The Mathematics of Tonal Music and Mutable Numbers

"This may have moved to that and that to this, yet still the sum remains the same."
Ovid: Metamorphoses, Book XV.

The aim of this document is to succinctly place the mathematics of mutable base numbers – and thereby the tonally organised music which they encapsulate – within a framework of established group theory. To achieve this end, mutable base numbers are shown to be an alternative numeration of the group of non-zero, positive rational numbers under the operation of multiplication. Consequently, this number system is taken to inherit the natural characteristics of the group: Closure, Identity, Inverses, Associativity and Commutativity.

$$(\mathbb{Q}^*, \times) \quad \text{where} \quad \mathbb{Q}^* = \mathbb{Q} > \text{zero}$$

[see Note 4]

Mutable Base Numbers

Generally in mathematic it is convenient to use number systems that map magnitudes to unique, unequivocal digit sequences – most often Indo-Arabic decimal place notation. Mutable Base Numbers (abbreviated MBN) take the opposite approach, finding in particular circumstances an advantage may accrue from a number system of equivocal character, that is, a number system containing a variety of different digit sequences for each unique value. While such an approach might at first sight appear chaotic or intractable; where the aim is to mathematically describe processes of change or transition within the material world, possessing a numerical vocabulary full of synonyms could turn out to be just what is required. Indeed, I believe this to be the case when trying to understand, mathematically, the fundamental nature and processes of western tonal music.

Mutable base numbers are a general or open form of place notation where not only are the columns allowed to hold unrestricted magnitudes, but also, the bases of columns may vary dynamically. This leads to a cornucopia of digit sequences; indeed, many large individual magnitudes may be expressed by tens, hundreds, and more, different mutable digit sequences. Though as will be seen below, it is only a limited subset of this abundance that has a direct correspondence with the harmonies and metres of tonally organised music. And inevitably there are some technical issues to be addressed regarding their written form.

A few example mutable numbers will illuminate their characteristics more effectively than many words, though a much fuller account and analytical examples may be found in *Journey to the Heart of Music* and the article *Music by Mutable Numbers*.

MBN $4_6 0_1 \quad = \ 6_4 0_1 \quad = \ 8_3 0_1 \quad = \ 12_2 0_1 \quad = \ 24_1 \quad = \ 2_{10} 4_1 \quad =$ Twenty-four

All the digit sequences immediately above represent the magnitude twenty four, and as mutable base numbers allow their column bases to vary at will, it is necessary to explicitly label every column with its base in the form of a subscript. Further, to save the chore of inventing an unending procession of numerical symbols, to represent the unlimited range of magnitudes mutable number columns and bases may encompass, it is convenient to use decimal numbers (consisting of one or more columns) and to interpret these decimal numbers as individual mutable number glyphs – whether used as column digits or column bases. Thus MBN 24_1 represents a single column of twenty four units as signified by its unitary subscript; while the column and base '12_2' represents a single column of twelve base two digits. All mutable base numbers begin by defining the unit with the subscript '$_1$' and this unit subscript is located in the number (string of digits) rather akin to a radix point – illustrated above and below. So crossing over into fractions, the radix positioned definition of the unit, now lies on the left side of the digit sequence.

MBN $0_1 1_2$ = Decimal 0.5 (1/2) MBN $0_1 1_4$ = Dec. 0.25 (1/4)

MBN $0_1 2_3$ = Dec. 0.666... (2/3) MBN $0_1 4_3$ = Dec. 1.333... (4/3)

Another example, a larger number, six hundred and ninety six, in a few of its mutable forms.

MBN $6_7 4_5 2_3 0_1$ = $29_8 0_3 0_1$ = $10_8 6_4 4_2 0_1$ = $6_{10} 9_{10} 6_1$ = Dec. 696

Notice that the notational convention adopted with Mutable Numbers is to derive the column base value via multiplication of bases (including fractions) from the unit, through the succession of column bases, up to, or down to, any particular column position.

For example, Decimal/Mutable hundreds column: $1_{10} 0_{10} 0_1$ $(1 \times 10 \times 10)$

or Decimal/Mutable one hundredths column: $0_1 0_{10} 1_{10}$ $(1 \times {}^1/_{10} \times {}^1/_{10})$

Applying this principle to the first two numbers above (–i.e. MBN $6_7 4_5 2_3 0_1$ and $29_8 0_3 0_1$) with column bases in parentheses and using decimal notation, we have:

	$0 \times (1) = 0$	$0 \times (1) = 0$
	$2 \times (3 \times 1) = 6$	$0 \times (3 \times 1) = 0$
	$4 \times (5 \times 3 \times 1) = 60$	$29 \times (8 \times 3 \times 1) = 696$
	$6 \times (7 \times 5 \times 3 \times 1) = 630$	
Totals:	$0 + 6 + 60 + 630 = 696$	$0 + 0 + 696 = 696$

Briefly, to conclude this section, all that remains is to unite whole and fractional Mutable Base Numbers in single digit sequences:

MBN $8_3 0_1 1_2$ = Dec. 24.5 MBN $24_1 1_3$ = Dec. 24.333...

MBN $6_4 0_1 1_4$ = Dec. 24.25 MBN $11_2 2_1 2_5$ = Dec. 24.4

Every mutable number has one, and only one, unit column, additional base one columns would be redundant.

The Connection Between Tonal Music and Mutable Numbers

From amongst the vast array of all possible mutable number digit sequences there is a subset of digit sequences which captures the essence of tonal music in a formal numeral system. This subset of mutable number digit sequences are those composed of a single most significant digit or a single least significant digit, followed by or preceded by, zero or more zero-digit columns. (Which is to say, digit sequences that equate to strings of factors multiplied together.) For example:

MBN $10_4\,0_{32}\,0_1$ = Dec. 1280 MBN $0_1 0_3 8_3$ = Dec. 0.888... , ($^8/_9$)

Mutable Base Numbers are manifest both as written numerals and physical waves, however our perception of these sound waves is subject to the processes of the human aural pathway and mind. Notwithstanding, the connection is close indeed, in that every chord in a tonal composition, and every ratio of harmonic progression between each of the chords in a tonal composition, can find an analogue in the form of a mutable base number, and similarly for every metre. Thus the progress through a tonal composition, from beginning to end, takes on the character of a sequential calculation. And, rather akin to the relationship between formal binary numbers and electrical potential in an electronic computer, so also there exists a similar relationship between this particular subset of mutable base numbers and the physical sound of tonally organised music. However, it must be noted that due to the approximations of scale temperaments and the vagaries of learned musical practices, that tonal music will occasionally struggle to precisely reflect the underlying mathematical logic[1].

Digits, Physical and Written

In physical form, the digits of Mutable Base Numbers are supplied by the tones/ratios of the acoustic Harmonic Series – there is an unlimited supply – and for convenience, as noted above, to save endlessly inventing new glyphs we label them with decimal numbers. And by extension, when writing fractional magnitudes in sound, the denominators are taken from the mirror reflection of the natural overtone series in the form of a descending sequence of matching intervals – the Undertone Series (Figures 1A & 1B).

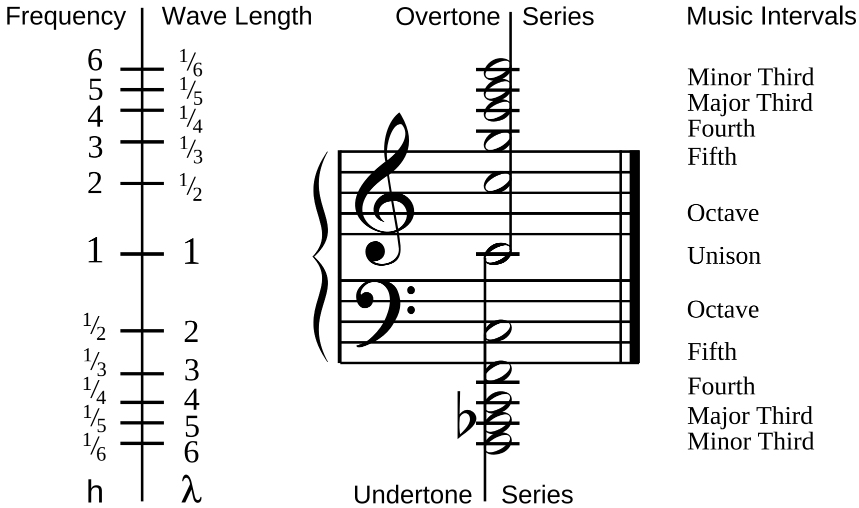


Figure 1A. The mirror reflection of the acoustic harmonic series in the pitch domain, emanating up and down from unity – middle c. (See Notes section for pitch nomenclature.)

Some music theorists in the nineteenth and early twentieth century toyed with the idea that the undertone series might be an observable physical fact[2]. Scientific investigation found this not to be so and sadly along with this misconception many of Dualism's more subtle and interesting insights were also tarnished.

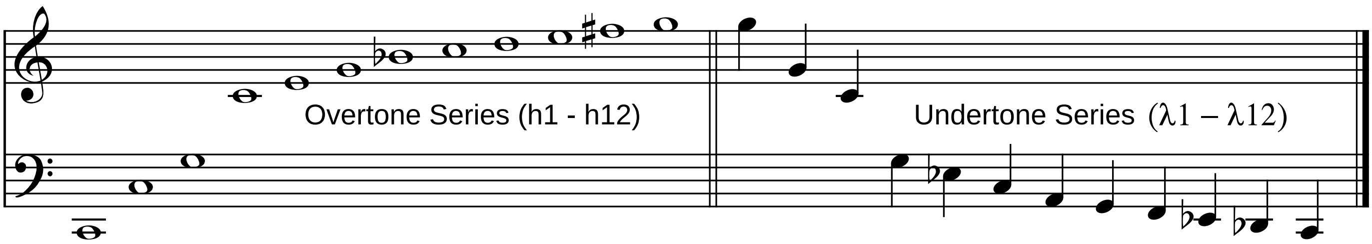


Figure 1B. An approximate representation of the Overtone and Undertone series within the limits of equal temperament and musical notation. The rising intervals of an octave, fifth, fourth, major third, etc... in the overtone series are mirrored as falling intervals in the undertone series.

The overtone series – the acoustic harmonic series – is formed from the natural whole number modes of vibration of a perfectly elastic physical object and the undertone series is constructed from its reflection in the pitch/frequency domain. These are the physical digits with which mutable numbers may be represent as sound, and although only the first twelve ascending and descending tones are illustrated in Figure 1B, there is no limit to them. So whereas decimal digits count from 0 – 9, after which a new column is initiated, mutable digits count from one to infinity, with new columns being initiated, at will, at any point along this limitless range.

The mutable zero digit is silent, the absence of physical sound. More strictly perhaps an empty space in the written digit sequence would be appropriate as zero is not a member of the group Q* but it is used for practical reasons: to avoid ambiguity and aid readability.

In written form decimal numbers are used to represent mutable digits but in their physical form, as sound, the 'note-digits' of the overtone and undertone series give objective being to mutable numbers. And notwithstanding the greater or lesser approximations of the equal-tempered scale to the whole number ratios of the acoustic harmonic series, our ears and aural cognition accept, indeed overlook, these deviations in their quest to understand and categorise tonal musical stimuli. For example, the equal-tempered fifth at the frequency ratio of 1:1.4983069... passes almost without notice for a 'perfect' fifth, and even when stretched considerably further away from the precise ratio of 2:3, will still be identified by the ear as a fifth – though a painfully out-of-tune fifth.

Nesting within Overtone & Undertone Series

A useful feature of the overtone and undertone series is that they are susceptible to nesting within themselves, rather in the manner of Russian dolls. That is, within an original 'parent' series an unending sequence of 'child' series can be constructed. This is a significant feature exploited by mutable base numbers. (Please mark the terminological distinction drawn between a <u>nested</u> series and a <u>nesting</u> series as it arises.) Figure 2 illustrates the first four possible nestings within both the overtone and undertone series.

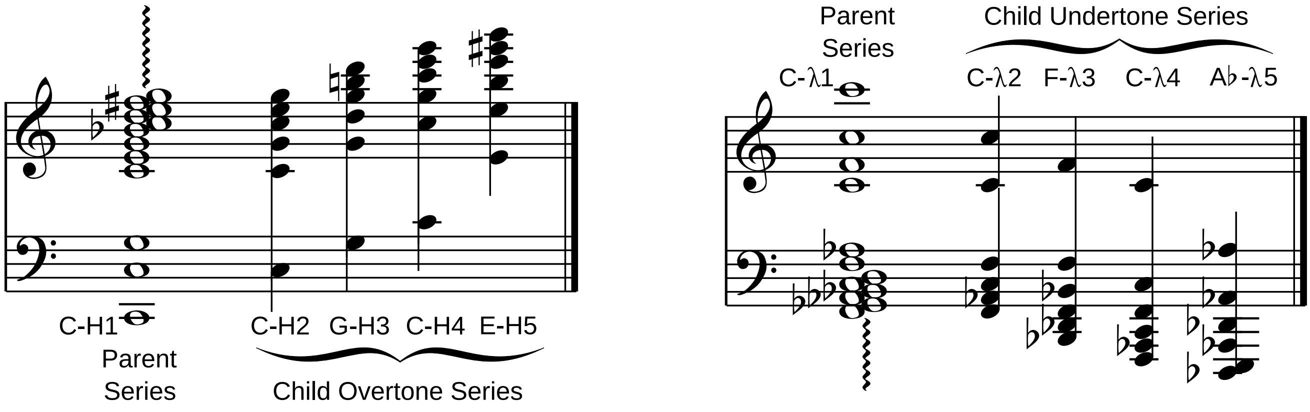


Figure 2. The the first four ‘child’ overtone and undertone series illustrated in quarter note values. The parent series are written in whole notes out to the eleventh partial with a wiggly line indicating their further limitless extension. The child series are written out to their fifth partials, which equally may be continued on indefinitely, with their fundamental tones labelled above or below.

In Figure 3 the whole number twelve and the fractions one-half, one-third, one-quarter and one-sixth are illustrated, with the written Mutable Base Number set above its physical (musical) equivalent. Notice that the fractions of the undertone series partition the overtone series, such that MBN $0_1 1_2$ ($^1/_2$) reaches down through the overtone series to the frequency level of MBN 6_1 (h6) and similarly the other fractions to h4 and h3, until MBN $0_1 1_6$ ($^1/_6$) on the right of Figure 3 reaches down to MBN 2_1 (h2). That is, an undertone series apportions the partials of an overtones series precisely, where their tones agree.

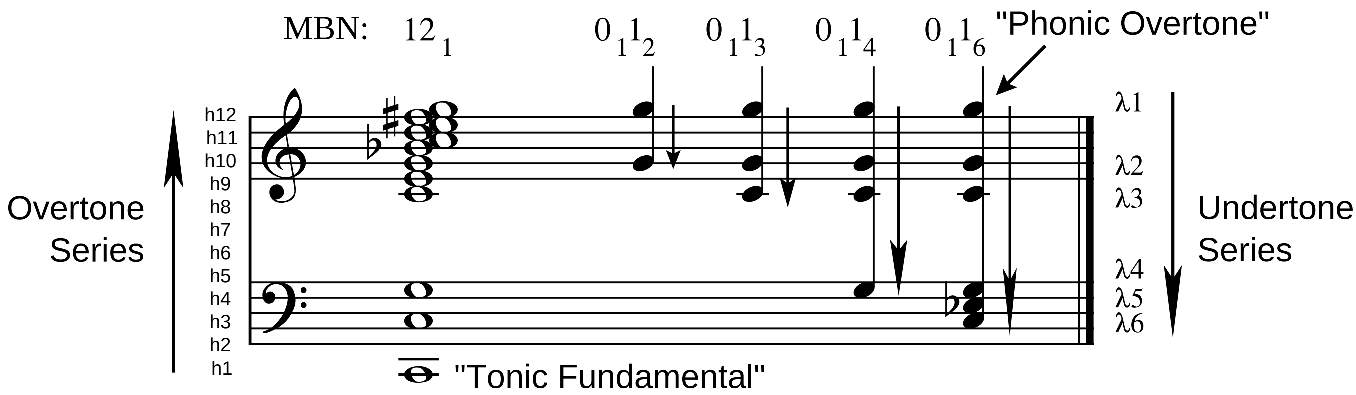


Figure 3. The whole number twelve with the fractions one-half, one-third, one-quarter and one-sixth partitioning the whole number appropriately.

Thus undertone series provide the facility of multiplication by fractions. Expressing this relationship in the terms introduced by Arthur von Öettingen (1836–1920), the overtone series multiplies the unit h1, the “tonic fundamental”, by whole frequency values – in this example up to twelve – while the undertone series multiplies its own unit frequency, the “phonic overtone”, in Figure 3 treble g^1 by a half, a third, a quarter and a sixth. What von Öettingen was pointing out with his use of the term the phonic overtone was that all the note-digits in an undertone series share a common partial (the phonic overtone) – analogous to the common fundamental shared by all the note-digits in an overtone series. Both the properties of multiplication by whole numbers and by fractions are required when interpreting tonal music as mutable base number processing. And here it might be helpful to keep in mind the over-arching scheme; that the chords in tonal music express positive whole mutable numbers while the proportions between chords are positive mutable number fractions. Thus the whole number chords in a tonal composition are transformed sequentially (–i.e. the harmonic progression) by the multiplicative action of the fractional proportions between them, all values of which are contained within the group $Q^* = (Q_{>0})$.

Columns in Mutable Numbers

In Figure 3 the magnitude twelve is represented physically by a fundamental tone plus eleven partials, which translates to a single column mutable base number (MBN 12_1) when written formally. Already the music staff is becoming quite crowded and adding more 'note-digits' would only make matters worse. What is needed is to use the power of multiplication, that is place notation, to streamline the process by introducing another column into this number. Physically, this is achieved by nesting one overtone series within another more fundamental overtone series – that is, the columns in formal mutable numbers translate to tiers of nesting in overtone and undertone series. Illustrated in Figure 4A & 4B.

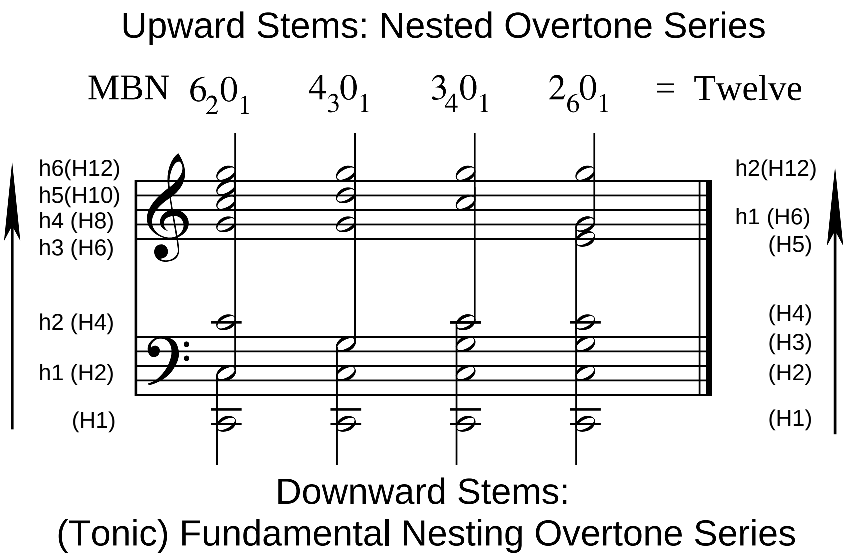


Figure 4A. Four mutable base digit sequences of magnitude twelve written formally above their physical manifestation as musical sound. The frequencies/ratios of the fundamental overtone series are delineated with an upper case 'H' while the nested overtone series are marked with a lower case 'h'. This H/h labelling is illustrated for the first and last chords in the figure. All the frequencies/ratios occurring in the upper nested overtone series are also present in the fundamental nesting series – indeed, even the note D in the second chord occurs as H9 in the underlying fundamental series.

For the moment we will stay with one tier of nesting but in principle the number of nested tiers, and thereby columns in written mutable numbers, is only limited by the magnitude itself. Also notice, the first two chords in Figure 4A are beginning to hint at another property of mutable numbers – that is, in their upper reaches they encompass different chords within expressions of the same magnitude. The nested overtone series within MBN $6_2 0_1$ forms a C major chord while that within MBN $4_3 0_1$ produces a rather bare G harmony; and, because both expressions represent the same magnitude (though with different internal structures) they share a common highest frequency, a feature which aurally smoothes the transition from one chord to the other. Common frequencies shared by adjacent chords such as these four notes on the top line of the treble staff (G-H12) are termed 'conjunctions'.

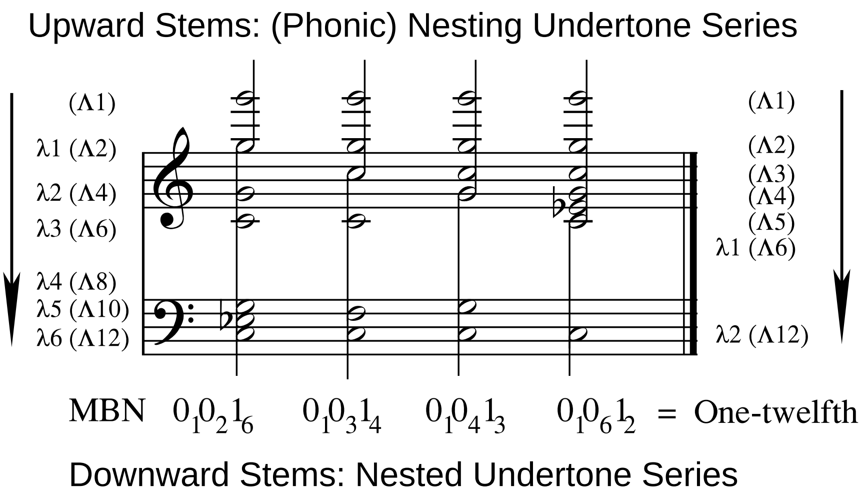


Figure 4B. Four mutable base digit sequences of magnitude one-twelfth written formally below their physical manifestation as musical sound. As with nested overtones series, nested undertone series are forever stepping in the footprints of their all-encompassing parent ‘phonic’ nesting series.

In Figure 4B a similar example of nesting is presented for undertone series. It is the rich variety of digit sequences possessed by mutable base numbers for the expression of individual magnitudes that is their particular advantage when it comes to encapsulating harmonic progression and metrical change in tonally organised music. Figure 5 contains three examples of multi-column numbers all (aurally!) of apparent equal magnitude.

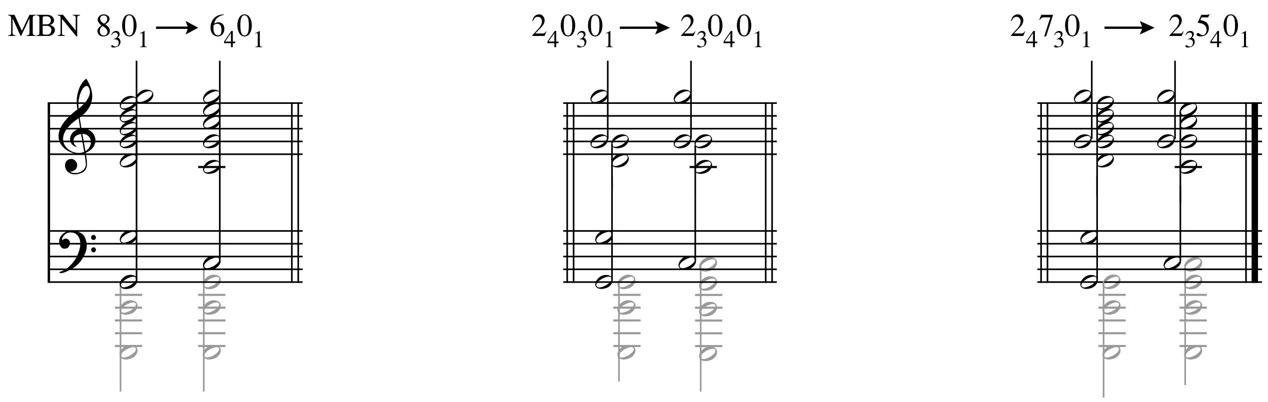


Figure 5. Three multi-column mutable number digit sequences each sharing the conjunction treble g^1. The third example on the left aurally equates forty five with forty four, included here to illustrate the care with which digit sequences beyond the ‘zero digit’ subset focused on in this document must be used.

Multiplication with Mutable Numbers

Now having explored a little of nesting in overtone and undertone series, we are in a position to mix the two elements together: multiplication by whole numbers and fractions. However, the nesting of overtone series within undertone series, or vice versa, is a little more difficult to represent in musical notation, because this mixed combination involves the nested series (the child series) turning back over the nesting series (the parent). In Figure 6 the nesting undertone series is written in whole notes while the nested overtone series is in half notes. Each example should be read left to right, beginning at the top of the whole note chord/series (undertone series) reading downward followed by ascending the half note chord/series, the overtone series.

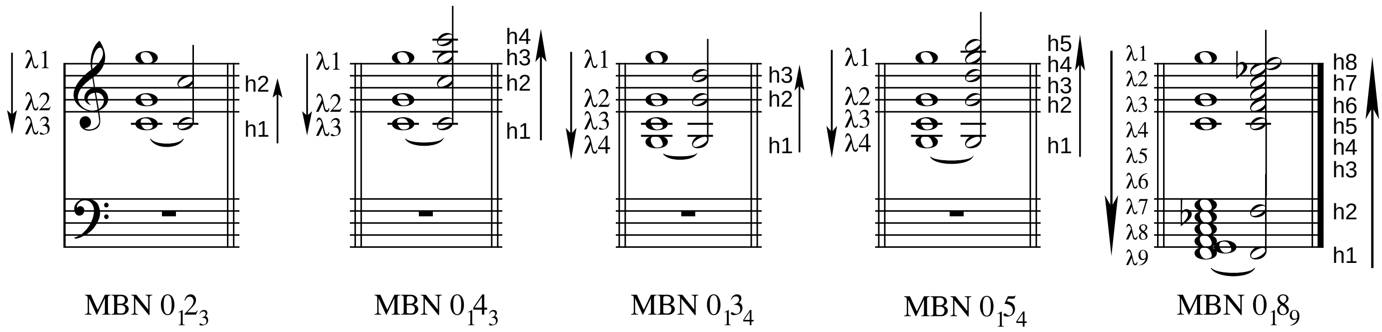


Figure 6. Five examples of overtone series nested within undertone series. First multiply the unit, the phonic overtone, by the (fractional) base value (or values if their is more than one column) by counting down the undertone series of the denominator, and then ascend the overtone series, -i.e. multiply by the digit magnitude, the numerator. Through this procedure the initial frequency, the unit, is modified in accordance with the proportion of the fraction.
From left to right: $^2/_3$ $^4/_3$ $^3/_4$ $^5/_4$ $^8/_9$.

The Full Cadence

Interestingly, by mixing overtone and undertone series together a hint of the proportions between the chords of tonal music, emerge. Through a physical process of stepwise movement, down and up undertone and overtone series, mathematical relationships are given material being. And with these elements introduced above, let us examine a very short snatch of tonal music – a full cadence.

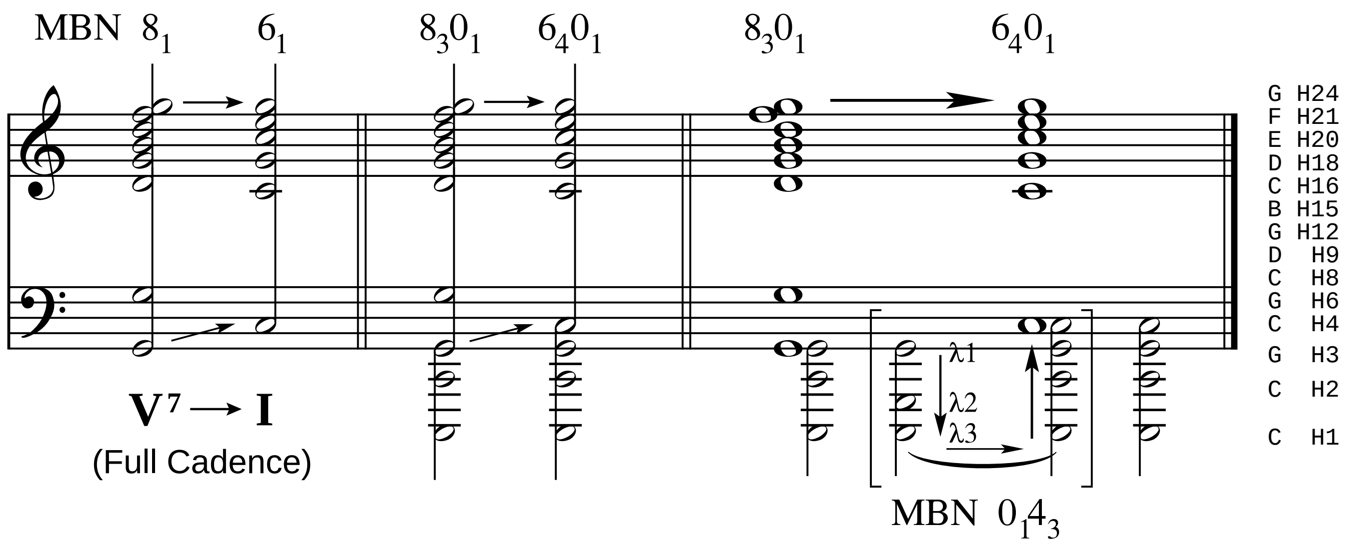


Figure 7A. The full cadence in its ascending form of dominant-seventh to tonic chords (V^7– I) labelled with mutable numbers above. The left hand measure contains the dominant-seventh and tonic chords in the form of two overtone series. These two chords are the actual acoustic stimuli. The middle measure contains the same chord progression set within the confines of an implied fundamental nesting (overtone) series H1–H24. The right hand measure interpolates the multiplication of the position of nesting (H3) by MBN $0_1 4_3$ thereby converting the dominant-seventh chord into the tonic chord. While in practical terms this one multiplication is sufficient because the overall sum of twenty-four remains constant between the two chord/digit sequences, both the proportion of the progression ($^4/_3$) and its group inverse ($^3/_4$) are employed in a calculation of symmetrical exchange:

Upper, Nested Series: MBN $8_1 \times 0_1 3_4 = 6_1$ Lower, Nesting Series: MBN $3_1 \times 0_1 4_3 = 4_1$

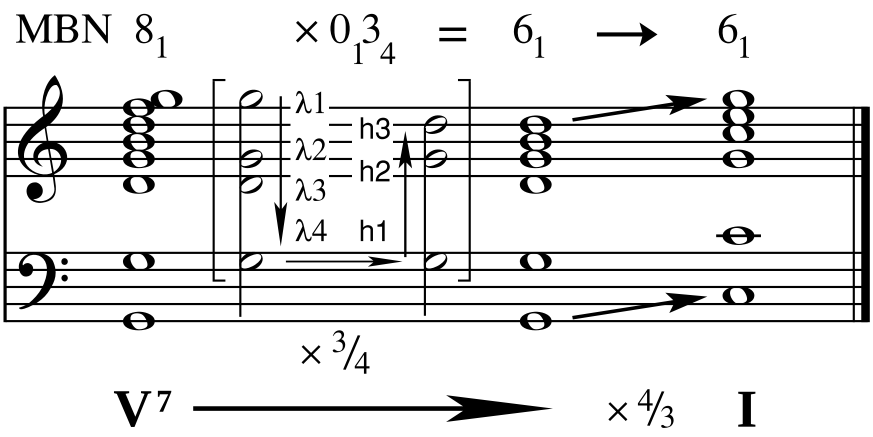


Figure 7B. Here the physical process of multiplication by $^3/_4$ (MBN $0_1 3_4$) in the upper nested overtone series shown in Figure 7A, is made explicit by descending four (base) steps in an undertone series (the denominator) followed by ascending three note-digits in an overtone series (the numerator), thereby reducing the eight note-digits on the left to six on the right.

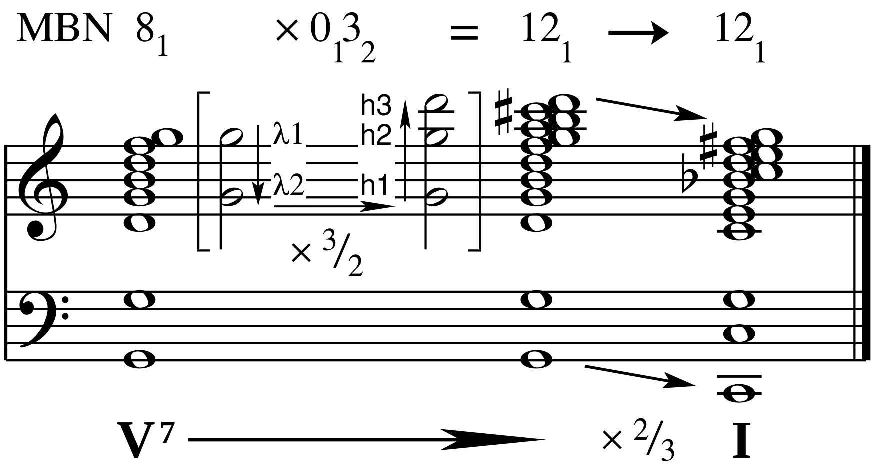


Figure 7C. An example of extending (rather than reducing) the upper nested overtone series in a perfect cadence, which would naturally be matched by its group inverse ($^2/_3$) operating on the underlying fundamental nesting series.

As well as combining the elements discussed separately above, Figure 7 has introduced other factors which may bear further consideration.

Points Arising

Firstly, mutable base numbers in their physical form of musical sound, imply a mathematical underpinning, rather than explicitly delineating such a structure. This mathematical structure, which may be inferred from the objective sound, can be construed from a property of waves: that any periodic wave form may be reduced to a fundamental frequency and zero or more of its whole number multiples.

Secondly, while in most of the examples given, a full set of partials (up to and including the conjunction partial) is written out explicitly in note-digits for completeness, in practice most chords in music are subsets of these collections. Of the six different tones present in the tonic chord in Figure 7C, normally, only three would appear in a score. (Though all would probably be present as partials at varying intensities and frequency levels, dependent on the circumstances of instrumentation, acoustics, etc.)

Thirdly, a common frequency (termed a ‘conjunction’) is taken to exist between any two adjacent chords. In the case of tonal music an inclusive and somewhat flexible approach has been adopted: In that this conjunction is taken to be a low or lowest common sum/frequency (–i.e. magnitude) shared by adjacent chords while still being equal to, or above, their highest written note. That is, the conjunction-sum is inclusive of the written notes in the chords it arises from, and normally encapsulates an inverse symmetry between the proportion of exchange and the changed number of note-digits, while also allowing for some degree of interpretation and choice.

Conjunctions

In Figure 7 the conjunction is the note treble g^1 on top of the treble staff. However, more often than not the conjunction frequency will be a common partial arising from adjacent chords and somewhat above the highest written notes. Thus the canvas musical sound upon which mutable numbers gain physical form is taken to be the whole acoustic spectrum available to the human ear and aural cognition. Indeed, because human (mammalian) aural perception presents the conscious sensation of a single tone distilled from the multi-frequency stimuli arising from the playing of one musical note – bundling up the many frequencies scientifically observable – the significance of the full spectrum of partials, where the majority of the energy normally lies, can be underestimated. Figure 8 illustrates this point.

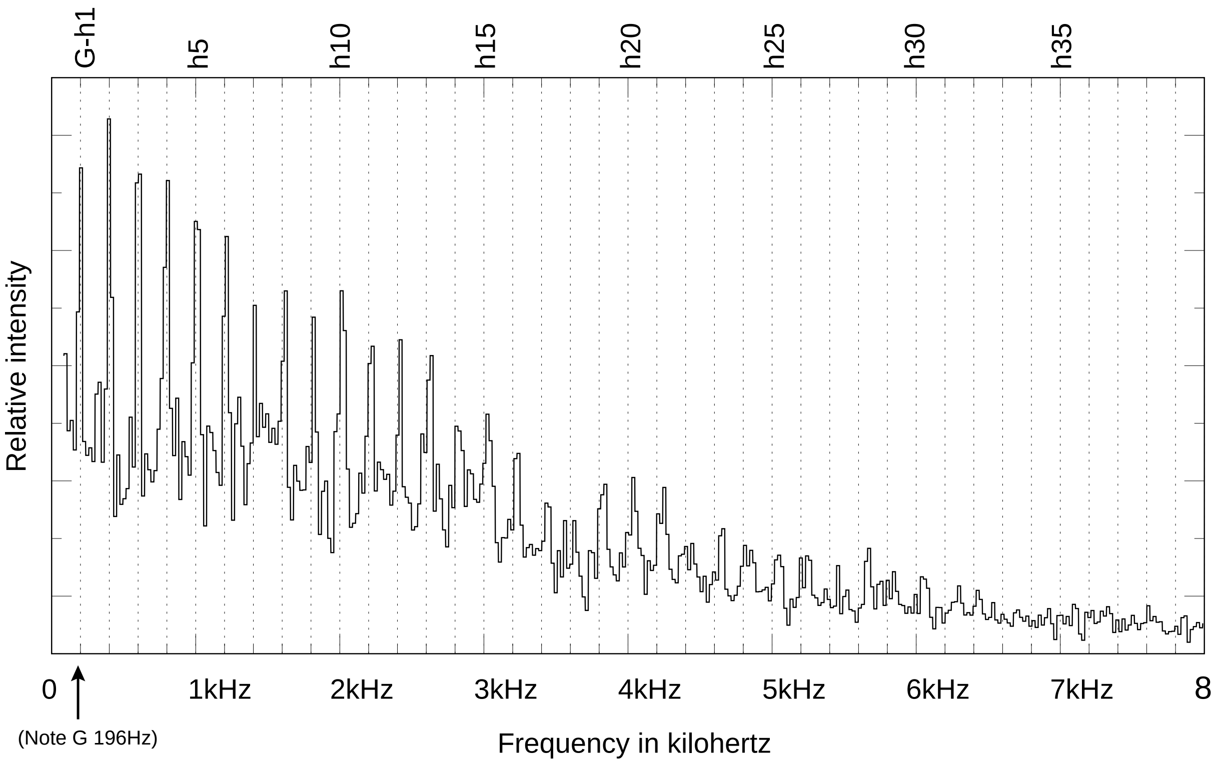
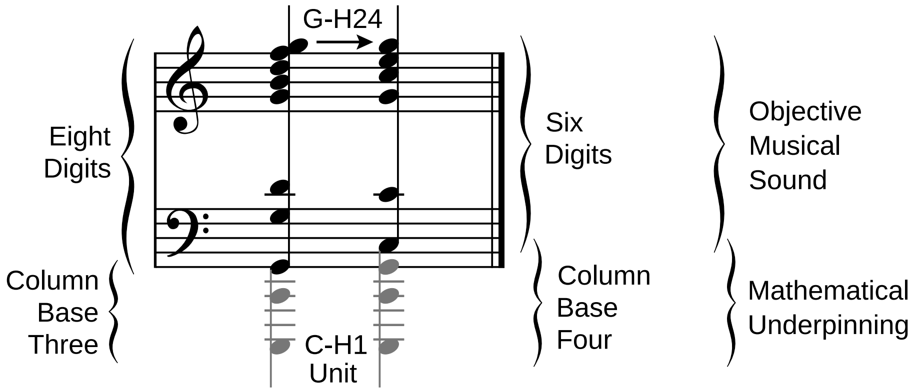


Figure 8. The frequency spectrum of the note tenor G played on a violoncello extended over the principal range of human hearing.

Please notice in Figure 8 that there is more energy in the second harmonic than the fundamental note itself and that the combined energy of the partials massively outweighs that of the (fundamental) tone alone. That said however, another counter balancing factor must also be considered: the nullifying impact of destructive interference brought about by the close spacing of the higher partials on the ear's basilar membrane. Effectively there is a not unlimited band of frequency headroom above a written chord from where an audibly available conjunction may be chosen. The breadth of this headroom is to some degree determined by the complexity of the chords and also the intensity with which individual notes within the chord are sounded. Notwithstanding these considerations, when viewed from the perspective of the (effective) acoustic palette – notes and overtones – any two adjacent chords in a musical composition, even those without a note in common, will in almost every case possess one or more linking frequencies. For the most part these conjunctions are to be found among the energetic lower partials (often doubled by actual written notes) and tracing downward from such conjunction frequencies leads, sooner or later, to a notional common fundamental frequency (H1). The unit. This principle is illustrated in Figure 9.



MBN $8_3 0_1$ $6_4 0_1$ = Twenty-four

Figure 9. The full cadence chord progression represented in back quarter notes with its implied mathematical underpinning shown in grey notes below.

At times another complicating factor can be the relative abundance of potential conjunctions, appearing both as written notes and partials. The example full cadence chords share two notes plus many partials, of which four are illustrated in Figure 10. These six conjunctions have been interpolated between the chords and are shown with diamond shaped note heads.

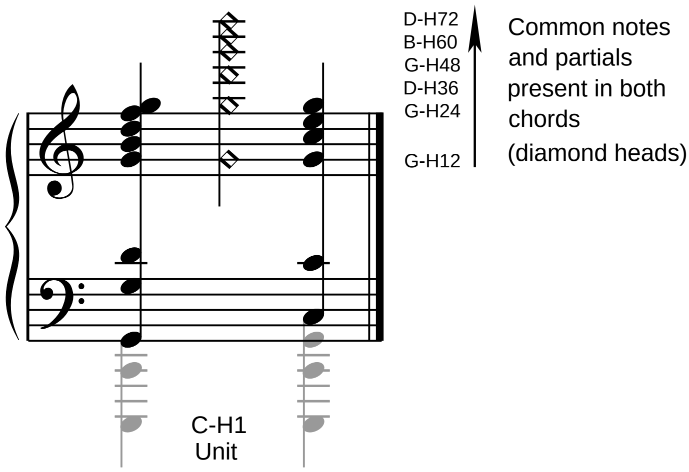


Figure 10. The first six potential conjunction frequencies common to both chords in the example full cadence (V^7–I). All of these conjunctions (and more) share the inverse symmetry of the 3:4 proportion of exchange: four notes for three, eight notes for six, twelve notes for nine, ..., etc.

The question arises: Which conjunction-sum is it? Twelve, twenty-four, thirty-six, ..., etc. Well, fundamentally, and in the spirit of mutable number abundance, it is all of them! Although using the definition already given – the lowest conjunction-sum equal to, or greater than, the highest written note – interprets this chord progression as delineating the magnitude twenty-four. However, in an extended ‘computation’ covering multiple chords – that is, a real musical composition – the actual path chosen through the sequence of possible conjunctions/sums is unlikely to change the eventual destination, only the route by which the number processing journeys. And notwithstanding the utility of defining a magnitude – some particular conjunction – it is likely the ear and aural cognition would probably register every common frequency apprehended between adjacent chords, including those within them. Indeed, with factors like direction of origin and commensurable volume remaining broadly constant, a succession of individual chords united by shared note and/or partial frequencies is, I would argue, likely to be interpreted by our hearing and aural cognition as a sequence of different, evolving inflections, emanating from a single continuing sound source.

Mutable Number Analysis

The full cadence illustrated in Figures 7 and 9 is a simple example of mutable numbers in action and while illuminating the basic principle, it also hides some of the complexity that can soon arise in more extended examples. In particular, the operation of multiplying the whole numbered magnitudes represented as chords in a musical composition, by the fractional proportions of exchange, can produce a gradual drift in the note relationships. That is with the chord progressions calculated sequentially by precise Just ratios of proportion, the note-frequency coordinates, as established at the beginning of a composition, may show some plasticity. One only has to consider the effect of a sequence of perfect fifth (2:3) chord exchanges to realise that the pitch of the composition will be tracking the spiral of fifths. This said however, the reflective symmetry between Just intervals around the tritone axis (see table in *Math of Exchange*) will operate to ameliorate this tendency overall, e.g. 2:3 = 702 cents while 3:4 = 498 cents. In practical terms, for the purposes of music analysis, it might be convenient simply to use an equal-tempered frequency grid and thereby any adjustment is made (although concealed) within the ratio of exchange between chords.

‘Summary’ Analysis

On the other hand, in terms of modelling a composition as an evolving calculation, it might be preferable to accept the precise outcomes of the pure ratios of exchange and aligned conjunctions, by allowing (and perhaps accounting for) the small frequency differences produced by the process. This is the approach taken in the music examples linked with *Journey to the Heart of Music*, with the conjunction frequencies taken to be exactly aligned and the proportions of exchange Just, the small differences that can arise – the flexing – are accounted for at the level of the fundamental[3] (e.g. H1 + 0.016) – the unit.

However, this latter approach is also impacted by the less than perfect match between the degrees of the scale and the lower reaches of the harmonic series. For example, while the Just fifth ratio 2:3 will turn H16 very nicely into H24, the Just fourth ratio 3:4 yields an ill-aligned H21.333... This imprecision may also be accommodated by the process of ‘unit accounting’; in this case by allowing the the unit to flex to the value H1.015873 approximately. This approach provides a practical or ‘summary’ method of mutable number analysis.

Whole Numbered 'Absolute' Analyses

For yet still further precision, the fundamental nesting series might be extended downward to a lower value for the H1 unit such that the pitch/frequency represented by H16 in the former example becomes H48 –i.e. the unit is reduced to one-third of its former value. So now the Just fifth ratio operating on H48 will yield H72 and the Just fourth ratio a nice whole H64. Naturally this procedure changes the musical relationship to the unit (no longer a power of two) and inflates the mutable number magnitude being represented; however, the outcome is a utterly precise integer calculation. (This 'absolute' interpretation or whole numbered analysis may be incorporated in the more practical 'summary' representation by means of real number values.) When such a full mutable number analysis is applied to a non-trivial composition the magnitudes generated can be enormous, yet never impossible. This is because any sequence of simple Just ratios will eventually find a position in the harmonic series (though probably an exceedingly remote one) where the outcome of every exchange will fall upon a whole numbered value.

When calculating such absolute analyses it becomes apparent that the placing of the initial nested series within its encompassing fundamental nesting series produces a value that enables all the proportions of exchange between chords to occur as an integer sequence, thus to some degree encapsulating or capturing the composition as a whole; and further, that the difference between the initial and final values reflects the product of all the proportions taken together. In the following section a short four measure snatch of music is analysed in 'summary' form using the processes already discussed; but if it were subjected to an 'absolute' analysis the column bases of its mutable numbers would be somewhat larger:

```
FundamentalSeriesH1->H? 63 - column base
4/7     (7:4)   (h1) =  36 - column base
4/3     (3:4)   (h1) =  48 - column base
16/12   (12:16) (h1) =  64 - column base
```

however, if all the piece's thirty-five measures are taken into account these column bases become a good deal larger still:

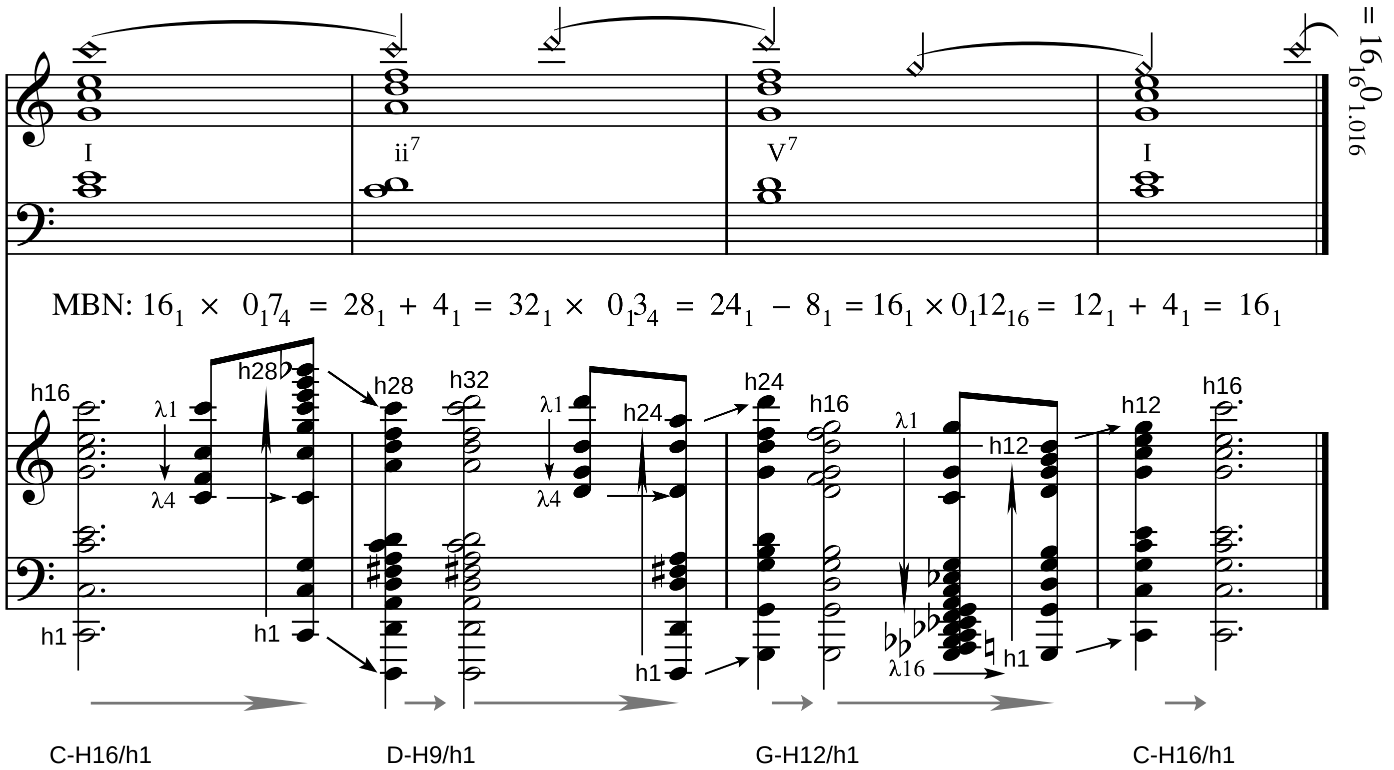
```
FundamentalSeriesH1->H? 25509168 - column base
9/16    (16:9)  (h1) =  14348907 - column base
4/3     (3:4)   (h1) =  19131876 - column base
2/3     (3:2)   (h1) =  12754584 - column base
```

Figure 11. A calculation of the nested fundamental frequecies for the sequence of proportions of exchange between the chords in the first four measures of J. S. Bach's Prelude No.1 from the Well-tempered Clavier. (Different proportions have been used between the first and second chords in the two analyses – $^{8}/_{7}$ and $^{9}/_{8}$ – for the complete analysis see *Bach's Calculation* – Ex_S3)

A Short Musical Example

Figure 12 illustrates the mutable number processing contained within the first four measures of J.S. Bach's Prelude No. 1 from Book I of the Well-Tempered Clavier. The magnitude of the numbers processed here are smaller than in analyses given elsewhere because the four measures are treated in isolation. The overall sums processed are written in formal mutable base numbers across the top of Figure 12 with the harmonies expressed by Bach's figuration consolidated into whole note chords placed on the upper two staves – labelled with the standard Weberian Roman numerals. Riding above the upper treble staff are diamond headed notes indicating the conjunction-sums crystallized by the harmonic motion.

MBN: $16_{16}0_1 = 28_9 0_{1.016} + 4_9 0_{1.016} = 32_9 0_{1.016} = 24_{12}0_{1.016} - 8_{12}0_{1.016} = 16_{12}0_{1.016} = 12_{16}0_{1.016} + 4_{16}0_{1.016}$



MBN: $16_1 \times 0_1 4_7 = 9_{1.016} \times 0_{1.016}4_3 = 12_{1.016} \times 0_{1.016}16_{12} = 16_{1.016}$

Unit: C-H1 C-H1+0.016 C-H1.016 C-H1.016

MBN: $16_{63}0_1 = 28_{36}0_1 + 4_{36}0_1 = 32_{36}0_1 = 24_{48}0_1 - 8_{48}0_1 = 16_{48}0_1 = 12_{64}0_1 + 4_{64}0_1 = 16_{64}0_1$

Decimal: 1008 = 1008 + 144 = 1152 = 1152 – 384 = 768 = 768 + 256 = 1024

Figure 12. Mutable Base Number processing in the first four measures of Prelude No.1, The Well-Tempered Clavier, by J.S. Bach, rendered in 'summary' format. The 'absolute' integer interpretation is given at the foot of the figure in mutable numbers and decimal.

Lying below these two relatively clear staves is the more detailed picture of mutable number processing:

Toward the bottom of Figure 12 is the notional unit C-H1 written out for each measure across the figure. In frequency terms the unit would be approximately 4Hz. It is the foundation of the fundamental nesting series, which can be imagined as enveloping the complete system and is formed by the least complex acoustic harmonic series capable of capturing all the objective written notes present in the composition. This implied mathematical underpinning is allowed a degree of plasticity informed by the harmonic motion of the objective musical sound – due to this being a 'summary' format analysis. And right away in the 'harmonic calculation' performed between the first and second measures the unit is impacted – an adjustment which is carried forward to subsequent harmonic exchanges. (In more extensive examples the overall effect of such adjustments, which can both expand or contract the unit, tends to roughly cancel out.)

The next line up from the unit shows, in mutable base numbers, the calculation taking place at the level of the fundamental series, that is, the dynamic changes of value in the bases of the second column of the mutable numbers displayed at the very top of the figure. As can be seen, the fractional term in each measure's calculation in this line of numbers is the group inverse of the proportion used in the line of numbers inbetween the two double staves.

The third line up from the unit level of Figure 12 contains the nesting points of the upper nested series in relation to the fundamental nesting series. Here both the upper and lower series are referenced, for example C-H16/h1. The capital 'H' signifies the position in the fundamental nesting series and the lower case 'h' the position in the nested series. Thus, the position of the conjunction-sum in the upper nested series at h16 in the first measure is equivalent to H256 in its underlying fundamental nesting series. Indeed, two hundred and fifty six is the magnitude of the mutable base number in the top left corner of Figure 12 (MBN $16_{16}0_{1}$).

Next up are the double staves containing the calculation directly connected to the objective musical sound. Beginning with the first measure, to the left, written as a dotted half note chord, is an abbreviated representation of the upper nested harmonic series beginning with C-H16/h1. This series contains both Bach's written notes for the first measure C major harmony, their implied underpinning and the conjunction-sum (a partial arising from two notes) set in diamond-headed notes. (Subsequent chord/series for the most part are also shown in this abbreviated form.) The conjunction-sum top frequency of this chord is also marked above as 'h16' and above that is the value written in mutable – MBN 16_{1}. This opening magnitude is multiplied by the fractional proportion $^{7}/_{4}$ th so as to produce the second measure's D minor harmony. To recap from earlier figures: from the conjunction-sum an undertone series descends by four note-digits, the denominator, and then rises through an overtone series by seven note-digits, the numerator. This process is sketched out by the beamed eighth note chords in the first measure. One might imagine the process as peeling off the four highest note-digits from the dotted half note chord/series and then grafting a new seven note-digit series on to the top of it. Having 'manufactured' the resultant magnitude (now h1 through h28) this freshly made chord/series is then shifted by the group inverse proportion ($^{4}/_{7}$) into position in the second measure – marked by angled arrows.

The second measure quarter note chord/series has the same treble c^1 top note-digit as the dotted half note chord/series with which measure one began. Treble c^1 is the conjunction-sum linking the C major and D minor harmonies of the first two measures of Figure 12. Having stepped smoothly into the second measure the extension of the following half note chord/series upward by four note-digits to treble d^1 is reflected by the addition of MBN 4_{1} in the mutable numbers written above. Only two of these four 'note-digits' have a normal representation on the staff, the other two are quarter tones. After the addition the resultant chord/ series is again subjected to multiplication by a fractional proportion. The following two beamed eighth note chord/series involving a descending undertone series and ascending overtone series accomplish the process of multiplication by $^{3}/_{4}$, so as to produce a chord/series encompassing twenty four note-digits. This is shifted into position at the start of the third measure, multiplication by $^{4}/_{3}$, so as to make the conjunction-sums shared between the second and third measures equal at treble d^1.

In the third measure the subtraction of eight note-digits from the initial magnitude expressed at the beginning of the measure is reflected in the half note chord/series being pared back to treble g^1, reducing its extent to sixteen note-digits. This sum is then further reduced by another multiplication by three quarters. Here the fraction $^{12}/_{16}$ has been used for illustrative purposes and a rather crowded complete undertone series of sixteen note-digits presented – which includes both low A_2 and low $G\#_2$ (written as A flat). The process is the same as before involving an undertone denominator series and an overtone numerator series. Only this time the half note chord/series is peeled right back to back h1 and rebuilt from the ground up entirely with new note-digits. The product, a series of twelve note-digits, is shifted into position using the group inverse ($^{16}/_{12}$) at the start of the fourth measure so that its conjunction-sum is in agreement with the end of the previous measure at treble g^1. From here all that remains is the addition of four note-digits (to prepare for the next calculation) which is shown in the final dotted half note chord/series of sixteen note-digits. The formal mutable base number reflecting this change has been placed on end for lack of space.

Taken in its entirety this short phrase is both a summary of, and the basis of, tonal composition during the period of common practice. Notice that the most significant digit of the mutable base number set out in the first measure is greatly enlarged – from 16 to 28 – by the harmonic exchange which marks its passage into the second measure. And that from this peak in the second measure, subsequent most significant digits gradually decline, reaching the end of the phrase at a level lower than at which it started. The most significant digits (that is to say the upper nested series) in mutable base numbers applied to music are where the association with the objective sound is most closely mirrored. Thus this short snatch of tonal composition, of musical calculation, closely embodies the familiar pattern of stress and relaxation followed in most typical musical phrases, and more broadly, compositions in general.

Mathematics and Computation in Tonal Music

The principal aim of this document is to demonstrate that Mutable Base Numbers, in and of themselves, lie within the compass of the Group of Positive Non-Zero Integers, thus forming a countable, infinite, commutative (Abelian) group; and further, as can be seen from the short example given in Figure 12 and elsewhere, that tonal musical compositions may therefore be construed, at least in principle, as a form of number processing in sound. The role played by mutable base numbers in this scenario is broadly similar to that performed by binary numbers in the electronic computer. The score may be characterised as a program run by musicians, with the aid of their instruments, that progressively computes a sequence of linked magnitudes from opening chord to final cadence. In executing this calculation of sums in the medium of musical sound, each chord and every proportion of harmonic and metrical exchange may be characterised as a multiplication of one Mutable Base Number by another.

Although I hope it is not so, quite possibly there is some fatal flaw or flaws, either in my knowledge or my argument, obvious to all but oblivious to me. If that is the case I apologise for detaining you so far. The ideas presented here and elsewhere have evolved, perhaps even matured, over some considerable time in my thinking about music. For good or no, I have given them my best shot and this document is where my arrow, eventually, came to rest.

Whether or not the concept of mutable base numbers, and the form of computation enabled by them, may be extended to fields other than music I am not competent to judge. However, notwithstanding, the conception of a mathematics emerging from the structure within, the idea of physical systems actually *being their own mathematics*, rather than being described by a math external to themselves, is an appealing thought.

Examples of the Group Axioms

Closure

MBN: $3_1 \times 2_1 = 6_1$ MBN: $3_1 \times 0_1 1_2 = 0_1 3_2$ ($= 1_1 1_2$)

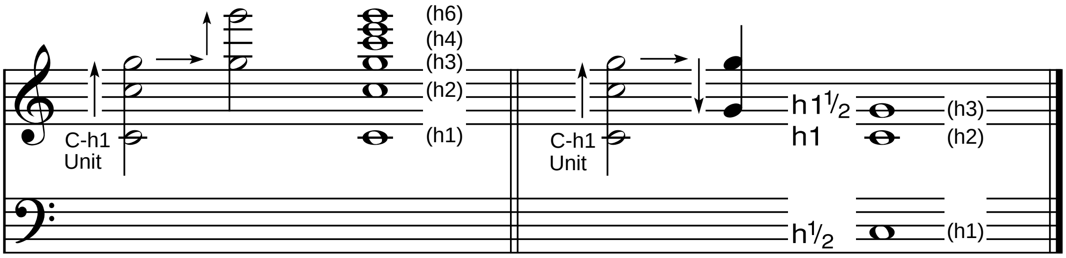


Figure 13. Closure: The product of any two mutable base numbers (group elements) multiplied together (combined by the group action) yields another mutable base number (group element).

Two examples of closure are illustrated in Figure 13. In the left hand measure two whole numbers are multiplied together by the straightforward procedure of stacking one ascending acoustic harmonic series on top of the other. As the second number/digit sequence/series begins on a harmonic of the first series all its overtones will coincide with the higher harmonics of the first number/digit sequence/series. The product of multiplying three by two is encapsulated by the whole note series on the right of this first measure, where the relationship between the unit middle c (C-h1) and high g^2 (G-h6) is explicitly drawn out.

In the right hand measure of Figure 13 the multiplication of three by a fraction is a little more involved as the nesting (i.e. multiplication) of an undertone series turns the system back upon itself. The descending undertone series is shown in quarter notes. From the unit, middle c, the mutable number three rises up to treble g^1 on top of the treble staff, and from this pitch the descending undertone series of one-half carries it down to middle g.

To make sense of this procedure, that is to relate the unit middle c to the product middle g, a whole note series built on tenor C (C-h$^1/_2$) has been drawn on the right. The logical harmonics of these whole note series are in parentheses and the actual values without parentheses. Looking at the process another way, it can be seen that if the ascending mutable number three had been transposed down an octave, that is halved in frequency, it would have come to rest in the same position as this whole note series built on tenor C.

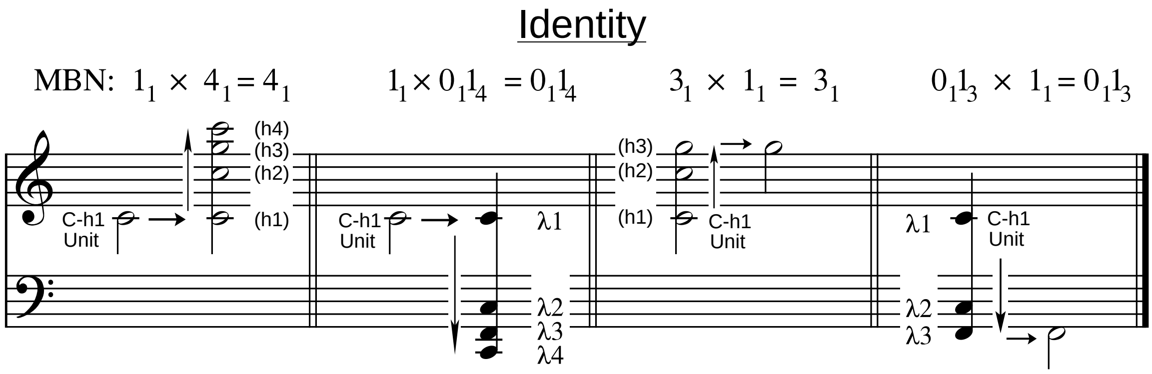


Figure 14. Identity: Any mutable base number (group element) is left unchanged under multiplication (the group action) by the unit, the identity element.

The mutable number unit, the identity element, is a single tone. Whether taken first or second in the multiplication process the unit does not change the product of multiplication from being the value of its companion term. In the left two measures of Figure 14 the unit is taken first and product of multiplication is the value of the co-term. On the right of Figure 14 there are two measures where the unit is the second of the terms in the multiplication and so appears as a single tone shifted up or down by the preceding values, without altering the product.

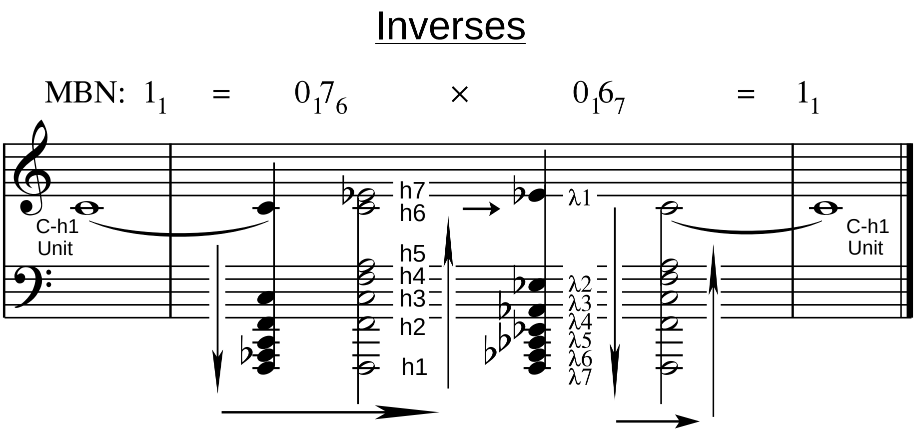


Figure 15. Inverses: For every mutable base number (group element) there is an inverse such that when multiplied together (combined by the group action) the product is unity (MBN 1_1) and unity is its own inverse.

In Figure 15 the left hand and right hand measures contain the unit. The central measure displays the multiplication of a given mutable number and its inverse, the fractions seven-sixth and six-seventh. From unity the divisor of the left hand fraction, written in quarter notes, reaches down to low F_2 beyond the bass staff in an undertone series. Followed by the numerator climbing back up in half notes to middle e-flat (d#), with its logical harmonics written out to its right. The second fraction in the expression, commencing from middle e-flat, again sees the divisor reaching down to low F_2 below the bass staff, before ascending back to middle c, the unit (MBN 1_1). Essentially the two inner series and two outer series cancel each other out leaving only the unit before and after.

Associativity

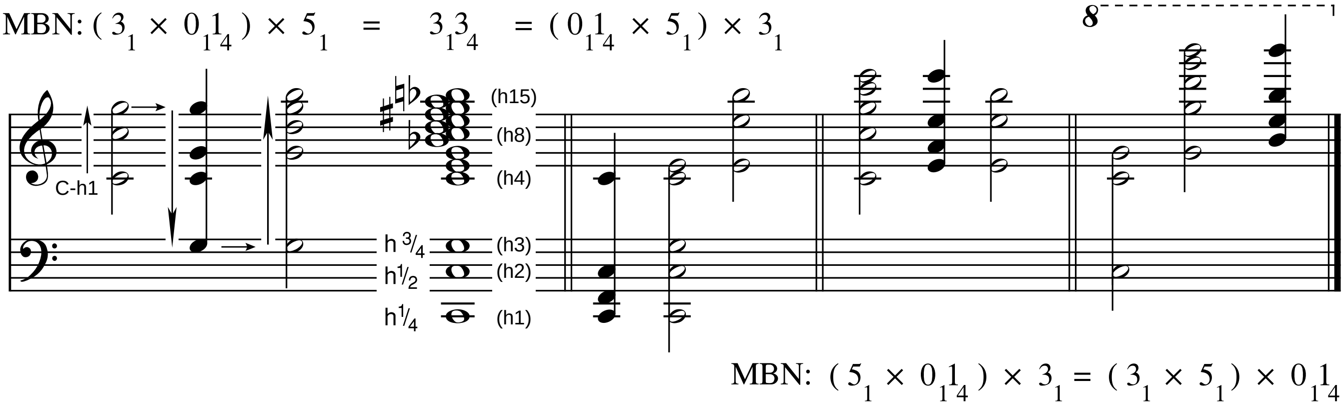


Figure 16. Associativity: For any three mutable base numbers (group elements) the precedence of combination with the group action (multiplication) does not change the final outcome.
That is: (x × y) × z = x × (y × z). Decimal: (3 × 0.25) × 5 = 3.75 = (0.25 × 5) × 3.

In Figure 16 three mutable numbers in both formal and physical form are multiplied together with varying precedence, each of which produces the same result. The left hand measure illustrates the process in some detail. The unit C-h1 is middle c. The magnitude three is presented in the form of an ascending acoustic harmonic series, which takes the system to treble g^1 on top of the treble staff. Treble g^1 (or more precisely the whole magnitude three) is then multiplied by one quarter in the form of a descending undertone series, taking the process to tenor G, in the top space of the bass staff. This is shown rather appropriately in quarter notes. (Again more precisely, notice that if the entire magnitude three ‘chord’ were shifted downward by the procedure it would come to rest in the same position as the bottom three whole notes illustrated on the right hand side of the first measure in Figure 16.) From tenor G the final multiplication by five takes the system to treble b^1 natural above the treble staff – an approximate frequency of 1000Hertz. Above the physical representation of this process as musical sound are written the concomitant formal mutable base numbers.

The ascending whole note series on the right hand side of this first measure encapsulates the complete process; in that the action of multiplying three by one quarter is incorporated in the series by reducing the fundamental tone to C-h¼. To the left the logical harmonics of this extended series are labelled and to its right the consequence of multiplying by one quarter is shown. Although obscured, also notice that the series contains both treble b^1 flat (a$\#^1$) and treble b^1 natural.

Moving right to the next measure in Figure 16 the precedence or parenthesis is transferred from the first and second terms in the original expression to the second and third. By taking these newly parenthesized terms first, associativity is demonstrated with a different route being traced from the unit middle c (C-h1), to the same product, treble b^1 natural. Again the formal mutable base numbers are written above the treble staff.

Further to the right in Figure 16 are two measures illustrating the three magnitudes multiplied with different precedence, labelled below the bass staff with formal mutable numbers. All four examples in Figure 16 begin on middle c and ends on treble b^1 natural, despite the re-ordering of precedence.

Commutativity

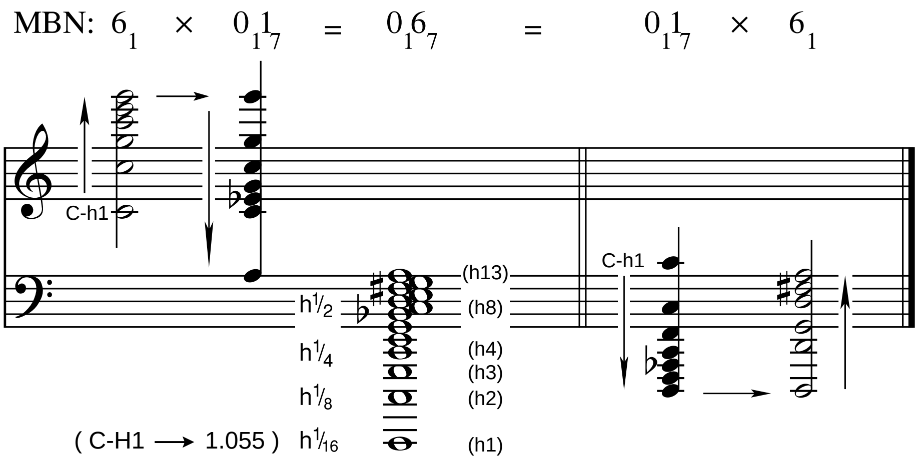


Figure 17. Commutativity: For any two mutable base numbers (group elements) the order in which the group action (multiplication) is applied does not change the outcome. That is, x × y = y × x ($6 \times {}^1/_7 = {}^1/_7 \times 6$).

Figure 17 follows a similar procedure to Figure 16. On the left hand side the unit is set at middle c from which, emanating upward, is the single column mutable number six, stretching up to high g^2 – An ascending acoustic harmonic series of six components, to which a descending undertone series of seven components is applied in the action of multiplication. That is to say from the tone high g^2 the descending undertone series reaches down to tenor A on the top line of the bass staff – the result.

In the right hand measure of Figure 17 the terms are reversed. Beginning again at the unit, middle c (C-h1), a descending undertone series (MBN $0_1 1_7$) reaches down to a low D_2 which is then multiplied by an ascending overtone series of six digits (MBN 6_1). The result, as before, culminates at tenor A.

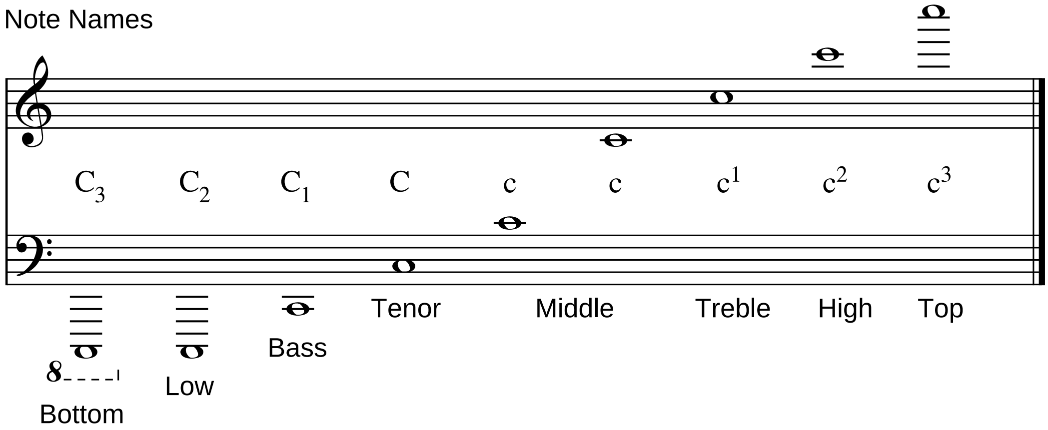
Above the treble staff the formal written representation of the physical (musical) sound is shown. And in the centre of Figure 17 there is a fundamental series, written in whole notes, relating the unit value to the product of multiplication. The fundamental tone of this series bottom C_3 (C-H1) is altered by the process of multiplication to an approximate value of C-H1.055, which is shown at the bottom left hand side. Such flexing of the frequency grid is a consequence of employing precise whole number ratios combined with exactly aligned conjunction frequencies[3]. That is:

$$6 \times {}^1/_7 \approx 13 \times {}^1/_{16} \times 1.055$$

Conclusion

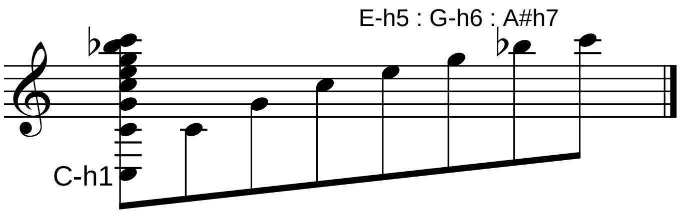
It appears to me, as a non-mathematician and assuming my understanding of these matters is broadly correct, that the mathematics presented in this document are well established and un-controversial. And further, that the formal written mutable base number system is little more than an alternative rewriting of the established regime of numerical expression. However, it is my hope that the nexus drawn here, and elsewhere, between tonally organised music at its most fundamental level of harmony and metre, and the basic mathematical machinery of a number system, may help to illuminate the essential nature of music.

Notes



The pitch nomenclature adopted in this document is shown above, one of the three schemes mentioned in the Harvard Dictionary of Music compounded with a verbal practice familiar to organ builders. The twelve ascending chromatic notes from bottom C_3 to bottom B_3 are spoken: bottom C, bottom C#, bottom D, etc... and written either as bottom C_3 or C_3 ; bottom $C\#_3$ or $C\#_3$, etc... This ascending octave based naming practice is applied throughout the compass of notes, and if required, may be extended further through the use of more super/subscripts. Also as amongst organ builders, notes are by preference named as sharps, for example A# rather than B-flat, but not exclusively so where the flattened form is more informative or convenient.

Note 1. An example of scale induced difficulties can be seen by comparing the 5:6 and 6:7 exchanges where $^6/_5$ equals 1.2 and $^7/_6$ 1.1666... Yet when these two different exchanges are translated into music, set within the grid of an equal-tempered scale, they produce identical values of 1.1892... – halfway between 5:6 and 6:7.



(At the standard pitch level of A=440Hz, E is 164.814...Hz, G is 195.998...Hz and A# is 233.082...Hz. Dividing E into G and G into A# produces 1.1892 – the value of an equal-tempered minor-third.) The ear accepts both exchanges as 'valid' chord progressions. There are perhaps two reasons for this: first and foremost the ear's ability and willingness to accommodate inaccuracies of intonation in its mission to 'understand', to categorise; and secondly familiarity, the effect of long exposure to equal-temperament.

Note 2. Although scientific research has demonstrated that undertone series are not generated by an oscillating body in the same way as an acoustic harmonic series would normally arise, it is interesting to notice that the encoding of nerve pulses signalling pitch in the mammalian hearing mechanism does take the form of an undertones series. That is, nerve pulses over a varying range of one, two, three, four, and more intervals of the period of a steady frequency are generated by the ear and flow through the aural pathway. Beament, J., *How We Hear Music,* (Boydell Press 2005) Section 9.9, p102.

Note 3. When I first began thinking about music in terms of being an evolving 'physical system', I was obsessed by the notion of everything emanating from an unchanging fundamental frequency. Relentlessly I viewed these systems from the bottom up and searched for evidence of there being some physical reality underpinning the extended nested harmonic series that I imagined were required in such a system; and rather like some of the Dualist commentators in the late nineteenth century I found little comfort. Years later and quite suddenly it dawned that I had grasped the wrong end of the stick: That it is the clear audible reality of objective musical sound that governs these systems and that the rock solid 'fundamental' foundations I had sought were indeed no more than projections of that musical realty down to a logical conclusion.

Note 4. Mistakenly I initially didn't explicitly excluded all negative rationals from Q*, I am grateful for this information.