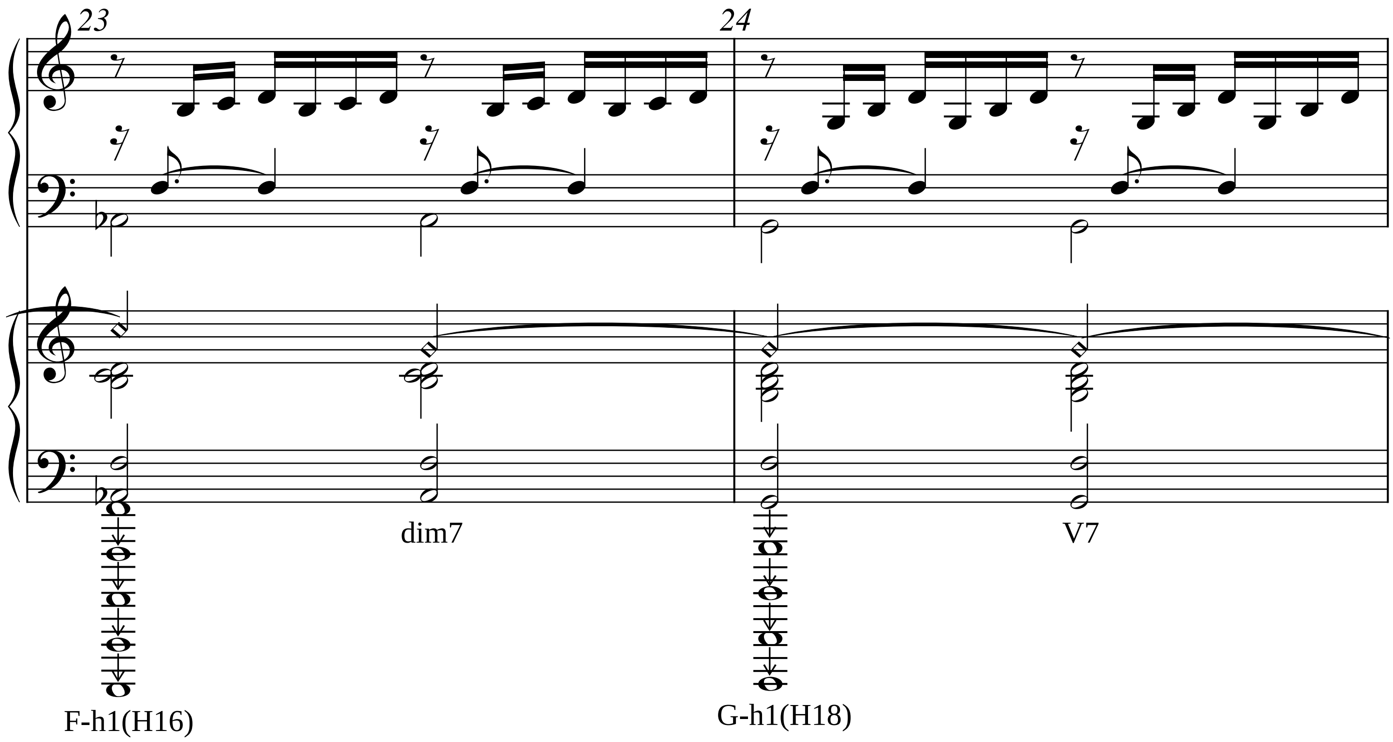
$= 24_{64}0_1$ $= 96_{16}0_1$



[196608000]
C-h24* ---> (493.2701841Hz)
E-h15~
C-h12~
A-h10~
F-h8~
F-h4~
C-h3
F-h2
F-h1(H64)[8192000]
(20.55292434Hz)
-- 4:1 -->
F-H1
(0.321139443Hz)

[196608000]
C-h96* ---> (493.2701841Hz)
D#h56~
C-h48~
A-h40~
C-h24~
F#h17~
F-h16
E-h15
D#h14
D-h13
C-h12
C#h11
A-h10
G-h9
F-h8
D#h7
C-h6
A-h5
F-h4
C-h3
F-h2
F-h1(H16)[2048000]
(5.138231087Hz)
-- 1:1 -->
F-H1
(0.321139443Hz)

$= 96_{16}0_1 - 24_{16}0_1 = 72_{16}0_1$ $= 64_{18}0_1$

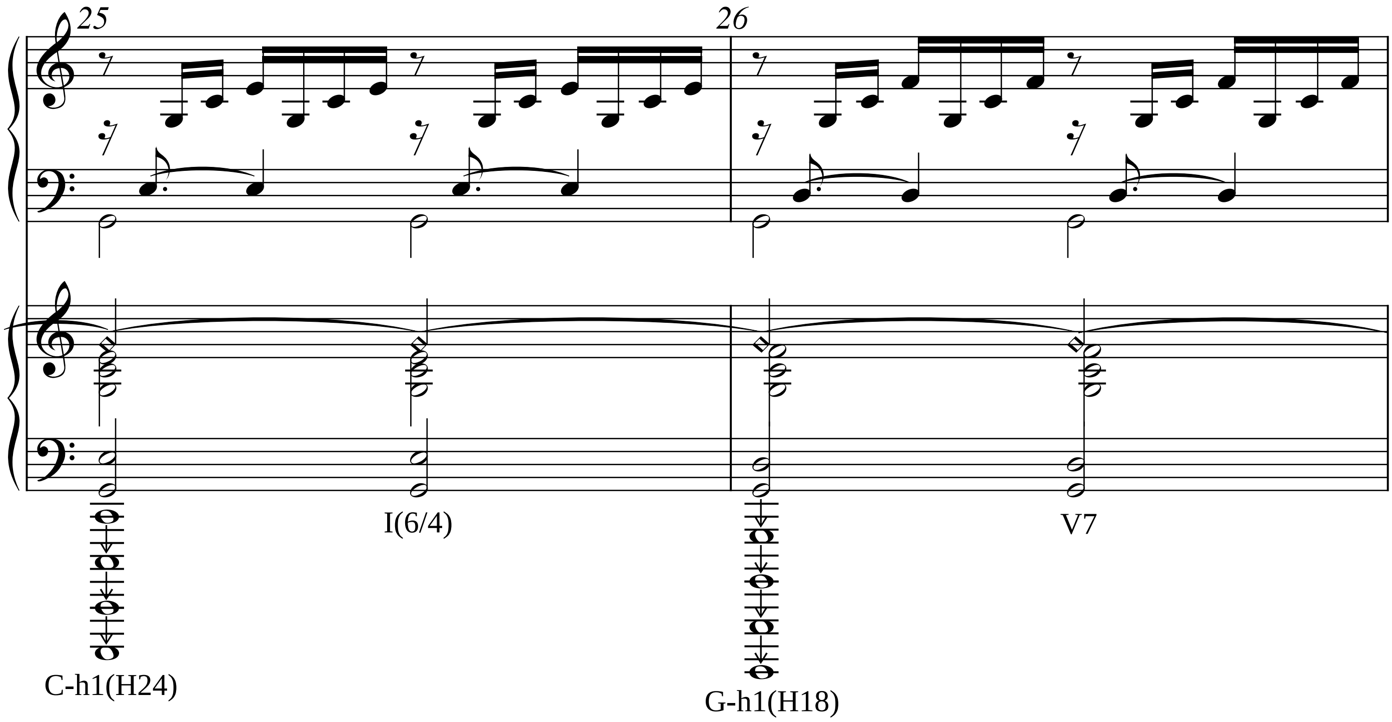


(493.2701841Hz)
[196608000]

C-h96* -24
G-h72* ---> (369.9526383Hz)
[147456000]
D-h54~
C-h48~
B-h44~
F-h32~
G#h19~
G-h18
F#h17
F-h16
E-h15
D#h14
D-h13
C-h12
C#h11
A-h10
G-h9
F-h8
D#h7
C-h6
A-h5
F-h4
C-h3
F-h2
F-h1(H16)[2048000]
(5.138231087Hz)
-- 8:9 -->
F-H1
(0.321139443Hz)

[147456000]
G-h64* ---> (369.9526383Hz)
D-h48~
B-h40~
G-h32~
F-h28~
G-h16~
F#h15
F-h14
E-h13
D-h12
C#h11
B-h10
D-h9
G-h8
F-h7
D-h6
B-h5
G-h4
D-h3
G-h2
G-h1(H18)[2304000]
(5.780509973Hz)
-- 3:4 -->
C-H1
(0.321139443Hz)

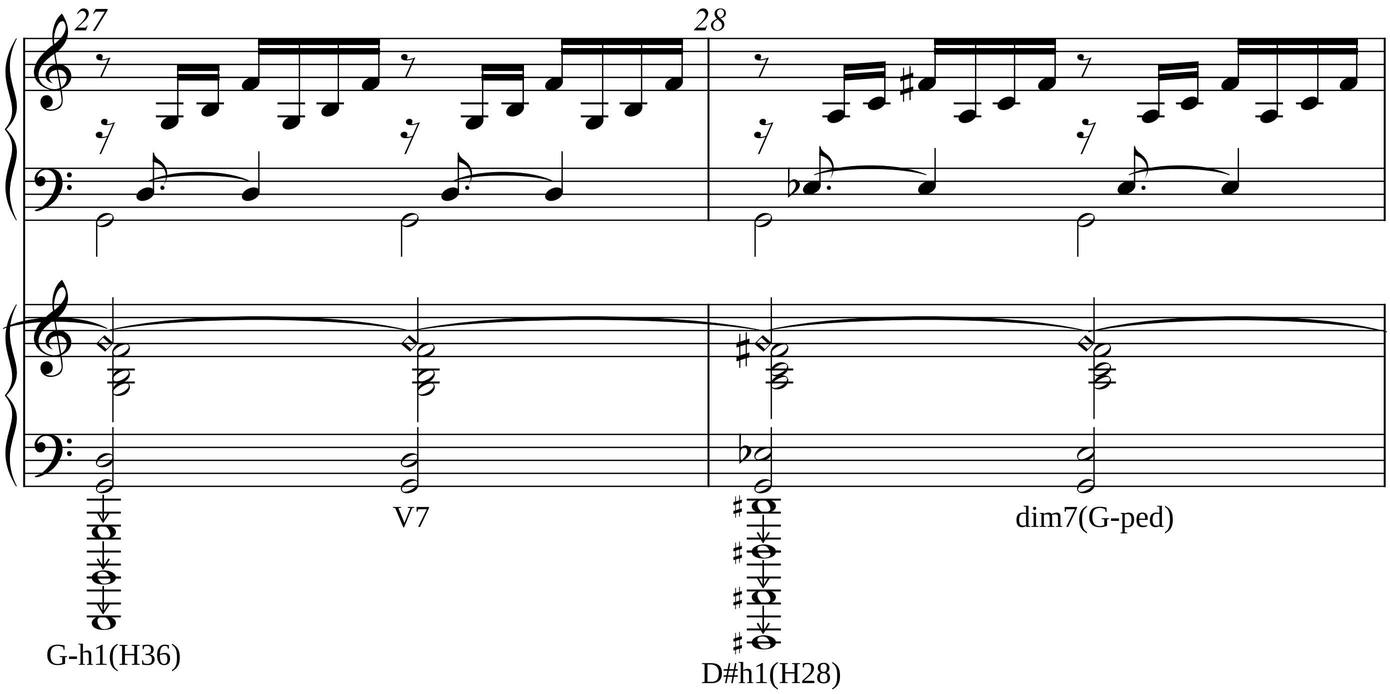
$= 48_{24}0_1$ $= 64_{18}0_1$



[147456000]
G-h48* ---> (369.9526383Hz)
E-h40~
C-h32~
G-h24~
E-h20~
G-h12~
F#h11
E-h10
D-h9
C-h8
A#h7
G-h6
E-h5
C-h4
G-h3
C-h2
C-h1(H24)[3072000]
(7.707346631Hz)
-- 4:3 -->
F-H1
(0.321139443Hz)

[147456000]
G-h64* ---> (369.9526383Hz)
F-h56~
C-h42~susp.
G-h32~
D-h24~
G-h16~
F#h15
F-h14
E-h13
D-h12
C#h11
B-h10
D-h9
G-h8
F-h7
D-h6
B-h5
G-h4
D-h3
G-h2
G-h1(H18)[2304000]
(5.780509973Hz)
-- 1:2 -->
F-H1
(0.321139443Hz)

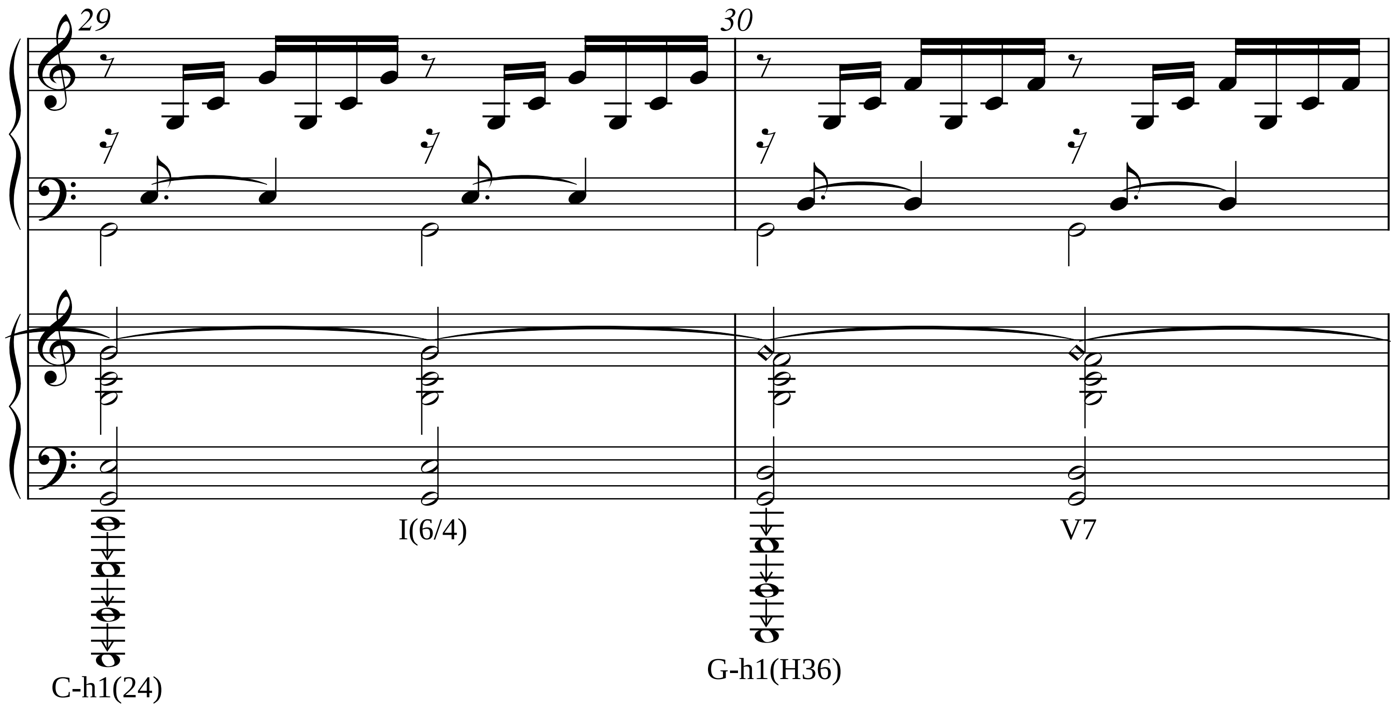
$= 32_{36}0_{1}$ $= 40_{28}0_{1}$



[147456000]
G-h32* ---> (369.9526383Hz)
F-h28~
B-h20~res.
G-h16~
D-h12~
G-h8~
F-h7
D-h6
B-h5
G-h4
D-h3
G-h2
G-h1(H36)[4608000]
(11.56101995Hz)
-- 5:4 -->
F-H1
(0.321139443Hz)

[147456000]
G-h40* ---> (369.9526383Hz)
F#h38~
C-h27
A-h22~
D#h16~
G-h10~
F-h9
D#h8
C#h7
A#h6
G-h5
D#h4
A#h3
D#h2
D#h1(H28)[3686400]
(9.248815957Hz)
-- 6:5 -->
C-H1
(0.330314856Hz)

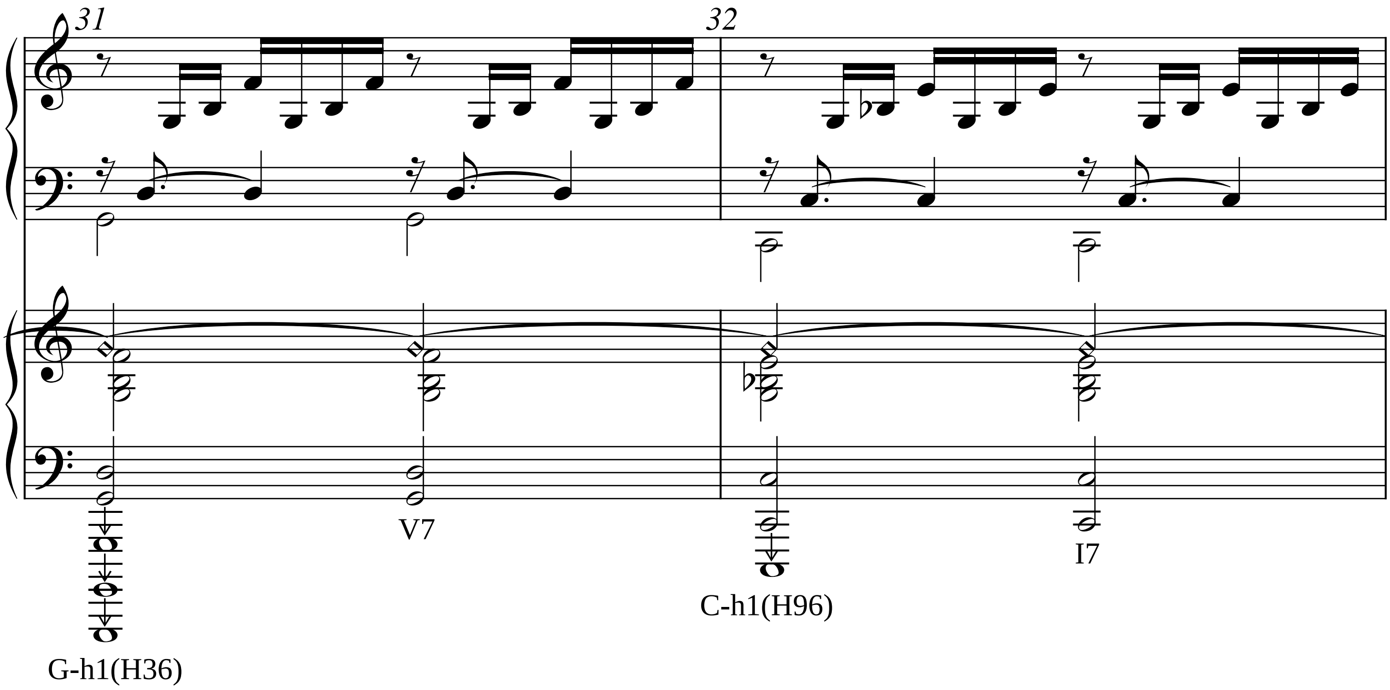
$= 48_{24}0_{1}$ $= 32_{36}0_{1}$



[147456000]
G-h48~ ---> (369.9526383Hz)
F-h45~
G-h24~
E-h20~
G-h12~
F#h11
E-h10
D-h9
C-h8
A#h7
G-h6
E-h5
C-h4
G-h3
C-h2
C-h1(H24)[3072000]
(7.707346631Hz)
-- 2:3 -->
F-H1
(0.321139443Hz)

[147456000]
G-h32* ---> (369.9526383Hz)
F-h28~
C-h21~susp.
G-h16~
D-h12~
G-h8~
F-h7
D-h6
B-h5
G-h4
D-h3
G-h2
G-h1(H36)[4608000]
(11.56101995Hz)
-- 1:1 -->
F-H1
(0.321139443Hz)

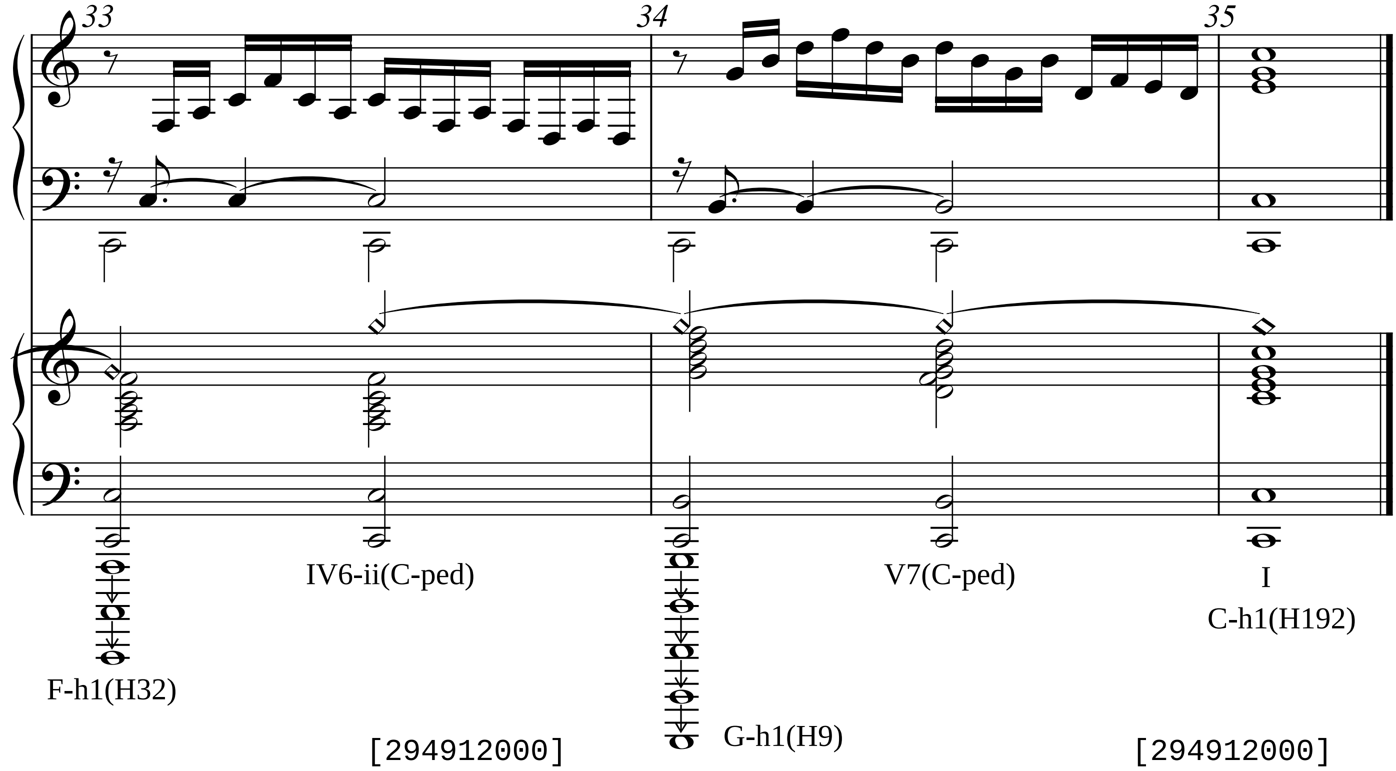
$=32_{36}0_1$ $= 12_{96}0_1$




```
                    [147456000]
G-h32* --->         (369.9526383Hz)
F-h28~
B-h20~
G-h16~
D-h12~
G-h8~
F-h7
D-h6
B-h5
G-h4
D-h3
G-h2
G-h1(H36)[4608000]
(11.56101995Hz)
-- 3:8 -->
F-H1
(0.321139443Hz)
```

```
                    [147456000]
G-h12* --->         (369.9526383Hz)
E-h10~
A#h7~
G-h6~
C-h4~
C-h2~
C-h1(H96)[12288000]
(30.82938652Hz)
-- 3:1 -->
F-H1
(0.321139443Hz)
```

$= 36_{32}0_1 + 36_{32}0_1 = 72_{32}0_1$ $= 256_{9}0_1$ $= 12_{192}0_1$



[294912000]
G-h72* ---> (739.9167655Hz)
G-h36* +36
F-h32~
C-h24~
A-h20~
F-h16~
D-h13~
C-h12~
C#h11
A-h10
G-h9
F-h8
D#h7
C-h6~
A-h5
F-h4
C-h3
F-h2
F-h1(H32)[4096000]
(10.27646217Hz)
-- 32:9 -->
F-H1
(0.321139443Hz)

G-h256* --->
F-h224~
D-h192~
B-h160~
G-h128~
G-h64
B-h40~
C-h21~
B-h20
A#h19
A-h18
G#h17
G-h16
F#h15
F-h14
E-h13
D-h12
C#h11
B-h10
A-h9
G-h8
F-h7
D-h6
B-h5
G-h4
D-h3
G-h2
G-h1(H9)[1152000]
(2.890254987Hz)
-- 3:64 -->
F-H1
(0.321139443Hz)

[294912000]
G-h12* (739.9167655Hz)
C-h8~
A#h7
G-h6~
E-h5~
C-h4~
G-h3
C-h2~
C-h1~(H192)[24576000]
(61.65877305Hz)
F-H1
(0.321139443Hz)

Calculation of Whole-Number Solutions

The Perl script `procalc` will find the lowest whole number sequence of note-ratios in an harmonic series compatible with a given scheme of harmonic progression conducted in Just proportions – 3:2, 5:6, 9:8, etc. The script takes as input a text file list of fractions or ratios, one per line, for example:

9/16	or	16:9
4/3		3:4
2/3		3:2
5/12		12:5
8/3		3:8
etc.....		

Usage: `[perl] procalc <textfile_of_ratios> [-h]`

Output is to STDOUT, which can be redirected to file,

```
[perl] procalc inputfile > outputfile
```

which would produce a text file list of note-ratios such as that below.

Pass: 18

FundamentalSeriesH1->H?			25509168
9/16	(16:9)	(h1) =	14348907
4/3	(3:4)	(h1) =	19131876
2/3	(3:2)	(h1) =	12754584
5/12	(12:5)	(h1) =	5314410
8/3	(3:8)	(h1) =	14171760
etc.....			

As you will no doubt have noticed this list is a calculation of the nested fundamental tones (h1) presented in the opening pages of the preceding example analysis of J.S. Bach's Prelude. Below the complete output of nested fundamental tones (h1) is given, multiplied by the value of their respective nested series, –i.e. the digit belonging to the most significant column. To the left, these 'conjunction-sums' have been broken down into prime factors.

The Perl script `bpf` (big prime factor, adapted from the *Perl Cookbook*) accepts as input a list of numbers and outputs that list with prime factors appended on the left. It dynamically handles integers large and small.

Usage: `[perl] bpf <textfile-list-of-numbers> [-h]`

Output:

```
503884800    2**10   3**9    5**2
335923200    2**11   3**8    5**2
223948800    2**12   3**7    5**2
```

Complete Whole Number Analysis

The table below is constructed from the output of the above scripts. The two 'sides' of each exchange have been notated –i.e. the 'out-going' and the 'in-coming' series, thus information relating to both mutable number digit sequences party to the exchanges and their shared sums are listed line by line.

For example, on the first line of the table, bar 1 – 2,

'Out-going:' 25509168 × 18 = 459165024 --------> 'In-coming:' 14348907 × 32 = 459165024

```
Bar|  Proportion  |    Nested h1  |  Nested Series up to Conjunction, | Prime Factors |
---|--------------|---------------|-----Reception Series & Total------|---------------|
 1.  FundamtalSeriesH1->H?25509168 ×  18 = 14348907 ×  32 = 459165024   2^5   3^15
 2.  9/16  (16:9)  (h1) = 14348907 ×  32 = 19131876 ×  24 = 459165024   2^5   3^15
 3.  4/3   (3:4)   (h1) = 19131876 ×  16 = 12754584 ×  24 = 306110016   2^6   3^14
 4.  2/3   (3:2)   (h1) = 12754584 ×  40 =  5314410 ×  96 = 510183360   2^6   3^13  5^1
 5.  5/12  (12:5)  (h1) =  5314410 ×  64 = 14171760 ×  24 = 340122240   2^7   3^12  5^1
 6.  8/3   (3:8)   (h1) = 14171760 ×  32 = 18895680 ×  24 = 453496320   2^9   3^11  5^1
 7.  4/3   (3:4)   (h1) = 18895680 ×  16 = 12597120 ×  24 = 302330880   2^10  3^10  5^1
 8.  2/3   (3:2)   (h1) = 12597120 ×  40 =  5248800 ×  96 = 503884800   2^10  3^9   5^2
 9.  5/12  (12:5)  (h1) =  5248800 ×  64 = 13996800 ×  24 = 335923200   2^11  3^8   5^2
10.  8/3   (3:8)   (h1) = 13996800 ×  16 =  9331200 ×  24 = 223948800   2^12  3^7   5^2
11.  2/3   (3:2)   (h1) =  9331200 ×  24 =  4665600 ×  48 = 223948800   2^12  3^7   5^2
12.  1/2   (2:1)   (h1) =  4665600 ×  48 =  3499200 ×  64 = 223948800   2^12  3^7   5^2
13.  3/4   (4:3)   (h1) =  3499200 ×  64 =  3499200 ×  64 = 223948800   2^12  3^7   5^2
14.  1/1   (1:1)   (h1) =  3499200 ×  64 =  6220800 ×  36 = 223948800   2^12  3^7   5^2
15.  16/9  (9:16)  (h1) =  6220800 ×  32 =  4147200 ×  48 = 199065600   2^15  3^5   5^2
16.  2/3   (3:2)   (h1) =  4147200 ×  40 =  3456000 ×  48 = 165888000   2^14  3^4   5^3
17.  5/6   (6:5)   (h1) =  3456000 ×  64 =  4608000 ×  48 = 221184000   2^16  3^3   5^3
18.  4/3   (3:4)   (h1) =  4608000 ×  32 =  6144000 ×  24 = 147456000   2^17  3^2   5^3
19.  4/3   (3:4)   (h1) =  6144000 ×  24 =  6144000 ×  24 = 147456000   2^17  3^2   5^3
20.  1/1   (1:1)   (h1) =  6144000 ×  32 =  8192000 ×  24 = 196608000   2^19  3^1   5^3
21.  4/3   (3:4)   (h1) =  8192000 ×  24 =  2048000 ×  96 = 196608000   2^19  3^1   5^3
22.  1/4   (4:1)   (h1) =  2048000 ×  96 =  2048000 ×  96 = 196608000   2^19  3^1   5^3
23.  1/1   (1:1)   (h1) =  2048000 ×  72 =  2304000 ×  64 = 147456000   2^17  3^2   5^3
24.  9/8   (8:9)   (h1) =  2304000 ×  64 =  3072000 ×  48 = 147456000   2^17  3^2   5^3
25.  4/3   (3:4)   (h1) =  3072000 ×  48 =  2304000 ×  64 = 147456000   2^17  3^2   5^3
26.  3/4   (4:3)   (h1) =  2304000 ×  64 =  4608000 ×  32 = 147456000   2^17  3^2   5^3
27.  2/1   (1:2)   (h1) =  4608000 ×  32 =  3686400 ×  40 = 147456000   2^17  3^2   5^3
28.  4/5   (5:4)   (h1) =  3686400 ×  40 =  3072000 ×  48 = 147456000   2^17  3^2   5^3
29.  5/6   (6:5)   (h1) =  3072000 ×  48 =  4608000 ×  32 = 147456000   2^17  3^2   5^3
30.  3/2   (2:3)   (h1) =  4608000 ×  32 =  4608000 ×  32 = 147456000   2^17  3^2   5^3
31.  1/1   (1:1)   (h1) =  4608000 ×  32 = 12288000 ×  12 = 147456000   2^17  3^2   5^3
32.  8/3   (3:8)   (h1) = 12288000 ×  12 =  4096000 ×  36 = 147456000   2^17  3^2   5^3
33.  1/3   (3:1)   (h1) =  4096000 ×  72 =  1152000 × 256 = 294912000   2^18  3^2   5^3
34.  9/32  (32:9)  (h1) =  1152000 × 256 = 24576000 ×  12 = 294912000   2^18  3^2   5^3
35.  64/3  (3:64)  (h1) = 24576000 ×  12                 = 294912000   2^18  3^2   5^3
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                                                    (see note below)
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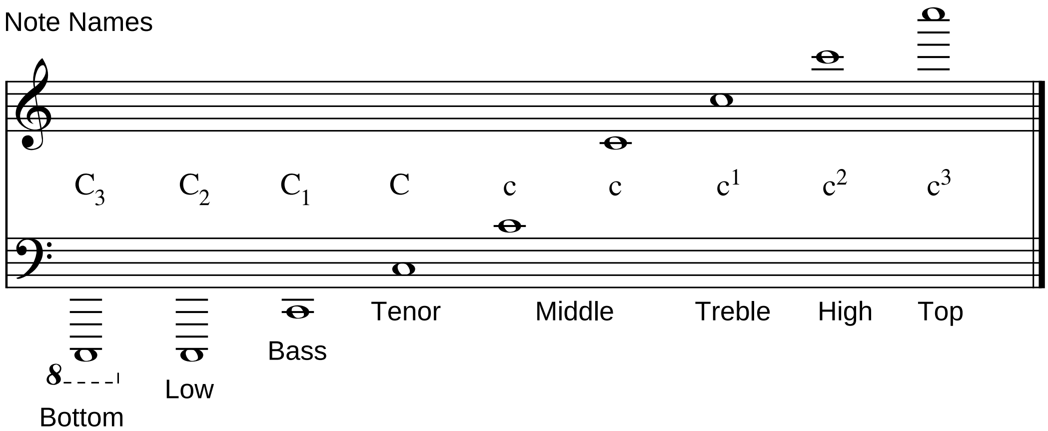
In the whole number analysis above there is an interesting pattern of decline in powers of three from beginning to end, accompanied by an broadly steady increase in the powers of two (and five). The pattern appears to be dependent on the exchange proportions and their sequence. Here sufficient 'threeness' is built up in the initial number to allow the sequence to run an integer course to the end, and over which, the store of 'threeness' is gradually eroded away by denominators containing three. Interestingly, the 'fiveness' is concealed in the first few numbers and emerges from the interaction of the proportions with the initial sum.

The table of absolute whole numbers given above is a concise version of the complete mutable base number description of Bach's Prelude, in which every note and ratio is accounted for exactly, and all relationships between notes and ratios made precisely explicit. By viewing this expansive domain of the underlying nesting series as a 'space' within which the Prelude's nested series is driven forward by the evolution of its own internal harmonic relationships, it becomes possible in turn to interpret the progress of the nested series through this space as being something akin to that of a musical worldline: A unique trace attributable to no other composition but that of Prelude No.1, Book No.1, *The Well-tempered Clavier* by J.S. Bach.

Notes



The pitch nomenclature adopted in this document is shown above, one of the three schemes mentioned in the Harvard Dictionary of Music compounded with a verbal practice familiar to organ builders. The twelve ascending chromatic notes from bottom C_3 to bottom B_3 are spoken: bottom C, bottom C#, bottom D, etc... and written either as bottom C_3 or C_3 ; bottom $C\#_3$ or $C\#_3$, etc... This ascending octave based naming practice is applied throughout the compass of notes, and if required, may be extended further through the use of more super/subscripts. Also as amongst organ builders, notes are by preference named as sharps, for example A# rather than B-flat, but not exclusively so where the flattened form is more informative or convenient.

Note 1. Or even, if wished, an arbitrarily close approximation to equal-temperament could be achieved – combined with a more or less constant amount of unit disturbance.