The Math of Exchange

This document brings together a new music example of tonal number processing, some recent thoughts about the mutable number model and two Perl scripts related to these topics. Some of the details of terminology and analysis are not fully explained here – where covered in documents elsewhere.

To sketch the overall concept: This or any tonal composition is viewed as an independent system of evolving relationships, existing within a two-dimensional pitch-duration space consisting of an all-encompassing fundamental nesting harmonic series. The gamut of possible relationships available to this nested system, the composition, are those contained within the positive, non-zero rational numbers. That is, all positive fractions and whole numbers, which, under the operation of multiplication, move the composition about the space that it inhabits. Additionally the view is taken that although something along the lines of equal temperament is the practical outcome under almost all performance scenarios, what the ear and mind actually extract from this approximation is the sense, and sensation, of rational Just relationships e.g. $^3/_2$, 6:5, etc. Thus to arrive at a truly meaningful understanding of tonal music, it is necessary to model the system, as does the ear, in relationships of whole numbers and fractions. Following this approach, embedding it within the physical fact of nature – the acoustic harmonic series – and informing it with the idea of nesting harmonic series within each other, leads on to the conception of tonal music being, effectively, a number system written in musical sound; and any individual piece, a computation of numbers.

The music example is J.S. Bach's lovely setting of the nativity chorale *O Jesulein Suss* (BWV493). In this example the mathematical machinery of applying the proportions of exchange between chords is explicitly worked out. The analysis is presented in a practical 'summary' format. That is, the nested harmonic series encapsulating the gist of the piece's chords –i.e. harmonic progression, is placed at a conveniently low position within its enfolding nesting harmonic series – the all-encompassing space it inhabits. An effect of using a 'summary' format is that a degree of mismatching will arise between the frequencies (computed with Just intervals of exchange) in the nested harmonic series and the whole numbered ratios of the fundamental nesting harmonic series. These mismatches are accounted for by allowing the fundamental nesting series to flex – the unit to expand and contract. Such a score based format provides considerable visual advantages.

Another source of variance is the effect of using Just intervals of exchange upon the overall pitch (rather than equal-tempered) resulting in a drift away from the opening frequency standard of middle C 256Hz. These changes are tracked by bracketed real number values in hertz attached to the analysis.

However, because only Just proportions consisting of simple ratios are involved in the sensible relationships between chords, it is possible to construct an 'absolute' mutable number analysis for any piece of tonal music. This is where all the stepping stones between nested series match integer ratios in the enfolding fundamental series. Consequently the unit is fixed. Mutable numbers produced by such analyses can be extremely large. For example, in an 'absolute' analysis of this piece the opening number for the chorale is a rather daunting MBN $10_{172186884}0_1$ which is getting on toward one and three quarter billion.

Example: the Score

At the top of the example are two double staves. The upper contains the source composition. The lower double staff displays both the conjunction frequencies in diamond-headed notes and below the conjunctions, nested harmonic series that encompass the harmonic progression of the piece. These nested harmonic series are not fully populated. They contain all the note-ratios from their nested fundamental frequency (h1) up to the bass note of the composition and thereafter only the note-ratios corresponding to the written music, then finally they are crowned with the conjunction note-ratio. This conjunction note-ratio is either a top written note or a low order harmonic arising from any one of the written notes. (Passing notes and other embellishments are ignored.) These conjunction frequencies are scientifically observable facts as much as any of the written notes, indeed they will normally possess more energy than the written frequencies themselves. However in contrast, while there may be some hints of the frequencies below the written bass notes of the upper staff, perhaps generated by the formant of instruments and accidents of acoustic resonance, for the most part what lies below this level is a mathematical contrivance implied by the observable fact of the music itself – but not physically present. Here also it might be mentioned, and again not a physical fact but a psycho-musical one perhaps, is the human perception of a 'root' to most chords: The somewhat disembodied sense of a fundamental frequency to most tonal collections of notes, particularly triads with their compression of the first few ratios in the harmonic series.

Example: the Computation of Numbers

The first numbers encountered below the two double staves are the frequencies of the various conjunctions (corresponding to diamond-headed notes) enclosed in brackets, given in hertz, beginning at the slightly lower scientific pitch – middle C equal to 256Hz. Throughout the analysis, these numbers are determined, sequentially, with Just intervals (to a level of accuracy) which inevitably leads to a degree of pitch instability.

Immediately below these explicit frequencies are columns delineating the nested harmonic series (matching the notes on the second double staff) in letters and numbers. The highest note-ratio in these columns represents the conjunction frequency – marked by asterisks – and they are the logical equivalent of the number in hertz appended above in brackets. So for example on the left: '(640Hz)' and 'E-h10*' refer to the same note/harmonic – the diamond-headed note in the pick-up beat. These conjunction note-ratios are connected with arrows across the page, correlating with the ties above the second double staff. The remainder of the nested harmonic series, below the conjunctions, contain the note-ratios corresponding to the written notes and a fully populated series below this down to the nested fundamental frequency (h1): which is also labelled with its position in the absolute, nesting, fundamental series, whose ratios are signified by a capital H. For example here at the first chord: 'C-h1 (H32)'. Below which, again in brackets, is the precise value in hertz, e.g. (64.0Hz).

From the preceding paragraphs it will have been gathered that a key terminological difference is drawn between a *nested* series and its underlying parent, the fundamental *nesting* series.

Now jumping down to the bottom of the page, there is again a succession of explicit frequency values in brackets referring to the absolute fundamental unit: C-H1. Here is recorded the ultimate effect of the process of moving from chord to chord in steps of Just intervals: Which is sometimes to shift the pitch of the whole system, as the mathematical underpinning adjusts to the reality of the music above. Immediately, the multiplication by $^5/_6$ inherent in the motion of the piece from the chord of C major to A minor forces the underlying fundamental series to flex (from 2Hz to 1.975 approximately); and after a brief return to its starting level at the next chord of E major, the fundamental unit settles back into more relaxed states over the succeeding pages. (As the H1 unit value is set at 2Hz at the start, the conjunction values in hertz are approximately double the mutable number magnitude being represented.)

Between the nested harmonic series above and the unit of the fundamental nesting series below, is the detailed working out of the math of exchange, as the composition passes from chord to chord. A complete explanation of how this math works is given in the document *The Mathematics of Tonal Music and Mutable Numbers*. However, briefly, it involves stepping down an undertone series (the reflection of the acoustic harmonic series in the pitch domain) by the number of note-ratios equal to the divisor in the proportion of exchange, and then from this position, climbing back up a normal overtone series by the number of note-ratio steps indicated by the numerator. This is how multiplication by a fraction is carried out.

Thus on the left hand side, beginning at the level of the *nested* fundamental note-ratio (C-h1) – which is also the *nesting* note-ratio C-H32 – this involves: First descending by six note-ratios of an undertone series, reaching down to F-a6 (10.666...Hz). [Here the symbol ‘a’ standing for ‘arithmonic’ has been used in place of lambda which is not easily accessible on this computer.] This process is equivalent to multiplying C-H32 by $^1/_6$. And secondly, from the lowly position of F-a6, a multiplication by $^5/_1$, which involves ascending an overtones series by five note-ratios, carrying the calculation up to A-h5, H27 in the fundamental nesting series. That is, tracing the arrows shown in the analysis, beginning at ‘C-a1 (H32)’ and finishing at ‘A-H27’ .

Thus the math is done, the proportion of 6:5 conveys the C major chord to A minor, and in the process the nested harmonic series is carried from H32 into A-H27. (Here also lies a noteworthy detail: there is scope for interpretation, in that the head of the ascending overtone series might have been ascribed to A-H26 rather than A-H27. More on this topic later.) Each succeeding chord is calculated in the same manner, with the ratios of exchange noted below in bold italic script, the whole procedure following a ‘snakes and ladders’ path across the page.

Most of the columns delineating nested series, but not the first few, display the effect of addition or subtraction. I suppose technically they all do, in the sense that the first few columns can be thought of as add zero. After these first three columns, numbers appear in bold type close to the conjunction arrows. For example, +4, –8, etc. These numbers are recording the changes in value that are being crystallized by successive steps in the harmonic motion from chord to chord. This is the process that lies at the heart of number processing encapsulated within tonal music.

There is a good deal of symmetry involved it this process. In the opening exchange from C major to A minor the proportion of exchange is six-is-to-five. The mechanism of exchange descends six note-ratios and then rises by five note-ratios, while inversely mirroring this, the nested series moves from ten note-ratios built on H32 to twelve note-ratios built on H27. The number of note-ratios in these nested series are inverse multiples of the proportion being exchanged. Thus the value or magnitude isolated by the procedure is closely related to some, or perhaps more strictly all, of the integral multiples of the proportion of the harmonic progression. For example, the value might be that defined at the level of E-h5 = E-h6, E-h10 = E-h12, B-h15 = B-h18, etc... an unending conjunction (harmonic) series positioned between the C major and A minor chords.

As mentioned else where, the rule of thumb for determining a value, is to take the conjunction series out to a point at which it encompasses the written chords it connects and coincides with an audible frequency, usually a low order acoustic harmonic of one of the written notes or occasionally the top note itself. Thus in this opening exchange the value is taken to be that of E-h10=E-h12, which if written in mutable base numbers looks like:

MBN $10_{32}0_1 = 12_{27}0_1$ or more precisely, $10_{32}0_1 = 12_{27}0_{0.987654321}$

which is equal to Decimal 320.

In regard to the number processing undertaken overall, it can be seen that a range of different magnitudes are computed, beginning from the round number decimal 320 (640Hz) or MBN $10_{32}0_1$ and finishing up with MBN $12_{32}0_1$ – decimal 365.3853216 (730.7706433Hz).

Over the first four chords the value computed remains unchanged at decimal 320, but is being repeatably defined by the harmonic progression between each chord pairing or digit sequence:

MBN: $10_{32}0_1$	$= 12_{27}0_1$	$= 8_{40}0_1$	$= 12_{27}0_1$
C major	= A minor	= E major	= A minor.

In measure two, where the A minor chord moves to D major, a new value is processed. To make the exchange viable in terms of the number of note-ratios matching the proportion of the harmonic motion, it is necessary to add four note-ratios to the A minor chord, from E-h12 to A-h16, raising its magnitude to MBN: $16_{27}0_1$ (Decimal 426.666...). This new value is crystallized by the transition to D major.

MBN: $16_{27}0_1 = 24_{18}0_1$
A minor = D major.

Then to reach the following G major chord a subtraction of eight is needed to find a commensurable conjunction (D-h16 = D-h12), yielding the sum MBN: $16_{18}0_1 = 12_{24}0_1$ (Decimal 284.444...). And so, in a like manner the computation proceeds on through the measures.

Example: the Minor Third

The nested series encompassing the A minor chords in measure one illustrates the treatment of the minor third. As can be seen the note itself (C-) is recorded in the series without a ratio attached. The view taken here with regard to the human aural perception of the interval, is that the minor third is a ‘colouring tone’ lying upon the basilar membrane a sufficient distance away from the prime and fifth so as not to induce a sensation of dissonance or irritation – except in combination with the prime in the lower reaches of the bass clef where the spacing on the membrane is more problematic – and, as a colouring tone it overrides or replaces what might ordinarily be ‘expected’, the fifth harmonic (h5) and/or its multiples –i.e. C#h10. Whether or not the ear and aural cognition can relate the interval of a minor third to some position in the harmonic series (as I suspect it might the major third) or alternatively, that the character of the interval is simply learned and remembered, or indeed both, I do not know. The interval does appear in near approximation explicitly (rather than the implicit h5–h6) in the harmonic series at h16–h19. And if the position of h19 were applied to the A minor chords in measure one it could be made to work with a $^{19}/_{16}$ proportion of exchange but would necessitate using a nested series set two octaves lower so as to accommodate middle C 256Hz. Indeed, if the piece were nested at a sufficiently high level within its enfolding nested series the regular proportion of $^{6}/_{5}$ could be used – as would be even more so the case in an ‘absolute’ analysis – however, such treatments are difficult to handle summarily.

Overall, I continue to favour the ‘colouring tone’ approach to the minor third, combined with a $^{6}/_{5}$ proportion of exchange, but remain open to the alternative interpretation. Nevertheless, though banished from participating in the underlying mathematics, the note would be noticed by the ear as an objective frequency connecting adjacent chords and so is marked with grey conjunction arrows in the score.

Example: Conclusion

The Mutable Number approach to tonal compositions conceives them as only knowing their own internal Just relationships. They are considered little worlds unto themselves, manoeuvring about an enfolding space consisting of an underlying fundamental nesting harmonic series; which, in a ‘summary’ analysis they lightly perturb as they go. At the final cadence, after computing thirty sums, the number processing concludes at a frequency of decimal 730.7706433Hz, which divided by the unit 1.90304855 equals, near enough, 384 – the note-ratio G-H384 – but I hope you will agree, MBN $12_{32}0_{1}$ better captures the essence of what the ear and mind perceive.

Example:*O Jesulein Suss* (BWV493)

The analysis is presented over the following five pages, and as mentioned above it is in ‘summary’ format. Nevertheless, the real number values given in hertz labelling the nested harmonic series follow the same trajectory as an would an absolute analysis given in integers. At the foot of this document a brief overview of the absolute analysis is given in decimal numbers.

Frequency:	(E-640Hz)			(A-853.333...)	(D-568.888...)	(G-758.5185184)	(D-1137.777...)	(D-568.888...)	(G-758.5185184Hz)	
Conjunction:	E-h10* →	E-h12* →	E-h8* →	A-h16* → +4	A-h24* -8 →	G-h16* → +4	D-h36* → +12	D-h16*	G-h16* → +4	G-h12* →
	C-h8 →	C-	B-h6	E-h12*	D-h16	D-h12*	G-h24*	D-h8* -8 →	D-h12*	E-h10
Nested Harmonic Series:	E-h5	E-h6	G#h5	C	A-h12	B-h10	C-h16	A-h6	G-h8	G-h6
	C-h4	C-	E-h4	A-h8	F#h10	G-h8	G-h12	F#h5	D-h6	C-h4
	G-h3	A-h4	E-h2	E-h6	D-h8	D-h6	E-h10	D-h4	B-h5	C-h2
	C-h2	E-h3	E-h1(H40)	A-h4	F#h5	G-h4	E-h5	A-h3	G-h4	C-h1(H32)
	C-h1(H32)	A-h2	(80.0)	A-h2	D-h4	D-h3	C-h4	D-h2	D-h3	(63.20987653)
Frequency:	(64.0Hz)	A-h1(H27)		A-h1(H27)	D-h2	G-h2	C-h2(H32)	D-h1(H36)	G-h2	
		(53.333...)		(53.333...)	D-h1(H18)	G-h1(H24)	C-h1(H16)	(77.111...)	G-h1(H24)	
					(35.555...)	(47.4074074)	(63.2098653)		(47.4074074)	
							(31.60493827)			

Math of Exchange:

C-a1(H32), C-a2, F-a3, C-a4, G#a5, F-a6 ↓ ; ↑ A-H27, F-h4, C-h3, F-h2, F-h1
C 6:5 A

A-H27 A-a1 / A-a2 ; E-H40 A-h2 A-h1
A 2:3 E

E-a1 E-a2 A-a3 ; A-H27 A-h1
E 3:2 A

A-a1 A-a2 D-a3 ; D-H18 D-h1
A 3:2 D

D-a1 D-a2 G-a3 ; G-H24 D-h3 G-h2 G-h1
D 3:4 G

G-a1 G-a2 C-a3 ; C-H16 C-h1
G 3:2 C

C-H32 C-h1
1:2

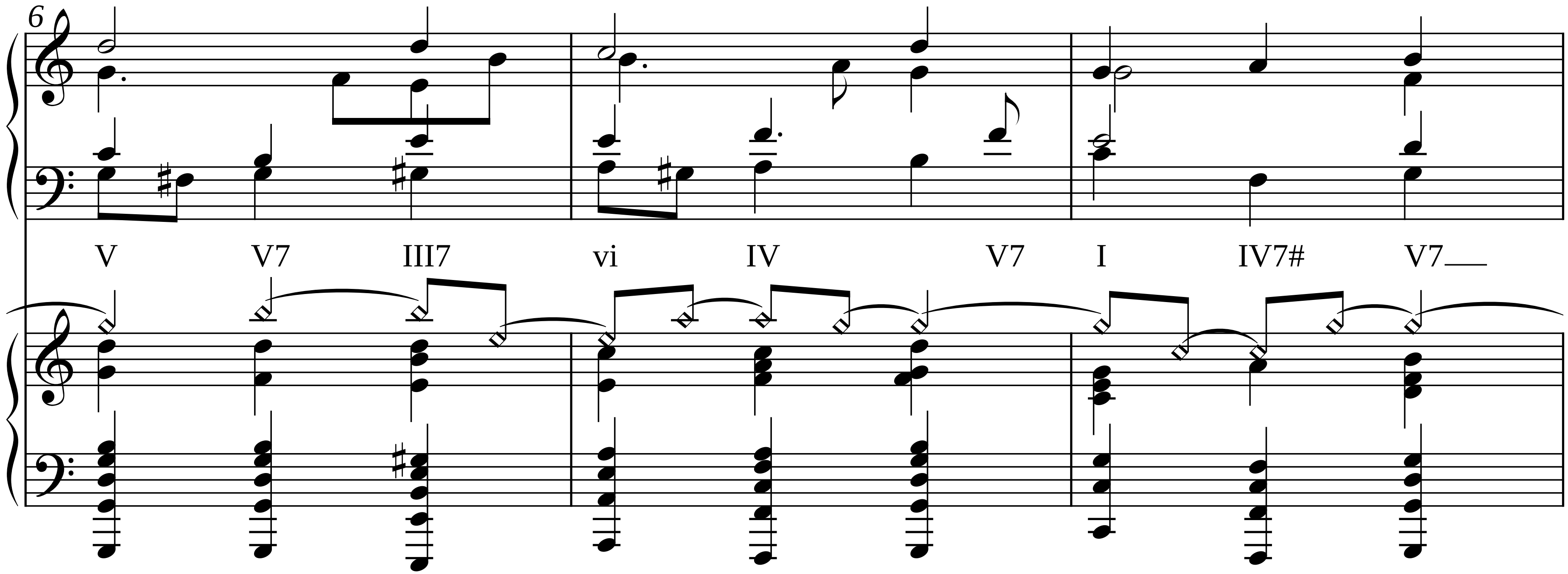
C-a1 C-a2 F-a3 C-a4 G#a5 F-a6 D-a7 C-a8 ; D-H36 C-h8 A#h7 G-h6 E-h5 C-h4 G-h3 C-h2 C-h1
C 8:9 D

D-a1 D-a2 G-a3 ; G-H24 G-h1
D 3:2 G

G-a1 G-a2 C-a3 ; C-H32 G-h3 C-h2 C-h1
G 3:4 C

C-a1 C-a2 F-a3 C-a4 →
C

Fundamental Frequency: C-H1 (2.0Hz) (1.9753086) (2.0) → (1.9753086) → (1.9753086) → (1.975308642) → C-H1 (1.975308642Hz)



(G-770.5585Hz) →

(B-948.148148)
B-h20* → +4
G-h16*
D-h12 →
G-h8
B-h5
G-h4
D-h3
G-h2
G-h1(H24)
(47.4074074Hz)

(E-632.0987653)
B-h24* -8
E-h16* →
D-h14
B-h12
E-h8
G#h5
E-h4
B-h3
E-h2
E-h1(H20)
(39.50617283)

(A-842.7983538)
A-h16* → +4
E-h12*
C-
A-h8
E-h6
A-h4
A-h2
A-h1(H27)
(52.67489711)

(G-758.5185184)
A-h20* -2
G-h18* →
C-h12
A-h10
F-h8
A-h5
F-h4
C-h3
F-h2
F-h1(H21)
(42.13991769)

(G-758.5185184)
G-h16* →
F-h14*
D-h12
G-h8
F-h7
B-h5
G-h4
D-h3
G-h2
G-h1(H24)
(47.4074074)

(C-505.6790123)
G-h12* -4
C-h8* →
G-h6
E-h5
C-h4
G-h3
C-h2
C-h1(H32)
(63.20987653)

(G-758.5185184)
G-h18* → +6
C-h12*
A-h10
F-h8
A-h5
F-h4
C-h3
F-h2
F-h1(H21)
(42.13991769)

→
G-h16*
B-h10
F-h7
D-h6
G-h4
D-h3
G-h2
G-h1(H24)
(47.4074074)

G-h3(H24)	G-a1
C-h2	G-a2
C-h1	C-a3
	G-a4
	D#a5
	C-a6

4:3 G

E-h5(H20)	E-a1
C-h4	E-a2
G-h3	A-a3
C-h2	
C-h1	

G 6:5 E

A-h4(A27)	A-a1
E-h3	A-a2
A-h2	E-a3
A-h1	A-a4
	F-a5

E 3:4 A

F-H21	F-a1
C-h3	F-a2
F-h2	A#a3
F-h1	F-a4
	C#a5
	A#a6
	G-a7
	F-a8

A 5:4 F

G-h9(H24)	G-a1
F-h8	G-a2
D#h7	C-a3
C-h6	
A-h5	
F-h4	
C-h3	
F-h2	
F-h1	

F 8:9 G

C-h4(H32)	C-a1
G-h3	C-a2
C-h2	F-a3
C-h1	

G 3:4 C

F-H21	F-a1
F-h1	F-a2
	A#a3
	F-a4
	C#a5
	A#a6
	G-a7
	F-a8

C 3:2 F

G-h9(H24)	G-a1
F-h8	G-a2
D#h7	C-a3
C-h6	
A-h5	
F-h4	
C-h3	
F-h2	
F-h1	

F 8:9 G

→

G 3:4

(C-H1 1.975308642Hz) → (1.950922115) (2.006662747) (1.975308642) → (2.006662747) (C-H1 1.975308642)

(G-758.5185184)
G-h12*
C-h8
E-h5
C-h4
C-h2
C-h1(H32)
(C-63.20987653Hz)

(C-1011.358025)
C-h16* → +4
G-h12*
E-h10
G-h6
C-h4
G-h3
C-h2
C-h1(H32)

(A-842.7983537)
C-h24* -4
A-h20*
F-h16
A-h10
F-h8
C-h6
A-h5
F-h4
C-h3
F-h2
F-h1(H21)
(42.13991769)

(D-1123.731138)
D-h32* → +8
A-h24*
F-
A-h14
C-h7
D-h4
A-h3
D-h2
D-h1(H18)
(35.11659807)

(G-749.1540922)
D-h24* -8
G-h16*
D-h12
G-h8
B-h5
G-h4
D-h3
G-h2
G-h1(H24)
(46.82213076)

(G-749.1540922)
G-h12*
E-h10
A#h7
C-h4
G-h3
C-h2
C-h1(H32)
(62.42950768)

(C-998.8721229)
C-h32* → +8
G-h24*
E-h20
G-h12
A#h7
C-h4
G-h3
C-h2
C-h1(H16)
(31.21475384)

C-H32
G-h3
C-h2
C-h1

3:4 C

C-a1(H32)
C-a2
F-a3

C 3:2 F

F-h2(H21)
F-h1

F-a1
F-a2
A#a3
F-a4
C#a5
A#a6

D-h5(H18)
A#h4
F-h3
A#h2
A#h1

F 6:5 D

D-a1
D-a2
G-a3

D 3:4 G

G-h4(H24)
D-h3
G-h2
G-h1

G-a1
G-a2
C-a3

C-h4(H32)
G-h3
C-h2
C-h1

G 3:4 C

C-a1
C-a2(H16)

C 2:1 C

C-a1
C-a2
F-a3

C 3:4

(C-H1.975308642Hz) (2.006662747) (1.950922115) → (C-H1.950922115Hz) →

(A-832.3934357)
C-h24* -4 →
A-h20* →
C-h12 →
A-h10
A-h5
F-h4
C-h3
F-h2
F-h1(H21)
(41.61967179)

A-h16* →
C-
A-h8
E-h6
A-h4
E-h3
A-h2
A-h1(H27)
(52.02458974)

(D-554.9289572)
A-h24* -8 →
D-h16
A-h12
D-h8
F#h5
D-h4
A-h3
D-h2
D-h1(H18)
(34.68305982)

(G-739.9052763)
G-h16* →
D-h12* +4
B-h10
G-h8
D-h6
G-h4
D-h3
G-h2
G-h1(H24)
(46.24407977)

G-h32* →
B-h20
G-h16
D-h12
G-h8
F-h7
D-h6
B-h5
G-h4
D-h3
G-h2
G-h1(H12)
(23.12203988)

(C-493.2701842)
G-h24* -8 →
C-h16
E-h10
C-h8
G-h6
E-h5
C-h4
G-h3
C-h2
C-h1(H16)
(30.82938651)

↑
F-h4(H21) F-a1
C-h3 F-a2
F-h2 A#a3
F-h1 F-a4

3:4 F

(C-H1.981889133)

A-h5(H27) A-a1
F-h4 A-a2
C-h3 D-a3
F-h2
F-h1

F 4:5 A

(1.926836657)

D-h2(H18) D-a1
D-h1 D-a2
G-a3

A 3:2 D

(1.926836657)

G-h4(H24) G-a1
D-h3 G-a2(H12) G-a1
G-h2 G-a2
G-h1 C-a3

D 3:4 G

(1.926836657)

G 2:1 G

(1.926836657)

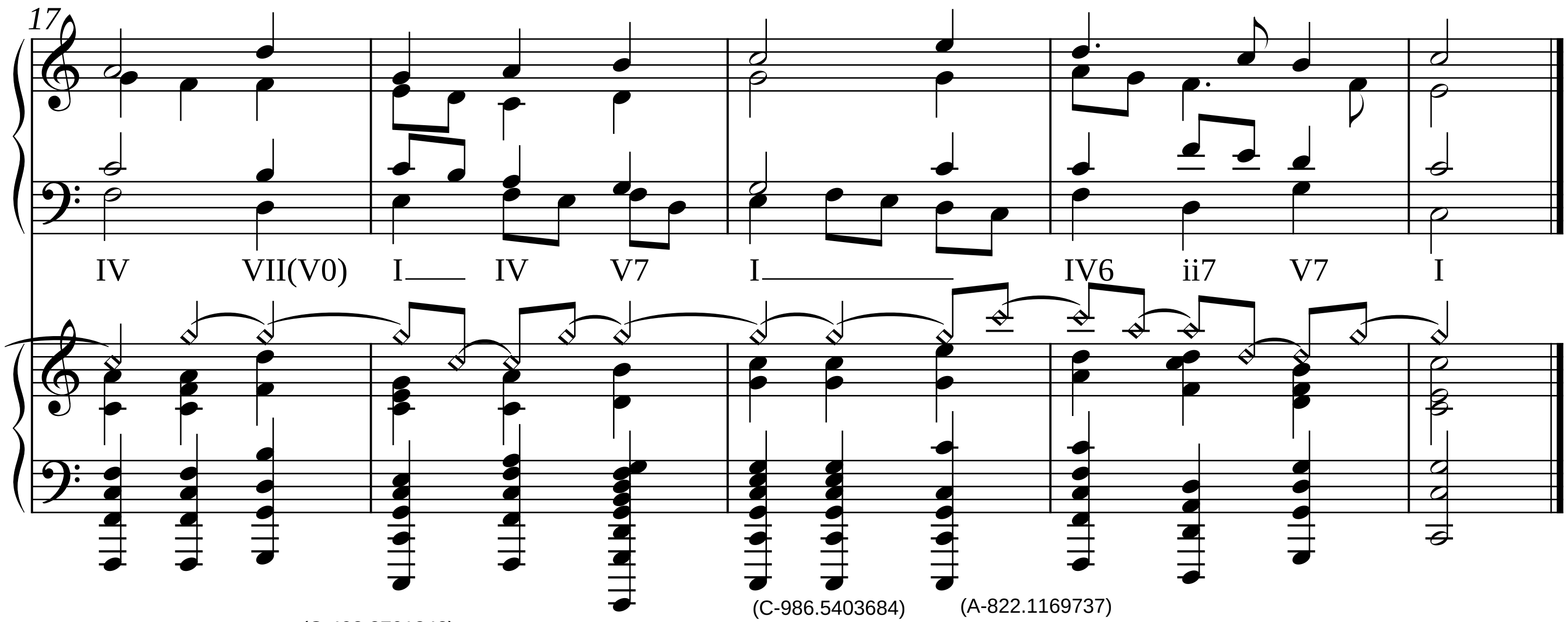
↑
C-h4(H32) →
G-h3
C-h2
C-h1

G 3:4 C

(C-H1.926836657Hz)

C-a1
C-a2
F-a3 →

C 3:4



(G-739.9052763Hz)
G-h18* → +6
C-h12*
A-h10
F-h8
C-h6
F-h4
C-h3
F-h2
F-h1(H21)
(41.10584868)

G-h16* →
D-h12
F-h7
B-h5
D-h3
G-h2
G-h1(H24)
(46.24407977)

(C-493.2701842)
G-h24* -8
C-h16* →
G-h12
E-h10
C-h8
E-h5
C-h4
G-h3
C-h2
C-h1(H16)
(30.82938651)

(G-739.9052763)
G-h18* → +6
C-h12*
A-h10
C-h6
A-h5
F-h4
C-h3
F-h2
F-h1(H21)
(41.10584868)

G-h32* →
B-h20
D-h12
G-h8
F-h7
D-h6
B-h5
G-h4
D-h3
G-h2
G-h1(H12)
(23.12203989)

(C-986.5403684)
C-h32* → +8
G-h24*
C-h16
G-h12
G-h6
E-h5
C-h4
G-h3
C-h2
C-h1(H16)
(30.82938651)

(A-822.1169737)
C-h24* -4
A-h20* →
D- →
A-h10
C-h6
F-h4
C-h3
F-h2
F-h1(H21)
(41.10584868)

(D-548.0779825)
A-h24* -8
D-h16 →
C-h14
F-
D-h4
A-h3
D-h2
D-h1(H18)
(34.2548739)

(G-730.7706433Hz)
G-h16* → +4
D-h12*
B-h10
F-h7
D-h6
G-h4
D-h3
G-h2
G-h1(H24)
(45.6731652)

G-h12*
C-h8
E-h5
C-h4
G-h3
C-h2
C-h1(H32)
(60.89755361)

F-H21
C-h3
F-h2
F-h1
3:4 F

F-a1
F-a2
A#a3
F-a4
C#a5
A#a6
G-a7
F-a8

G-H24
F-h8
D#h7
C-h6
A-h5
F-h4
C-h3
F-h2
F-h1
F 8:9 G

G-a1
G-a2
C-a3
C-H16
C-h1
G 3:2 C

C-a1
C-a2
F-a3
C 3:4 F

F-H21
C-h3
F-h2
F-h1

F-a1
F-a2
A#a3
F-a4
C#a5
A#a6
G-a7
F-a8
D#a9
C#a10
B-a11
A#a12
G#a13
G-a14
F#a15
F-a16

G-H12
F-h8
D#h7
C-h6
A-h5
F-h4
C-h3
F-h2
F-h1
F 16:9 G

G-a1
G-a2
C-a3
G 3:4 C

C-H16
G-h3
C-h2
C-h1

C-a1
C-a2
F-a3
C 3:4 F

F-h4(H21)
C-h3
F-h2
F-h1

F-a1
F-a2
A#a3
F-a4
C#a5
A#a6
F 6:5 D

D-h5(H18)
A#h4
F-h3
A#h2
A#h1

D-a1
D-a2
G-a3
D 3:4 G

G-h4(H24)
D-h3
G-h2
G-h1

G-a1
G-a2
C-a3

C-h4(H32)
G-h3
C-h2
C-h1
G 3:4 C

(C-H1.957421366)(1.926836657) → (1.957421366) (1.926836657) → (1.957421366) (1.90304855) → (C-H1.90304855Hz)

Symmetrical Pitch Distribution of Just Intervals

The table below illustrates the symmetrical pitch-spacing of Just intervals around the tritone axis. Therefore given a matched distribution of these intervals in the root harmonic motion of a piece (which is rarely the case) then the pitch disturbance to the fundamental nesting series in summary format would, overall, cancel out. For example, if every fifth was matched by a fourth, every major third by a minor sixth, etc.; or put another way, adding the pitch divergence of the Just intervals, in cents, modulus 100 yields zero. That is:

0 + 112 + 204 + 316 + 386 + 498 + 590 + 610 + 702 + 814 + 884 + 996 + 1088 mod 100 = 0.

Comparative Twelve Tone Scales (Philosophical Pitch)								
Notes Letter	Equal Temperament		Just Temperament			Harmonic Series		
	Frequency	Cents	Frequency	Cents	Ratio	Frequency	Cents	Ratio
C	256Hz	0	256Hz	0	1:1	256Hz	0	1:1
C#	271.22255	100	273.0666...	112	16:15	272.0 (h17)	105	17:16
D	287.35028	200	288.0	204	9:8	288.0 (h9)	204	9:8
D#	304.43701	300	307.2	316	6:5	304.0 (h19)	297.5	19:16
E	322.53977	400	320.0	386	5:4	320.0 (h5)	386	5:4
F	341.71898	500	341.333...	498	4:3	336.0 (h21)	471	21:16
F#	362.03864	600	360.0 364.0888	590 610	45:32 64:45	352.0 (h11) 368.0 (h23)	551 628	11:8 23:16
G	383.56657	700	384.0	702	3:2	384.0 (h3)	702	3:2
G#	406.37462	800	409.6	814	8:5	400.0 (h25)	773	25:16
A	430.53891	900	426.666...	884	5:3	416.0 (h13)	841	13:8
A#	456.14008	1000	455.111... 460.8	996 1018	16:9 9:5	448.0 (h7)	969	7:4
B	483.26358	1100	480.0	1088	15:8	480.0 (h15)	1088	15:8
C	512.0	1200	512.0	1200	2:1	512.0	1200	2:1

Attribution of Note-Ratios (A27 or A26)

Earlier in the document the issue of ascribing the nested fundamental ratio (h1) to a particular position within its encompassing nesting series was raised. The issue arises between the first two chords in the piece, the step from C major to A minor, where the nested fundamental frequency C-h1 (64Hz) is transformed into A-h1 (53.333...Hz). Clearly there is no whole numbered ratio in the fundamental series based on C-H1 (2Hz) to match the figure 53.333...Hz. Therefore the fundamental series – the pitch-duration space in which the piece exists – must adjust to accommodate the physical reality of the music. The choices are to flatten the unit to approximately C-H1 (1.975Hz) which will make H27 equal to 53.333... or sharpen to approximately C-H1 (2.05Hz) to capture the figure with A-H26. Either is possible, but as A-H27 is the nearer; that is a reasonable choice.

More broadly attributing notes and intervals to positions in the harmonic series is somewhat problematic. The default attribution adopted in this numerical model of tonal music is, simply to ascribe them to their the nearest, lowest approximation.

Note:	C	C#	D	D#	E	F	F#	G	G#	A	A#	B	C
Ratio:	1	17	9	19	5	21	11	3	25	13	7	15	2

The result encompasses a range: The majority mesh well or well enough with Just intervals: C, G, E, D, B, C# and D#; while others not so well: A#, F#, F and G#. (And the overall well-meshing of the twelve-tone scale with the harmonic series is a reflection of that ‘sweet spot’ or better ‘sweet compromise’ that Rameau described as ‘nature’s gift’.) Though not shown above, this default attribution also includes their multiples by powers of two. For example, G-H3 includes G-H6, G-H12, etc., and A-H13 extends to A-H26, A-H52 and beyond.

In making such an attribution some interesting points emerge. It can be seen in the above list that eight notes fall on prime ratios – C, G, E, A#, F#, A#, C# and D# – and the balance on compound ratios. While the prime ratios are unique – with those low enough in the harmonic series for the ear to discern yielding equally unique sensations – the compound ratios suggest some further internal structure. For example, B-H15 (3 × 5) might be considered to be derived from either G-H3 or E-H5 – either as the fifth nested harmonic of G-H3 or the third nested harmonic of E-H5, that is B-h5×3 or B-h3×5.

```
B-h5(H15)          B-h3(H15)
G-h4(H12)          E-h2(H10)
D-h3 (H9)          E-h1 (H5)
G-h2 (H6)              (H4)
G-h1 (H3)              (H3)
     (H2)              (H2)
     (H1)              (H1)
```

Equally, F-H21 may be derived from G-H3 or A#H7, and G#H25 from E-H5.

Now, turning outward and upward from this internal structure, to where there is already an unlimited range of multiples by powers of two for assigned primes like C-H2, H4, H8, ... and G-H3, H6, H12, ... ; here similarly, all other primes may be involved as multipliers as well. For example, from the left-hand list:

D-H3×3, B-H3×5,

continues:

F-H3×7, A-H3×3×3, C#H3×11, E-H3×13, F#H3×3×5, etc.

And what emerges from this continuation of the nested series built on G-H3 is a slightly different perspective on note-ratios from the default attribution built on C-H1 – bring us full circle to the question of whether the value 53.333...Hz is attributed to A-H26 or A-H27. While A-H26 derives from the extension of the default attribution, A-H27 has its roots in the dominant so to speak, G-H3, and is indeed one of the staging posts along the spiral of fifths.

Another example of this broadening phenomenon is presented by F#H3×3×5, H45. The default assignment comes at F#H11 in the harmonic series, which multiples out to F#H44 and beyond. Yet here conveniently next door is F#H45, derived from G-H3 and E-H5, the critical ratio for determining the Just intervals of an augmented fourth and diminished fifth.

Generalising this view of note-ratio attribution between pitch classes and the harmonic series leads to the conception of the harmonic series (and its reflection, the undertone series) being constructed out of a multi-stranded sequence of relationships consisting of interlaced threads of nested series built upon each and every prime number ratio.

To help illuminate and illustrate this concept, and to aid the practical determination of which higher order ratios belong to which pitch classes, two Perl scripts script accompany this document (`ths and ths2`). Examples of their output are attached in the final pages below.

Rising further up the harmonic series, that is expanding the distance between the *nesting* fundamental H1, and the *nested* fundamental h1, the number of candidate note-ratios available for use increases greatly. The complete music example examined in this document has been set with its nested series situated at the lowest possible level that can be sustained throughout by a single encompassing fundamental nesting series. However, it is entirely feasible to raise the nested series' position in the underlying series to any convenient higher point. This would of course magnify the values being processed – and taken to the limit would result in an 'absolute' analysis. The advantage of a higher position stems from the vastly greater range of note-ratios available to choose amongst, and thus, consequently, the fundamental nesting series would be ever more lightly disturbed by the harmonic motion.

The rather clunky default attribution of note letters (or pitch classes) to the opening ratios of the harmonic series brings to mind the analogy of it being not unlike a 'Mercator' projection of the twelve note scale onto the 'curving surface' that the harmonic series presents; and that by rising ever higher in its ranks an improving, enmeshment is possible. However, it is also well to notice that notwithstanding the delights of closer approximations where fit, the the principal structural relationships of chords I, II and V can sit comfortably together upon the same unit value, even in summary format, as illustrated in the workings of the example piece.

Perl Scripts

Building on the above idea of an underlying fundamental harmonic series being made up of many threads of prime ratio based nested series, provides an opening for a link to be made between the prime factors of ratios and pitch classes. Using an algorithm that reduces the row ratios (i.e. 1:2, 1:3, 1:4, etc. and their inverses) in the table of harmonic series to prime factors, will allow two conversion tables, one for overtone series and the other for undertone series, to be used to relate the individual prime factors to specific pitch classes. With collections of these prime factors being interpreted as steps along the threads of individual nested series, tracing a path from nested fundamental up to any row ratio within their trajectory. And further, the pitch class sums of such collections, modulus twelve, allow for the ultimate pitch class and therefore note letter to be assigned.

(Harmonic series are based on C = 1 and pitch classes on C = 0.)

Thus in an overtone series for example: B 1:30

Prime Factors	Conversion to Pitch Classes			Addition of Pitch Classes
? 1:2 × 3 × 5	2 => 12	3 => 7	5 => 4	(12 + 7 + 4) mod 12 = 11 = B

and for example in an undertone series: C# 30:1

? 2 × 3 × 5:1	2 =>12	3 => 5	5 => 8	(12 + 5 + 8) mod 12 = 1 = C#

An immediately interesting aspect of this algorithm is that it appears to connect multiplication with addition: Multiplication of a note-ratio's prime factors with the addition of pitch classes, a feature which perhaps exudes something of a rather logarithmic essence.

Also of note was the exclusion of compound numbers from the algorithm. For example computing the note letter for the harmonic series ratio 1:9 or 9:1 (D and A#) put the focus on its being the dominant of the dominant, or sub-dominant of the sub-dominant, so to speak:

? 1:3 × 3	3 => 7	3 => 7		(7 + 7) mod 12 = 2 = D
? 3 × 3:1	3 => 5	3 => 5		(5 + 5) mod 12 = 10 = A#

and going a step further down the line to calculate the note letter for 1:27 and its inverse, touched on earlier, for the overtone and undertone series respectively:

? 1:3 × 3 × 3	3 => 7	3 => 7	3 => 7	(7 + 7 + 7) mod 12 = 9 = A
? 3 × 3 × 3:1	3 => 5	3 => 5	3 => 5	(5 + 5 + 5) mod 12 = 3 = D#

However having reached A 1:27 by this route, the algorithm also works out that A 1:26 is derived from the (admittedly rather imprecise) attribution of note letter A to be the default attribution of prime ratio 1:13.

Indeed what the algorithm is also highlighting, along with the connection between multiplication and addition, is the multiple trajectories of individual nested harmonic series within the body of an all encompassing fundamental series.

Gradually as the algorithm is run out to ever more remote regions and ratios, the number of note letter attributions increased and even begin to intermingle with their neighbours. Logically I suppose if the process of chasing dominants, as above, were continued far enough through to the end of the spiral of fifths, the long thread of this particular nested series would have removed itself by one Pythagorian comma frequency distance from its origins.

Running the Scripts

As there is space, here below is the help text from the Perl script ths which hopefully may be of some use. This script produces a Table of Nested Harmonic Series or its reflection in the pitch/frequency domain. (The script ths2 produces a combined table of nested overtone and undertone series.) You may wish or need to append a `.pl` suffix to the script names depending on you system.

```
Usage: [perl] ths [ -r<number> -s<number> -t -w<number> -h ]
```

Output is to STDOUT, which can be redirected to file, for example:

```
perl ths -r36 -s100 -t -w30 > outputfile
```

Which produces a text file in the current directory consisting of an undertone series from the 1/100th row to the 1/135th row with a table width equal to that of the 1/30th row, amounting to 80 text characters overall.

The -r (rows) option controls the number of rows in the table, no maximum. Default without option: 60 rows, an A4 page full.

The -s option sets the start position within the series. This is useful for examining some particular portion of an overtone or undertone series. For example, say, the region between 1/968 and 1/1026, illustrated below. The default is 1, –i.e. start at the beginning. (With -r set to 1 a single ratio can be examined.)

The -t option is required to produce an undertone series, otherwise (–i.e. with no -t option) the script produces an overtone series by default. (Not applicable to `ths2`)

The -w option governs the overall width of the table by limiting its growth beyond the given row’s length. The default is 24, minimum 9. Larger values produce a wider spread, to obtain a full width table add the row value and start valve together to determine this width option. (Less than half full width restricts the identification of prime numbers.)

The -h option produces allegedly useful prose.

Tables of Nested Harmonic Series

(Output from Perl Script ths2.pl)

```
B      1:30  X X X   X X       X         X                             X ..30
*      1:29  X                                                       X ..29
A#     1:28  X X   X     X             X                           X ..28
A      1:27  X   X           X                                   X ..27
A      1:26  X X                     X                         X ..26
G#     1:25  X       X                                       X ..25
G      1:24  X X X X   X   X       X                       X ..24
F#     1:23  X                                           X ..23
F#     1:22  X X                 X                     X ..22
F      1:21  X   X       X                           X ..21
E      1:20  X X   X X         X                   X ..20
D#     1:19  X                                   X ..19
D      1:18  X X X     X     X                 X ..18
C#     1:17  X                               X ..17
C      1:16  X X   X       X               X ..16
B      1:15  X   X   X                   X ..15
A#     1:14  X X         X             X ..14
A      1:13  X                       X ..13
G      1:12  X X X X   X           X ..12
F#     1:11  X                   X ..11
E      1:10  X X     X         X ..10
D      1:9   X   X           X ..9
C      1:8   X X   X       X ..8
A#     1:7   X           X ..7
G      1:6   X X X     X ..6
E      1:5   X       X ..5
C      1:4   X X   X ..4
G      1:3   X   X ..3
C      1:2   X X ..2
*      1:1   X ..1
C      2:1   X X ..1/2
F      3:1   X   X ..1/3
C      4:1   X X   X ..1/4
G#     5:1   X       X ..1/5
F      6:1   X X X     X ..1/6
D      7:1   X           X ..1/7
C      8:1   X X   X       X ..1/8
A#     9:1   X   X           X ..1/9
G#    10:1   X X     X         X ..1/10
F#    11:1   X                   X ..1/11
F     12:1   X X X X   X           X ..1/12
D#    13:1   X                       X ..1/13
D     14:1   X X         X             X ..1/14
C#    15:1   X   X   X                   X ..1/15
C     16:1   X X   X       X               X ..1/16
B     17:1   X                               X ..1/17
A#    18:1   X X X     X     X                 X ..1/18
A     19:1   X                                   X ..1/19
G#    20:1   X X   X X         X                   X ..1/20
G     21:1   X   X       X                           X ..1/21
F#    22:1   X X                 X                     X ..1/22
F#    23:1   X                                           X ..1/23
F     24:1   X X X X   X   X       X                       X ..1/24
E     25:1   X       X                                       X ..1/25
D#    26:1   X X                     X                         X ..1/26
D#    27:1   X   X           X                                   X ..1/27
D     28:1   X X   X     X             X                           X ..1/28
*     29:1   X                                                       X ..1/29
C#    30:1   X X X   X X       X         X                             X ..1/30
```

Output from Perl Script ths.pl

```
   (Showing the range of values generated by nested series for C, circa 1:1000)
C     1:968   X X   X       X     X                     X                 ..968
B     1:969   X   X                           X   X                       ..969
*97   1:970   X X     X         X                                         ..970
*     1:971   X                                                           ..971
B     1:972   X X X X   X     X     X           X                 X       ..972
*139  1:973   X           X                                               ..973
*487  1:974   X X                                                         ..974
C     1:975   X   X   X               X   X                   X           ..975
*61   1:976   X X   X       X               X                             ..976
*     1:977   X                                                           ..977
*163  1:978   X X X     X                                                 ..978
*89   1:979   X                   X                                       ..979
C     1:980   X X   X X   X     X       X           X               X     ..980
*109  1:981   X   X           X                                           ..981
*491  1:982   X X                                                         ..982
*     1:983   X                                                           ..983
*41   1:984   X X X X   X   X       X                       X             ..984
*197  1:985   X       X                                                   ..985
*29   1:986   X X                             X                       X   ..986
*47   1:987   X   X       X                           X                   ..987
C     1:988   X X   X                 X           X             X         ..988
*     1:989   X                                           X               ..989
C     1:990   X X X   X X     X X X       X     X       X               X ..990
*     1:991   X                                                           ..991
*31   1:992   X X   X       X               X                             ..992
*331  1:993   X   X                                                       ..993
*71   1:994   X X         X             X                                 ..994
*199  1:995   X       X                                                   ..995
*83   1:996   X X X X   X           X                                     ..996
*     1:997   X                                                           ..997
*499  1:998   X X                                                         ..998
*37   1:999   X   X           X                                   X       ..999
C     1:1000  X X   X X     X   X                   X         X           ..1000
C#    1:1001  X           X       X   X                                   ..1001
*167  1:1002  X X X     X                                                 ..1002
*59   1:1003  X                               X                           ..1003
*251  1:1004  X X   X                                                     ..1004
*67   1:1005  X   X   X                   X                               ..1005
*503  1:1006  X X                                                         ..1006
*53   1:1007  X                                   X                       ..1007
C     1:1008  X X X X   X X X X     X   X   X   X     X     X       X     ..1008
*     1:1009  X                                                           ..1009
*101  1:1010  X X     X         X                                         ..1010
*337  1:1011  X   X                                                       ..1011
*23   1:1012  X X   X             X                     X X               ..1012
*     1:1013  X                                                           ..1013
C#    1:1014  X X X     X             X                         X         ..1014
*29   1:1015  X       X   X                                           X   ..1015
*127  1:1016  X X   X       X                                             ..1016
*113  1:1017  X   X           X                                           ..1017
*509  1:1018  X X                                                         ..1018
*     1:1019  X                                                           ..1019
C     1:1020  X X X X X X       X   X     X   X     X                   X ..1020
*     1:1021  X                                                           ..1021
*73   1:1022  X X         X             X                                 ..1022
*31   1:1023  X   X               X                                       ..1023
C     1:1024  X X   X       X               X                             ..1024
*41   1:1025  X       X                                       X           ..1025
C     1:1026  X X X     X     X                 X X               X       ..1026
```

Absolute Analysis in Brief

(based on output from procalc.pl)

```
Ratio of Exchange|Fundamental Series| Nested Series  |    Number Processing
-----------------------------------------------------------------------------------
FundamentalSeriesH1->H? 172186884  × 10 = 1721868840
5/6     (6:5)   (h1) =  143489070  × 12 = 1721868840
3/2     (2:3)   (h1) =  215233605  ×  8 = 1721868840
2/3     (3:2)   (h1) =  143489070  × 12 = 1721868840 + (4 × 143489070) = 2295825120
2/3     (3:2)   (h1) =   95659380  × 24 = 2295825120 – (8 ×  95659380) = 1530550080
4/3     (3:4)   (h1) =  127545840  × 12 = 1530550080 + (4 × 127545840) = 2040733440
2/3     (3:2)   (h1) =   85030560  × 24 = 2040733440 + (12 × 85030560)
2/1     (1:2)   (h1) =  170061120  × 12 = 2040733440                    = 3061100160
9/8     (8:9)   (h1) =  191318760  × 16 = 3061100160 – (8 × 191318760) = 1530550080
2/3     (3:2)   (h1) =  127545840  × 12 = 1530550080 + (4 × 127545840) = 2040733440
4/3     (3:4)   (h1) =  170061120  × 12 = 2040733440
3/4     (4:3)   (h1) =  127545840  × 16 = 2040733440 + (4 × 127545840) = 2550916800
5/6     (6:5)   (h1) =  106288200  × 24 = 2550916800 – (8 × 106288200) = 1700611200
4/3     (3:4)   (h1) =  141717600  × 12 = 1700611200 + (4 × 141717600) = 2267481600
4/5     (5:4)   (h1) =  113374080  × 20 = 2267481600 – (2 × 113374080) = 2040733440
9/8     (8:9)   (h1) =  127545840  × 16 = 2040733440
4/3     (3:4)   (h1) =  170061120  × 12 = 2040733440 – (4 × 170061120) = 1360488960
2/3     (3:2)   (h1) =  113374080  × 12 = 1360488960 + (6 × 113374080) = 2040733440
9/8     (8:9)   (h1) =  127545840  × 16 = 2040733440
4/3     (3:4)   (h1) =  170061120  × 12 = 2040733440 + (4 × 127545840) = 2550916800
2/3     (3:2)   (h1) =  113374080  × 24 = 2720977920 – (4 × 113374080) = 2267481600
5/6     (6:5)   (h1) =   94478400  × 24 = 2267481600 + (8 ×  94478400) = 3023308800
4/3     (3:4)   (h1) =  125971200  × 24 = 3023308800 – (8 × 125971200) = 2015539200
4/3     (3:4)   (h1) =  167961600  × 12 = 2015539200
1/2     (2:1)   (h1) =   83980800  × 24 = 2015539200 + (8 ×  83980800) = 2687385600
4/3     (3:4)   (h1) =  111974400  × 24 = 2687385600 – (4 × 111974400) = 2239488000
5/4     (4:5)   (h1) =  139968000  × 16 = 2239488000
2/3     (3:2)   (h1) =   93312000  × 24 = 2239488000 – (8 ×  93312000) = 1492992000
4/3     (3:4)   (h1) =  124416000  × 12 = 1492992000 + (4 × 124416000) = 1990656000
1/2     (2:1)   (h1) =   62208000  × 32 = 1990656000
4/3     (3:4)   (h1) =   82944000  × 24 = 1990656000 – (8 ×  82944000) = 1327104000
4/3     (3:4)   (h1) =  110592000  × 12 = 1327104000 + (6 × 110592000) = 1990656000
9/8     (8:9)   (h1) =  124416000  × 16 = 1990656000
2/3     (3:2)   (h1) =   82944000  × 24 = 1990656000 – (8 ×  82944000) = 1327104000
4/3     (3:4)   (h1) =  110592000  × 12 = 1327104000 + (6 × 110592000) = 1990656000
9/16    (16:9)  (h1) =   62208000  × 32 = 1990656000
4/3     (3:4)   (h1) =   82944000  × 24 = 1990656000 + (8 ×  82944000) = 2654208000
4/3     (3:4)   (h1) =  110592000  × 24 = 2654208000 – (4 × 110592000) = 2211840000
5/6     (6:5)   (h1) =   92160000  × 24 = 2211840000 – (8 ×  92160000) = 1474560000
4/3     (3:4)   (h1) =  122880000  × 12 = 1474560000 + (4 × 122880000) = 1966080000
4/3     (3:4)   (h1) =  163840000  × 12 = 1966080000
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From the left: The first two columns contain the Just proportions that propel the composition's harmonic progression from start to finish. The third column gives the connection points where the nested series is attached to the integer ratios in the fundamental nesting series. Thus, whereas in the summary format, on page 6, the first chord/nested series attaches at C-h1(H32), in this absolute analysis it is C-h1(H172186884). The fourth column contains the number of note-ratios in each nested series and the value of the top ratio – the conjunction. The fifth group of figures details the number processing necessarily undertaken to allow the nested series to step from one to the next. The number processing either raises or lowers the value computed in the previous exchange so as to realign the nested series in preparation for the next up-coming interaction. These values then match the sum computed on the next line of the table. The whole integer precise.