Extract from ‘Per non mi dir ch’io moia’

Pitch Instability and Mutable Base Numbers

This short extract from a madrigal by Michaelangelo Rossi comes from the Early Music Sources website (www.earlymusicsources.com/) where it accompanies an informative video on the subject of Just Intonation. I do not wish to say anything about the information presented in the video *Just Intonation in the Renaissance* as this excellent material, professionally produced, speaks for itself. Rather I have added mutable numbers to their analysis of a strict application of Just Intonation to these measures (pages 5 & 6), in order to examine and discuss the effects of using simple ratios both within chords themselves and between the adjacent harmonic series hosting them. A practice fundamental to the Mutable Base Number model of tonal music. And in so doing, hopefully, further illuminate the nature of each unique trajectory of whole numbers and sums computed by any given tonal composition. (This ‘absolute’ whole number data is usually displayed at the end of analyses, here on pages 8 & 9.) Needless to say, watching the above video is highly recommended, and will be most helpful in understanding the argument presented and illustrated in this document and elsewhere: *That mutable numbers are a close, but idealised, abstraction of the physical reality of tonal music; a reality compromised as it always must be by physics on the one hand and cultural musical practice on the other, yet despite the approximation and aberration inherent in live musical performance, the model captures, as does the ear and mind, the essential sense and logic of the system.*

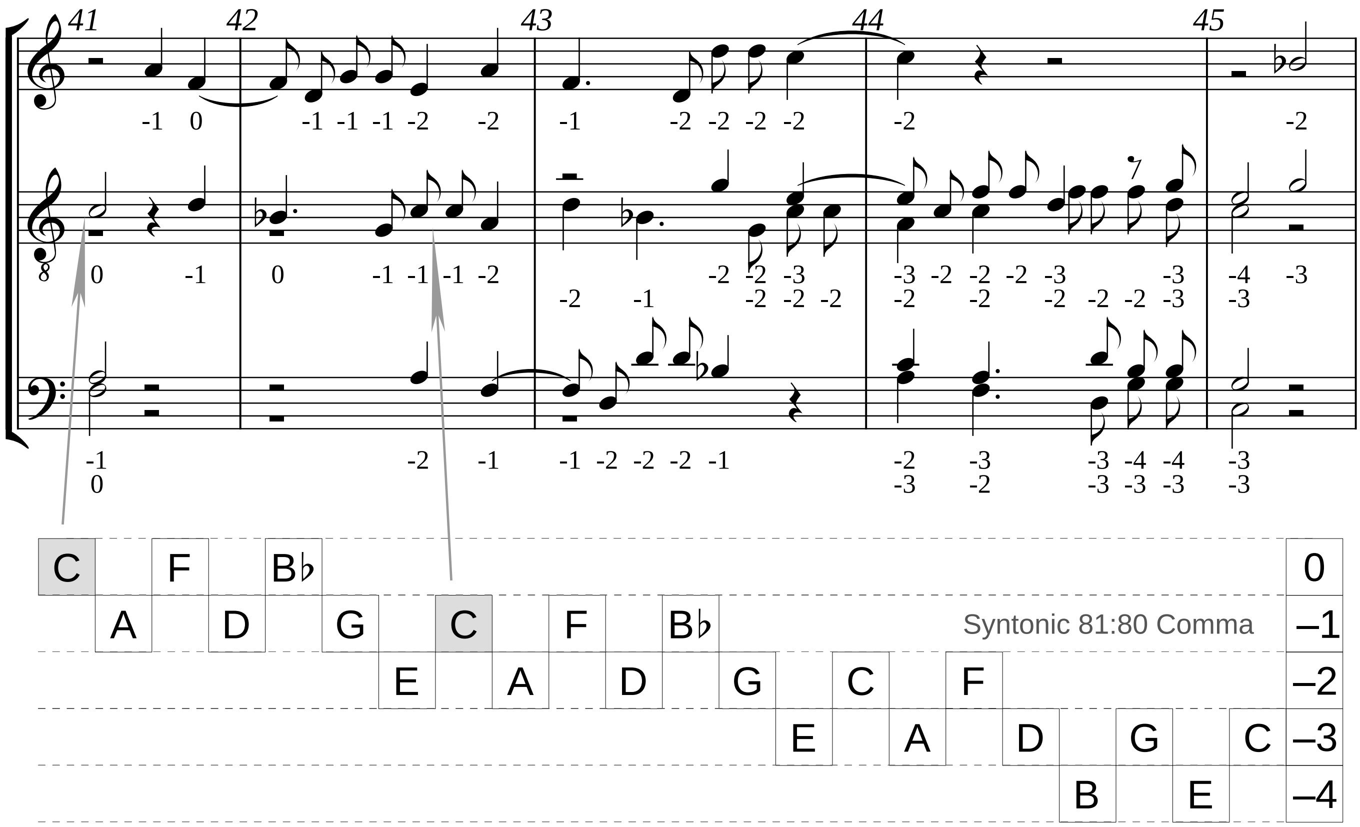


Figure 1. The extract from ‘Per non mi dir ch’io moia’ with the syntonic levels filled by sequences of rising fourths rather than falling fifths for illustrative purposes. Notice in particular that the note C in the alto part in measures 41 and 42 is located on different syntonic levels. Also the harmonic progression (F major, D minor, Bb major, etc..) traced out in an overlapping chain of groups of three squares, bears testament to the connecting role of common frequencies between (all but one of) the chords, as detailed in the mutable number analysis below.

The Syntonic Comma

A comma in music refers to a discrepancy that arises between different means of calculation. The syntonic comma (or didymic comma, from Didymos, Greek philosopher and theorist) comes from the difference between calculating in perfect 2:3 fifths and pure 1:2 octaves combined with a Just 4:5 major third. And it is at the centre of one of the conundrums inherent to the intersection between the practical demands of tonal music making and the physical nature of sound. If Just Intonation is rigidly applied to music in performance we garner the reward of sweetly pure intervals at the cost of compromising the pitch stability of the composition – illustrated in Figure 1. by numbers 0 through –4 syntonic commas.

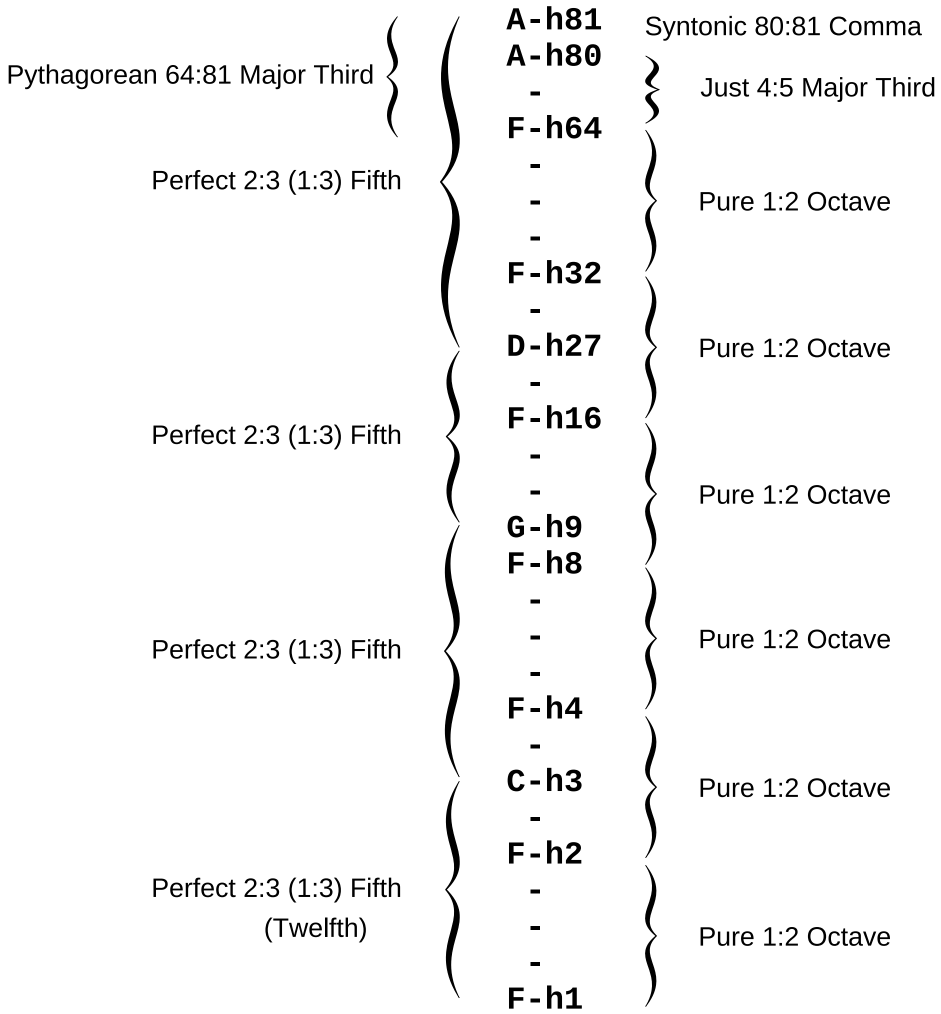


Figure 2. Two different ways of calculating the interval of a major third. The difference between the Pythagorean 64:81 major third and the Just 64:80 (4:5) major third forms the Syntonic 80:81 Comma.

The first chord in the extract, measure 41, is F major (notes† tenor F, tenor A and middle c) and there are two different ways of calculating the pitch of tenor A relative to the root of the triad. On the left in Figure 2. is the Pythagorean method, where the note is generated by a sequence of perfect 2:3 fifths – displayed as twelfths rising up the harmonic series to A-h81. On the right the same note is generated by an alternative method using octaves and a true Just Intonation 4:5 major third which calculates that the note is A-h80! Between the two methods of calculation lies a small gap, the syntonic comma, which is approximately 22 cents or a bit less than one quarter of a semitone. Expressed as a ratio, ascending it is 80:81 or descending 81:80.

† Please see Appendix III (page 11) for the nomenclature conventions used in this document.

Returning to Figure 1. In the first chord of F major, if the note middle c is assigned a pitch of 256Hz – syntonic level zero in the example – the root of the triad, F, calculated by a Just Intonation perfect 2:3 fifth will be:

(middle c) $256 \times 2/3 = 170.666...$Hz. (tenor F)

And the note tenor A, placed a Just 4:5 major third above F will be:

(tenor F) $170.666... \times 5/4 = 213.333...$ Hz. (tenor A)

Leading on from this F major chord the harmonic progression is: D minor, Bb major, G minor, A minor.

So now proceeding in a similar manner from tenor A, in steps of Just intervals, it is possible to calculate the pitches of the notes in these succeeding chords. Middle d is a perfect 3:4 fourth above A:

(tenor A) $213.333... \times 4/3 = 284.444...$ Hz. (middle d)

And from middle d both the succeeding notes, B-flat and G can be calculated Justly, either through the two steps of multiplying by a descending major 5:4 third and a descending minor 6:5 third, or by stepping directly to G by multiplying by a descending perfect 3:2 fifth. Please try the two steps on a calculator, here is the direct route:

(middle d) $284.444... \times 2/3 = 189.629...$ Hz. (tenor G)

Next from tenor G in the alto part it is only an ascending step of a perfect 3:4 fourth to reach the next note, middle c – providing the minor third in the A minor chord:

(tenor G) $189.629... \times 4/3 = \underline{252.8395061}$ Hz. (middle c)

A conundrum! Having begun at middle c 256Hz, by means of a few precise Just interval steps the alto part has reached the lower pitch of middle c 252.8395061Hz one measure later. How flat relative to 256Hz is this? One syntonic comma:

(–1 syntonic comma) $252.8395061 \times 81/80 = 256$ Hz. (middle c, level zero)

In the space of one measure the alto part has sunk approximately 3.2Hz, or 22cents, or $^{1}/_{4}$ of a semitone and over the five measures from 41 to 45 the pitch of middle c is reduced to near 246.6Hz or $^{3}/_{4}$ of a semitone. For the whole story, a table containing the complete example with the frequencies of all notes, rendered in hertz, is provided at the foot of this document.

It is clear from these calculations that Just Intonation, applied rigorously in performance, must affect the pitch stability of a composition. Indeed, in some more complex instances, to maintain pure intervals would involve adjusting the pitch of notes, in flight, and/or making awkward steps between notes. These difficulties are particularly high-lighted in the performance of contrapuntal polyphonic music where the horizontal motion of the voices is a crucial feature. Whereas in contrast a melody riding above or within a chord-laden accompaniment may appear to escape more lightly, but even though less apparent, there is no hiding place from the inescapable mathematics. Just Intonation and pitch stability are incompatible. The situation might be characterised as being analogous to that of trying to stick a square grid Mercator projection map upon the curved surface of a globe – something has to give!

Having establish that pitch instability is inherent to Just Intonation, there is some small glimmer of relief to be found in the fact that it is not confined to declining pitch. Equally, depending upon the intervals involved and their disposition within a composition, it is possible for a piece to ascend in pitch. The archetypal example being a repeated pattern of rising perfect 2:3 fifth and falling perfect 4:3 fourth describing the spiral of fifths out to the Pythagorean Comma and beyond: Each interval raising the pitch by approximately two cents. However, if these two intervals of a perfect fifth and fourth are both ascending or descending they would cancel out relative to the octave, the Just fifth being 702 cents and the fourth 498 cents, together they equal 1200 cents, an octave:

$$2:3 \times 3:4 = 6:12 = 1:2 \qquad 3/2 \times 4/3 = 12/6 = 2/1$$

Thus, given a judicious mixture of Just intervals, ascending and/or descending, it would be possible for a composition to oscillate around a stable pitch centre – theoretically. (I have not yet found a non-trivial composition that achieves this stability, but no doubt there probably is one out there somewhere!)

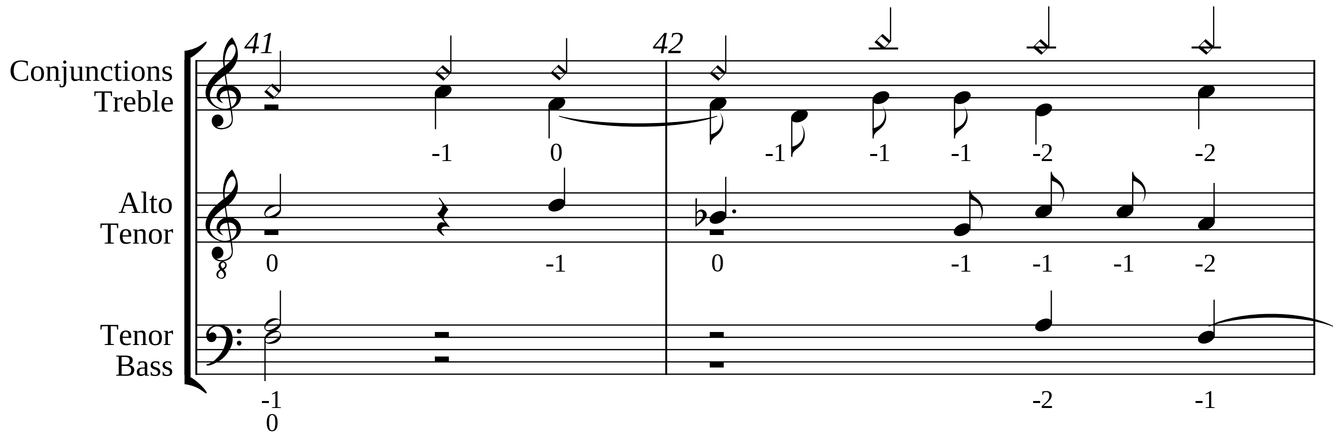
Just Intonation and Mutable Numbers

This is the crucial point: *At its most fundamental level, viewed through the lens of Mutable Base Numbers, tonal music is ordered by harmony* – the contrapuntal aspects are secondary. For myself, brought up on the English renaissance school and amongst organists and organ builders this conclusion is profoundly difficult and disappointing. One of the attributes of the ear is the ability to simultaneously distinguish and track multiple sound sources emanating from a variety of different directions. Clearly derived from an important evolutionary imperative. I suspect that our delight in polyphonic music generally and counterpoint in particular is linked to the exercise of this facility of hearing. However in tonally organised music the joyous exertion of this faculty, through the apprehension of elaborate figuration or intricate contrapuntal motion between independent voices can, at most, outline an underlying harmonic structure, while on other occasions merely ornament it. Counterpoint, the contrapuntal dimension of music, like timbre, is not foundational, tonal organisation is. And in pursuit of the truly fundamental, mutable numbers show scant regard for the delightful, the exquisite, in tonal musical art. Rather the model delineates a mundane sequence of discrete states or sums, that notwithstanding, are the ground upon which the whole edifice stands – *and that ground is rarely level:* For mutable numbers describe a unique whole number topography for each and every distinct tonal composition that incorporates the pitch instability dictated by Just Intonation.

There is another faculty of the ear of the highest importance, its ability to categorise and remember sound stimuli. From such libraries of remembered sensations aural cognition apprises and evaluates sound stimuli, fitting them, as best it can, into an existing palette of aural experience. Over time, apperceptively, new aural memories are created, while established memories are constantly burnished and occasionally adjusted. In particular, musical stimuli – the major or minor triad, bare fifth, the seventh chord, etc. – become deeply embedded, making categorical identification from mere resemblance (and expectation) possible. The ear and tool chain of aural cognition bring this integrated physical and mental apparatus to the task of understanding musical sound, and in so doing is able to extract a perception of the ideal form out of the wayward variability of audible stimuli. *Though we hear an equal-tempered 4:5.039684307 major third, a sensation more acerbic than sweet, we comprehend the meaning of a pure 4:5 relationship; and, through the plasticity of ear and mind, reconcile the conflicting demands of melody, pitch and ratio, thereby transcending fractured musical reality in a synthesis combining contrapuntal integrity with harmonic structure (mutable base numbers).* And all this with more ease than I might catch the sentence spoken in a Geordie or West Country dialect.

Mutable Number Analysis

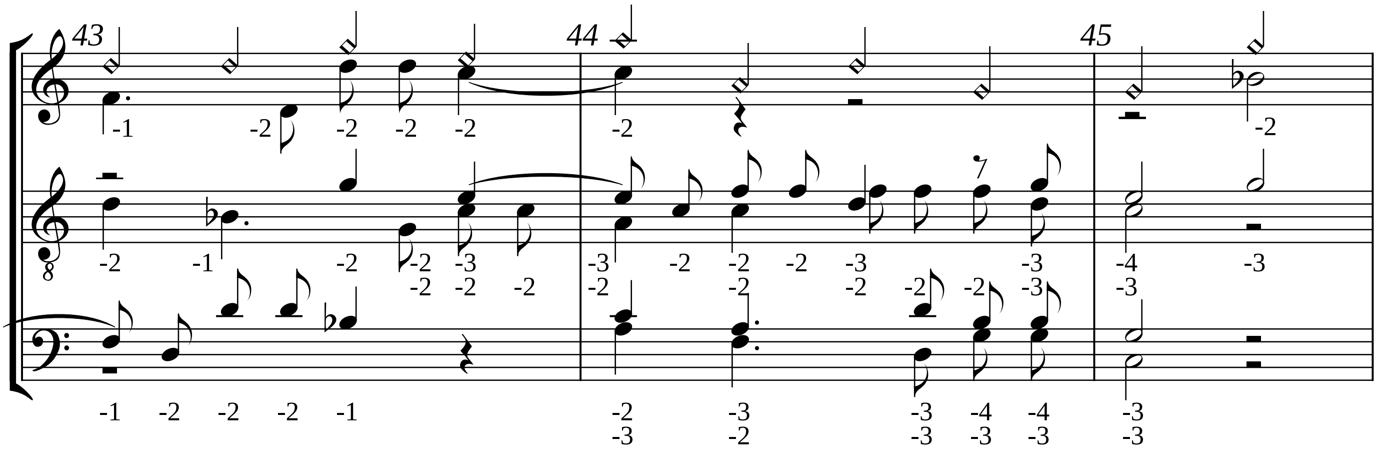
The mutable number analysis is set out in summary form. The minor third in chords is notated without ratio‡, in line with the comments made on page five of *The Math of Exchange*, and more generally, it has been endeavoured to keep the nested series as short as possible. To this end, in two places – both G minor chords – upward steps have incorporated an extra octave. Additionally, from measures 43 to 45 the full whole number values of nested fundamentals – h1[Hn] – have been omitted due to lack of space, however, these values can be read from the Mutable Base Numbers lower down on the page. Diamond-headed conjunction frequencies have been appended to the upper staff. [An alternative version of this analysis is presented on page 12.]



				B-h40*--->	B-h18* -2	
Conjunctions		D-h16*---->	D-h20*--->	D-h24* +16	A-h16*--->	A-h20--->
& Partials *	A-h10*--->	A-h12~ +4	F-h12~	G-h16~	E-h6~	A-h10~
	C-h6~	F~	D-h10~	A#~	C~	A-h5~
Notes ~	A-h5~	D-h8~	A#h8~	G-h8~	A-h4~	F-h4~
	F-h4~	C-h7	G#h7	F-h7	E-h3	C-h3
	C-h3	A-h6	F-h6	D-h6	A-h2	F-h2
	F-h2	F#h5	D-h5	B-h5	A-h1[H437400]	F-h1[H349920]
	F-h1[H354294]	D-h4	A#h4	G-h4	(52.6749Hz)	(42.1399Hz)
	(42.666...Hz)	A-h3	F-h3	D-h3	5:4-->	6:5-->
	6:5-->	D-h2	A#h2	G-h2	[H1]	[H1]
	[H1]	D-h1[H295245]	A#h1[H236196]	G-h1[H196830]		
		(35.555...Hz)	(28.444...Hz)	(23.703...Hz)		
		5:4-->	6:5-->	9:20-->		
		[H1]	[H1]	[H1]		

MBN: $10_{354294}0_1 = 12_{295245}0_1$
$+\ 4_{295245}0_1$
$= 16_{295245}0_1 = 20_{236196}0_1 = 24_{196830}0_1$
$+\ 16_{196830}0_1$
$= 40_{196830}0_1 = 18_{437400}0_1$
$-\ 2_{437400}0_1$
$= 16_{437400}0_1 = 20_{349920}0_1 =$

‡ Values in the fundamental nesting series may be calculated –i.e. F~ H2834352 in the D minor chord, measure 41.



A-h24* -8		G-h32*->	G-h12*-2	A-h16*->	A-h20-10	D-h16*->	D-h12*-4	G-h12*->	G-h8*
D-h16*--->	D-h20*->	D-h24~+8	E-h10*->	E-h12*+4	A-h10*->	A-h12*+4	G-h8~->	G-h6* +6	A#~
F~	F-h12~	G-h16~	C-h8~	C~	F-h8~	F~	D-h6~	E-h5~	G-h4~
D-h8~	D-h10~	A#~	E-h5~	E-h6~	C-h6~	D-h8~	B-h5~	C-h4~	D-h3
F~	A#h8~	G-h8~	C-h4~	C~	A-h5~	F~	G-h4~	G-h3~	G-h2
D-h4~	G#h7	F-h7	G-h3	A-h4~	F-h4~	D-h4~	D-h3	C-h2~	G-h1
A-h3	F-h6	D-h6	C-h2	E-h3	C-h3	A-h3	G-h2	C-h1	92.5Hz
D-h2	D-h5	B-h5	C-h1	A-h2	F-h2	D-h2	G-h1	61.7Hz	m:n-->
D-h1H291600	A#h4	G-h4	62.4Hz	A-h1	F-h1	D-h1	46.2Hz	2:3-->	[H1]
(35.1166Hz)	F-h3	D-h3	6:5-->	52.0Hz	41.6Hz	34.7Hz	3:4-->	[H1]	
5:4-->	A#h2	G-h2	[H1]	5:4-->	6:5-->	3:4-->	[H1]		
[H1]	A#h1[Hn]	G-h1		[H1]	[H1]	[H1]			
	28.1Hz	23.4Hz							
	6:5-->	3:8-->							
	[H1]	[H1]							

$$= 24_{291600}0_1$$
$$- \ 8_{291600}0_1$$
$$= 16_{291600}0_1 = 20_{233280}0_1 = 24_{194400}0_1$$
$$+ \ 8_{194400}0_1$$
$$= 32_{194400}0_1 = 12_{518400}0_1$$
$$- \ 2_{518400}0_1$$
$$= 10_{518400}0_1 = 12_{432000}0_1$$
$$+ \ 4_{432000}0_1$$
$$= 16_{432000}0_1 = 20_{345600}0_1$$
$$- 10_{345600}0_1$$
$$= 10_{345600}0_1 = 12_{288000}0_1$$
$$+ \ 4_{288000}0_1$$
$$= 16_{288000}0_1$$

$$= 16_{288000}0_1 = 12_{384000}0_1$$
$$- \ 4_{384000}0_1$$
$$= \ 8_{384000}0_1 = \ 6_{512000}0_1$$
$$+ \ 6_{512000}0_1$$
$$= 12_{512000}0_1 = 8_{768000}0_1$$

[See Appendix IV, page 12 for Re-numbered Analysis]

The Whole Tone Exchange

The whole tone exchange in measure 42, between G minor and A minor chords, presents some interesting choices, each one with advantages and disadvantages. In the analysis above the sesquinona 9:10 minor whole tone exchange is used (combined with an octave leap) so as to keep the mutable numbers in step with what I shall term the ‘syntonic calculation’ –i.e. Just, sequentially calculated, voices or parts. The other two possible exchanges are the major 8:9 whole tone or the septimal 7:8 whole tone, illustrated in Figure 3.

```
Missing Conjunction   A-h36*      A-h36*--->  A-h16* -8   |  G-h32*===>  G-h14* -6
                      D-h24*      D-h24* +12  A-h8*--->   |  D-h24* +8   A-h8*--->
Missing Note          D-h12~      G-h16~      E-h6~       |  G-h16~      E-h6~
                                  A#~         C~          |  A#~         C~
                                  G-h8~       A-h4~       |  G-h8~       A-h4~
                                  F-h7        E-h3        |  F-h7        E-h3
                                  D-h6        A-h2        |  D-h6        A-h2
                                  B-h5        A-h1        |  B-h5        A-h1
                                  G-h4        5:4--->     |  G-h4        5:4--->
                                  D-h3        [H1]        |  D-h3        [H1]
                                  G-h2                    |  G-h2
                                  G-h1                    |  G-h1
                                  8:18--->                |  7:16--->
                                  [H1]                       [H1]
```

Figure 3. The sesquioctava 8:9 major whole tone exchange and the sesquiseptima 7:8 septimal whole tone exchange (plus octaves).

So, which exchange best describes the aural experience? To begin, the minor 9:10 whole tone, as used in the analysis in measure 42 above, allows the extract to follow the syntonic calculation and so illustrates the point that mutable numbers follow the logic of Just Intonation and in so doing incorporate the pitch instability inherent in using simple ratios of exchange. On the other hand, although the symmetry of the exchange is perfectly regular – B-h40 exchanged for B-h18, and both partials extant, generated by G-h8~ and E-h6~, it would be difficult for the ear to distinguish this conjunction against the background of partials generated by the minor thirds in both chords, A#~ and C~ respectively. Thus it is mathematically correct but in practical terms implausible.

The major 8:9 whole tone exchange fares little better. In this particular case the fifth is missing from the G minor chord – though D is present in the preceding chord so perhaps its influence may linger somewhat. Yet without an objective middle d in the G minor chord there will be no third harmonic generated at the level of A-h36* to form a conjunction with the A minor chord. The partial D-h24* is present to receive the preceding exchange, generated as a third harmonic by G-h8~, but the first objective A partial only arises from the same source at A-h72*. Again, at a rate of A-h72* exchanged for A-h32* there is perfect 8:18 symmetry, but whether the ear could discern such a partial against the strength of partials generated by the octave G-h8~ and G-h16~ is questionable.

The ear must work with what is audibly available, and what is there is a link between a strong G-h32* harmonic and a rather ill-matched third harmonic generated by middle c natural, the minor third in the A minor chord. This receiving end of the conjunction for a sesquiseptima 7:8 (16) exchange will be way sharp of a perfect alignment, and this is indicated by the use of ==> rather than ---> in Figure 3. Elsewhere

(*Chopin: Prelude 20*) I have described ‘improper’ conjunctions, in a mathematical context. This is where the ear is using a linking frequency that does not apportion elements between the nested series party to the exchange symmetrically. However, in this case there is nothing ‘improper’ mathematically about the symmetry of a 7:16 exchange between G-h32* exchanged for G-h14* – two groups of sixteen elements and two groups of seven elements. The problem lies in the alignment of the conjunction frequency, and so perhaps the term ‘auxiliary’ or perhaps even ‘awkward’ conjunction might be used. And interestingly, adding to this upward thrust, the ascending broader step of a septimal whole tone exchange more than compensates for the downward syntonic pressure, pushing C~ in the A minor chord up to 260Hz, providing an example of upward pitch mobility.

As suggested above, the ear and aural cognition work hard to make sense of auditory stimuli, and in the A minor chord a third harmonic of middle c at the rate of G-h14.27* generated by a tempered C natural or G-h14.4* from a Just one is, probably, grist to the mill of aural cognition when considering the strength of the incoming G-h32* conjunction generated by the G minor chord and bare octave. Additionally, learned habits, memories and expectations will doubtless play a part too. The ear accepts, learns and remembers stimuli, matching roughly right transitions to better formed prototypes, acquired and confirmed over time and experience, in its unending quest to make sense of aural reality.

Overall, the argument is not that the ear follows the mathematical propriety of mutable numbers rigidly, after all it has its own work to do conditioned by eons of evolutionary development, only that it mostly does – sometimes bending the rules towards the edges in its struggle to understand wayward aural stimuli. Therefore an appreciation of the underlying mathematical logic involved in the cognition of tonal music may prove informative, perhaps even useful, in understanding music at its deepest structural level.

Appendix I - Absolute Whole Number Analyses

Sesquinona 9:10 (20) Minor Whole Tone Exchange

Bar	Chord, Proportion			h1 × Nested Series	Value Computed	Prime Factors		
41	F	FundamentalSeriesH1->H?		354294 × 10 = 295245 × 12 =	3542940	2**2	3**11	5
	D	5/6	(6:5)	(h1) = 295245 × 16 = 236196 × 20 =	4723920	2**4	3**10	5
42	Bb	4/5	(5:4)	(h1) = 236196 × 20 = 196830 × 24 =	4723920	2**4	3**10	5
	G	5/6	(6:5)	(h1) = 196830 × 40 = 437400 × 18 =	7873200	2**4	3**9	5**2
	A	20/9	(9:20)	(h1) = 437400 × 16 = 349920 × 20 =	6998400	2**7	3**7	5**2
	F	4/5	(5:4)	(h1) = 349920 × 20 = 291600 × 24 =	6998400	2**7	3**7	5**2
43	D	5/6	(6:5)	(h1) = 291600 × 16 = 233280 × 20 =	4665600	2**8	3**6	5**2
	Bb	4/5	(5:4)	(h1) = 233280 × 20 = 194400 × 24 =	4665600	2**8	3**6	5**2
	G	5/6	(6:5)	(h1) = 194400 × 32 = 518400 × 12 =	6220800	2**10	3**5	5**2
	C	8/3	(3:8)	(h1) = 518400 × 10 = 432000 × 12 =	5184000	2**9	3**4	5**3
44	A	5/6	(6:5)	(h1) = 432000 × 16 = 345600 × 20 =	6912000	2**11	3**3	5**3
	F	4/5	(5:4)	(h1) = 345600 × 10 = 288000 × 12 =	3456000	2**10	3**3	5**3
	D	5/6	(6:5)	(h1) = 288000 × 16 = 384000 × 12 =	4608000	2**12	3**2	5**3
	G	4/3	(3:4)	(h1) = 384000 × 8 = 512000 × 6 =	3072000	2**13	3	5**3
45	C	4/3	(3:4)	(h1) = 512000 × 12 = 768000 × 8 =	6144000	2**14	3	5**3
	G	3/2	(2:3)	(h1) = 768000		2**11	3	5**3

Sesquioctava 8:9 (18) Major Whole Tone Exchange

Bar	Chord	Proportion	h1 × nested series	Value Computed	Prime Factors
41	F	FundamentalSeriesH1->H?	8748 × 10 = 7290 × 12 =	87480	2**3 3**7 5
41	D	5/6 (6:5)	(h1) = 7290 × 16 = 5832 × 20 =	116640	2**5 3**6 5
42	Bb	4/5 (5:4)	(h1) = 5832 × 20 = 4860 × 24 =	116640	2**5 3**6 5
42	G	5/6 (6:5)	(h1) = 4860 × 36 = 10935 × 16 =	174960	2**4 3**7 5
42	A	18/8 (8:18)	(h1) = 10935 × 16 = 8748 × 20 =	174960	2**4 3**7 5
42	F	4/5 (5:4)	(h1) = 8748 × 20 = 7290 × 24 =	174960	2**4 3**7 5
43	D	5/6 (6:5)	(h1) = 7290 × 16 = 5832 × 20 =	116640	2**5 3**6 5
43	Bb	4/5 (5:4)	(h1) = 5832 × 20 = 4860 × 24 =	116640	2**5 3**6 5
43	G	5/6 (6:5)	(h1) = 4860 × 32 = 12960 × 12 =	155520	2**7 3**5 5
43	C	8/3 (3:8)	(h1) = 12960 × 10 = 10800 × 12 =	129600	2**6 3**4 5**2
44	A	5/6 (6:5)	(h1) = 10800 × 16 = 8640 × 20 =	172800	2**8 3**3 5**2
44	F	4/5 (5:4)	(h1) = 8640 × 10 = 7200 × 12 =	86400	2**7 3**3 5**2
44	D	5/6 (6:5)	(h1) = 7200 × 16 = 9600 × 12 =	115200	2**9 3**2 5**2
44	G	4/3 (3:4)	(h1) = 9600 × 8 = 12800 × 6 =	76800	2**10 3 5**2
45	C	4/3 (3:4)	(h1) = 12800 × 12 = 19200 × 8 =	153600	2**11 3 5**2
45	G	3/2 (2:3)	(h1) = 19200		2**8 3 5**2

Sesquiseptima 7:8 (16) Septimal Whole Tone Exchange

Chord,	Proportion	h1 × Nested Series	Value Computed-	Prime Factors
F	FundamentalSeriesH1->H?	275562 × 10 = 229635 × 12 =	2755620	2**2 3**9 5 7
D	5/6 (6:5)	(h1) = 229635 × 16 = 183708 × 20 =	3674160	2**4 3**8 5 7
Bb	4/5 (5:4)	(h1) = 183708 × 20 = 153090 × 24 =	3674160	2**4 3**8 5 7
G	5/6 (6:5)	(h1) = 153090 × 32 = 349920 × 14 =	4898880	2**6 3**7 5 7
A	16/7 (7:16)	(h1) = 349920 × 8 = 279936 × 10 =	2799360	2**8 3**7 5
F	4/5 (5:4)	(h1) = 279936 × 10 = 233280 × 12 =	2799360	2**8 3**7 5
D	5/6 (6:5)	(h1) = 233280 × 16 = 186624 × 20 =	3732480	2**10 3**6 5
Bb	4/5 (5:4)	(h1) = 186624 × 20 = 155520 × 24 =	3732480	2**10 3**6 5
G	5/6 (6:5)	(h1) = 155520 × 32 = 414720 × 12 =	4976640	2**12 3**5 5
C	8/3 (3:8)	(h1) = 414720 × 10 = 345600 × 12 =	4147200	2**11 3**4 5**2
A	5/6 (6:5)	(h1) = 345600 × 16 = 276480 × 20 =	5529600	2**13 3**3 5**2
F	4/5 (5:4)	(h1) = 276480 × 10 = 230400 × 12 =	2764800	2**12 3**3 5**2
D	5/6 (6:5)	(h1) = 230400 × 16 = 307200 × 12 =	3686400	2**14 3**2 5**2
G	4/3 (3:4)	(h1) = 307200 × 8 = 409600 × 6 =	2457600	2**15 3 5**2
C	4/3 (3:4)	(h1) = 409600 × 12 = 614400 × 8 =	4915200	2**16 3 5**2
G	3/2 (2:3)	(h1) = 614400		2**13 3 5**2

Appendix II - Note Values in hertz, All Solutions

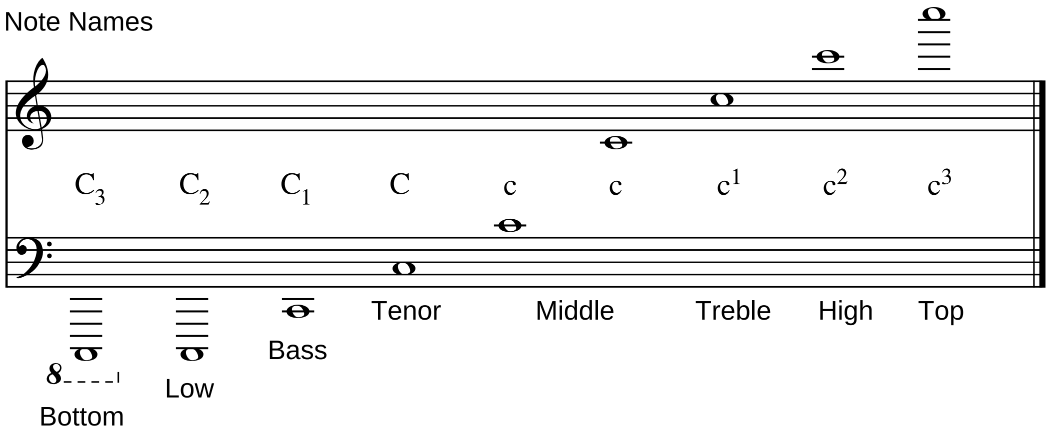
Presented over the following two pages is a table of the frequencies, in hertz, listing all the outcomes generated by using the three different exchanges: the sesquiseptima 7:8 septimal whole tone, the sesquioctava 8:9 major whole tone and the sesquinona 9:10 minor whole tone exchanges, plus the sequential in-line syntonic calculation. (Initial frequency middle c = 256Hz.)

Please notice that the values in the sesquinona and syntonic sequences are effectively identical.

Proportion	Nested h1, Hz	Note, Hz	Note, Hz	Note, Hz	Note, Hz
(41)					
Just 1:1	F 42.6666666	F 170.666...	A 213.333...	C 256.000...	
Syntonic		F 170.666...	A 213.333...	C 256.000...	
Just 6:5	D 35.555...	D 284.444...	F 341.333...	A 426.666...	
Syntonic		D 284.444...	F 341.333...	A 426.666...	
(42)					
Just 5:4	A#28.444...	A#227.555...	D 284.444...	F 341.333...	
Syntonic		A#227.555...	D 284.444...	F 341.333...	
Just 6:5 Calc.	G 23.703...	G 189.629...	A#227.555...	G 379.259...	
Syntonic Calc.		G 189.629...	A#227.555...	G 379.259...	
1. Just 7:16 Calc.	A 54.17989417	A 216.7195767	C 260.063492	E 325.079365	
2. Just 8:18 Calc.	A 53.333...	A 213.333...	C 256.000...	E 319.999...	
3. Just 9:20 Calc.	A 52.67489711	A 210.6995884	C 252.8395061	E 316.0493827	
Syntonic Calc.		A 210.6995884	C 252.8395061	E 316.0493827	
1. Just 5:4 Calc.	F 43.34391534	F 173.3756614	A 216.7195767	A 433.4391534	
2. Just 5:4 Calc.	F 42.666...	F 170.666...	A 213.333...	A 426.666...	
3. Just 5:4 Calc.	F 42.13991769	F 168.5596708	A 210.6995884	A 421.3991769	
Syntonic Calc.		F 168.5596707	A 210.6995884	A 421.3991768	
(43)					
1. Just 6:5 Calc.	D 36.11992945	D 144.4797178	F 173.3756614	D 288.9594356	F 346.7513227
2. Just 6:5 Calc.	D 35.555...	D 142.222...	F 170.666...	D 284.444...	F 341.333...
3. Just 6:5 Calc.	D 35.11659807	D 140.4663923	F 168.5596707	D 280.9327846	F 337.1193415
Syntonic Calc.		D 140.4663922	F 168.5596707	D 280.9327845	F 337.1193414
1. Just 5:4 Calc.	A#28.89594356	A#231.1675485	D 288.9594356	F 346.7513227	
2. Just 5:4 Calc.	A#28.444...	A#227.555...	D 284.444...	F 341.333...	
3. Just 5:4 Calc.	A#28.09327846	A#224.7462277	D 280.9327846	F 337.1193415	
Syntonic Calc.		A#224.7462276	D 280.9327845	F 337.1193414	
1. Just 6:5 Calc.	G 24.07995297	G 192.6396238	A#231.1675485	G 385.2792475	D 577.9188713
2. Just 6:5 Calc.	G 23.703...	G 189.629...	A#227.555...	G 379.259...	D 568.888...
3. Just 6:5 Calc.	G 23.41106538	G 187.288523	A#224.7462276	G 374.5770461	D 561.8655691
Syntonic Calc.		G 187.288523	A#224.7462276	G 374.577046	D 561.865569
1. Just 3:8 Calc.	C 64.21320791	C 256.8528316	E 321.0660395	C 513.7056633	
2. Just 3:8 Calc.	C 63.20987654	C 252.8395062	E 316.0493827	C 505.6790123	
3. Just 3:8 Calc.	C 62.42950768	C 249.7180307	E 312.1475384	C 499.4360614	
Syntonic Calc.		C 249.7180307	E 312.1475383	C 499.4360613	
(44)					
1. Just 6:5 Calc.	A 53.51100659	A 214.0440264	C 256.8528316	E 321.0660395	C 513.7056634
2. Just 6:5 Calc.	A 52.67489712	A 210.6995885	C 252.8395062	E 316.0493827	C 505.6790124
3. Just 6:5 Calc.	A 52.02458973	A 208.0983589	C 249.7180307	E 312.1475384	C 499.4360614
Syntonic Calc.		A 208.0983589	C 249.7180307	E 312.1475384	C 499.4360614

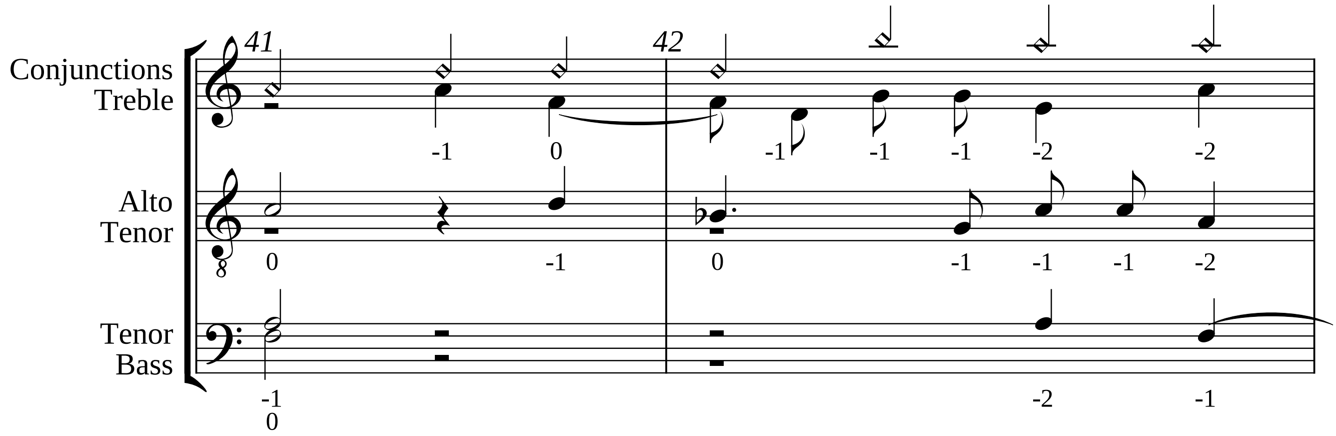
(table continued)					
Proportion	**Nested h1, Hz**	**Note, Hz**	**Note, Hz**	**Note, Hz**	**Note, Hz**
1. Just 5:4 Calc.	F 42.80880527	F 171.2352211	A 214.0440264	C 256.8528316	F 342.4704422
2. Just 5:4 Calc.	F 42.13991769	F 168.5596708	A 210.6995884	C 252.8395061	F 337.1193415
3. Just 5:4 Calc.	F 41.61967179	F 166.4786872	A 208.0983589	C 249.7180307	F 332.9573743
Syntonic Calc.		F 166.4786871	A 208.0983589	C 249.7180307	F 332.9573742
1. Just 6:5 Calc.	D 35.67400439	D 142.6960176	F 171.2352211	D 285.3920351	F 342.4704421
2. Just 6:5 Calc.	D 35.11659808	D 140.4663923	F 168.5596708	D 280.9327846	F 337.1193416
3. Just 6:5 Calc.	D 34.68305982	D 138.7322393	F 166.4786871	D 277.4644786	F 332.9573743
Syntonic Calc.		D 138.7322392	F 166.4786871	D 277.4644785	F 332.9573742
1. Just 3:4 Calc.	G 47.56533919	G 190.2613568	B 237.826696	D 285.3920351	G 380.5227135
2. Just 3:4 Calc.	G 46.82213077	G 187.2885231	B 234.1106539	D 280.9327846	G 374.5770462
3. Just 3:4 Calc.	G 46.24407976	G 184.976319	B 231.2203988	D 277.4644786	G 369.9526381
Syntonic Calc.		G 184.976319	B 231.2203987	D 277.4644785	G 369.952638
(45)					
1. Just 3:4 Calc.	C 63.42045225	C 126.8409045	G 190.2613568	C 253.681809	E 317.1022612
2. Just 3:4 Calc.	C 62.42950769	C 124.8590154	G 187.2885231	C 249.7180308	E 312.1475385
3. Just 3:4 Calc.	C 61.65877302	C 123.317546	G 184.9763191	C 246.6350921	E 308.2938651
Syntonic Calc.		C 123.317546	G 184.976319	C 246.635092	E 308.293865
1. Just 2:3 Calc.	G 95.13067838	G 380.5227135	A#456.6272562		
2. Just 2:3 Calc.	G 93.64426154	G 374.5770462	A#449.4924554		
3. Just 2:3 Calc.	G 92.48815953	G 369.9526381	A#443.9431657		
Syntonic Calc.		G 369.952638	A#443.9431656		

Appendix III - Notes



The pitch nomenclature adopted in this document is shown above, one of the three schemes mentioned in the Harvard Dictionary of Music compounded with a verbal practice familiar to organ builders. The twelve ascending chromatic notes from bottom C_3 to bottom B_3 are spoken: bottom C, bottom C#, bottom D, etc... and written either as bottom C_3 or C_3 ; bottom $C\#_3$ or $C\#_3$, etc... This ascending octave based naming practice is applied throughout the compass of notes, and if required, may be extended further through the use of more super/subscripts. Also as amongst organ builders, notes are by preference named as sharps, for example A# rather than B-flat, but not exclusively so where the flattened form is more informative or convenient.

Appendix IV - Example Re-Numbered Analysis

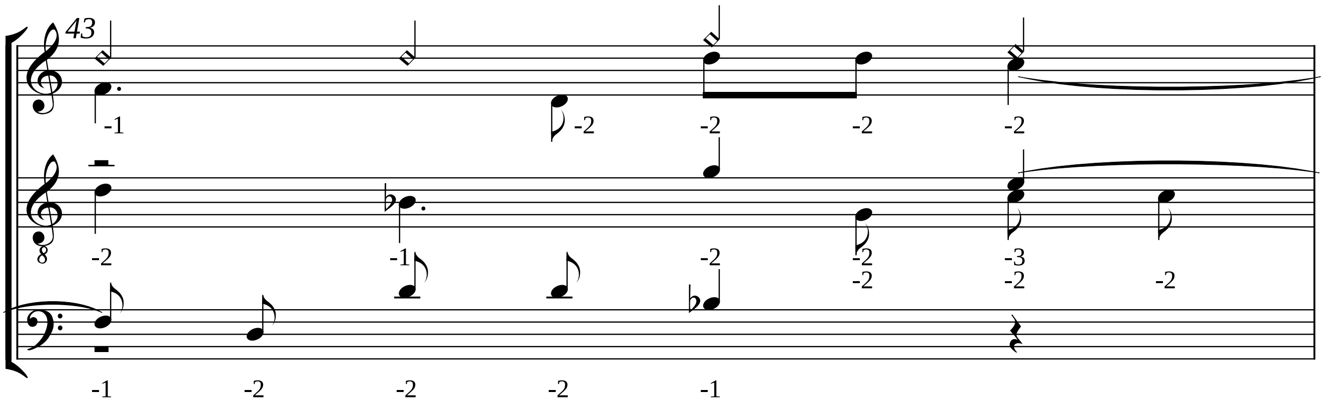


				B-h40*--->	B-h18* -2	
Conjunctions		D-h16*---->	D-h20*--->	D-h24* +16	A-h16*--->	A-h20--->
& Partials *	A-h10*--->	A-h12~ +4	F-h12~	G-h16~	E-h6~	A-h10~
	C-h6~	F~	D-h10~	A#~	C~	A-h5~
Notes ~	A-h5~	D-h8~	A#h8~	G-h8~	A-h4~	F-h4~
	F-h4~	C-h7	G#h7	F-h7	E-h3	C-h3
	C-h3	A-h6	F-h6	D-h6	A-h2	F-h2
	F-h2	F#h5	D-h5	B-h5	A-h1[H437400]	F-h1[H349920]
	F-h1[H354294]	D-h4	A#h4	G-h4	(52.6749Hz)	(42.1399Hz)
	(42.666...Hz)	A-h3	F-h3	D-h3	5:4-->	6:5-->
	6:5-->	D-h2	A#h2	G-h2	[H1]	[H1]
	[H1]	D-h1[H295245]	A#h1[H236196]	G-h1[H196830]		
		(35.555...Hz)	(28.444...Hz)	(23.703...Hz)		
		5:4-->	6:5-->	9:20-->		
		[H1]	[H1]	[H1]		

$$
\begin{aligned}
\text{MBN:}\quad 10_6 0_{59049} 0_1 &= 12_5 0_{59049} 0_1 \\
&\;\; + \;\; 4_5 0_{59049} 0_1 \\
&= 16_5 0_{59049} 0_1 = 20_4 0_{59049} 0_1 \\
&\qquad = 20_6 0_{39366} 0_1 = 24_5 0_{39366} 0_1 \\
&\qquad\qquad = 24_9 0_{21870} 0_1 \\
&\qquad\qquad + 16_9 0_{21870} 0_1 \\
&\qquad\qquad = 40_9 0_{21870} 0_1 = 18_{20} 0_{21870} 0_1 \\
&\qquad\qquad\qquad = 18_5 0_{87480} 0_1 \\
&\qquad\qquad\qquad - \;\; 2_5 0_{87480} 0_1 \\
&\qquad\qquad\qquad = 16_5 0_{87480} 0_1 = 20_4 0_{87480} 0_1 \\
&\qquad\qquad\qquad\qquad = 20_6 0_{58320} 0_1
\end{aligned}
$$

--->

In these pages the mutable number analysis (given on pages 5 & 6) has been re-numbered – though of course the sums remain unchanged – so as to provide an extra intermediate column in each digit sequence incorporating the ratios of exchange employed between adjacent chords –e.g. first exchange 6:5 10_6... = 12_5...



-----> A-h24* -8
D-h16*-------------->
F~
D-h8~
F~
D-h4~
A-h3
D-h2
D-h1H291600
(35.1166Hz)
5:4-->
[H1]

D-h20*------------->
F-h12~
D-h10~
A#h8~
G#h7
F-h6
D-h5
A#h4
F-h3
A#h2
A#h1[H233280]
28.1Hz
6:5-->
[H1]

G-h32*-------------->
D-h24~ +8
G-h16~
A#~
G-h8~
F-h7
D-h6
B-h5
G-h4
D-h3
G-h2
G-h1[H194400]
23.4Hz
3:8-->
[H1]

G-h12* -2
E-h10*----------->
C-h8~
E-h5~
C-h4~
G-h3
C-h2
C-h1[H518400]
62.4Hz
6:5-->
[H1]

---> $= 24_{5}0_{58320}0_{1}$
$- \ 8_{5}0_{58320}0_{1}$
$= 16_{5}0_{58320}0_{1}$ ----> $= 20_{4}0_{58320}0_{1}$
$= 20_{6}0_{38880}0_{1}$ ----------> $= 24_{5}0_{38880}0_{1}$
$= 24_{3}0_{64800}0_{1}$
$+ \ 8_{3}0_{64800}0_{1}$
$= 32_{3}0_{64800}0_{1}$ ----------> $= 12_{8}0_{64800}0_{1}$
$= 12_{6}0_{86400}0_{1}$
$- \ 2_{6}0_{86400}0_{1}$
$= 10_{6}0_{86400}0_{1}$ -->


```
      A-h16*--------->  A-h20 -10          D-h16*-------->  D-h12* -4        G-h12*----->  G-h8*
 ---> E-h12* +4         A-h10*--------->   A-h12* +4        G-h8~--------->  G-h6* +6      A#~
      C~                F-h8~              F~               D-h6~            E-h5~         G-h4~
      E-h6~             C-h6~              D-h8~            B-h5~            C-h4~         D-h3
      C~                A-h5~              F~               G-h4~            G-h3~         G-h2
      A-h4~             F-h4~              D-h4~            D-h3             C-h2~         G-h1[H768000]
      E-h3              C-h3               A-h3             G-h2             C-h1[H512000] 92.5Hz
      A-h2              F-h2               D-h2             G-h1[H384000]    61.7Hz        m:n-->
      A-h1[H432000]     F-h1[H345600]      D-h1[H288000]    46.2Hz           2:3-->        [H1]
      52.0Hz            41.6Hz             34.7Hz           3:4-->           [H1]
      5:4-->            6:5-->             3:4-->           [H1]
      [H1]              [H1]               [H1]
```

$$= 12_5 0_{86400} 0_1$$
$$+ \quad 4_5 0_{86400} 0_1$$
$$= 16_5 0_{86400} 0_1 \qquad = 20_4 0_{86400} 0_1$$
$$= 20_6 0_{57600} 0_1$$
$$- 10_6 0_{57600} 0_1$$
$$= 10_6 0_{57600} 0_1 \qquad = 12_5 0_{57600} 0_1$$
$$= 12_3 0_{96000} 0_1$$
$$+ \quad 4_3 0_{96000} 0_1$$
$$= 16_3 0_{96000} 0_1 \qquad = 12_4 0_{96000} 0_1$$
$$= 12_3 0_{128000} 0_1$$
$$- \quad 4_3 0_{128000} 0_1$$
$$= \quad 8_3 0_{128000} 0_1 \qquad = \quad 6_4 0_{128000} 0_1$$
$$= \quad 6_2 0_{256000} 0_1$$
$$+ \quad 6_2 0_{256000} 0_1$$
$$= 12_2 0_{256000} 0_1 = \quad 8_3 0_{256000} 0_1$$

Because the mutable numbers describing tonal musical compositions are made up of many small prime factors there is considerable scope for generating lush under-storeys of detail within the digit sequences applicable to these sums. There are many possible mutable number interpretations lying between the ‘high entropy’ version presented on pages 5 & 6 and ‘lower entropy’ configurations incorporating gradually increasing numbers of zero filled, lower base value columns:

for example, Measure 41, beat 1: MBN: $10_2 0_3 0_3 0_3 0_3 0_3 0_3 0_3 0_3 0_3 0_3 0_3 0_1$

or, Measure 45, beat 3: MBN: $8_2 0_2 0_2 0_2 0_2 0_2 0_2 0_2 0_2 0_2 0_2 0_3 0_5 0_5 0_5 0_1$